Adv. Appl. Prob. **21**, 938–940 (1989)
Printed in N. Ireland

A CIRCULAR PROPERTY OF THE OCCURRENCE OF SEQUENCE PATTERNS IN THE FAIR COIN-TOSSING PROCESS

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Let $\Omega = \{0, 1\}$ and for each integer $n \geqq 1$ let $\Omega_n = \Omega \times \Omega \times \cdots \times \Omega$ (n-tuple) and $\Omega_n^k = \{(a_1, a_2, \cdots, a_n) \mid (a_1, a_2, \cdots, a_n) \in \Omega_n \text{ and } \sum_{i=1}^n a_i = k\}$ for all $k = 0, 1, \cdots, n$. Let $\{X_m\}_{m \geqq 1}$ be a sequence of i.i.d. random variables such that $P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2}$. Any element A in Ω_n is called a sequence pattern and for each sequence pattern A, let T_A be the first occurrence time of A (with respect to the process $\{X_m\}_{m \geqq 1}$) which is defined by $T_A(X_1, X_2, \cdots) = \inf \{m \mid (X_{m-n+1}, \cdots, X_m) = A\}$. For any two distinct sequence patterns A and B, A occurs (stochastically) after B (denoted by $B \ll A$) if $P(T_A > T_B) > \frac{1}{2}$. In this paper, we prove that if $n \geqq 4$, $k = 1, 2, 3 (n-3, n-2, n-1)$, and $n \neq 2k$, then there is an arrangement $\{B_1, B_2, \cdots, B_{\binom{n}{k}}\}$ of Ω_n^k such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{k}} \ll B_1$. Albeit we are not able to prove the statement for any $k \neq n/2$, from our proof for special cases we strongly believe that the statement is true for any $n \geqq 4$ and $k \neq n/2$. This new result reveals a circular property of the first occurrence among the sequence patterns in Ω_n^k and also provides us with an understanding of the regularity of the fair coin-tossing process. We start with the following notation and lemmas.

For each sequence pattern $A = (a_1, a_2, \cdots, a_n)$ and $B = (b_1, b_2, \cdots, b_n)$ (not necessarily distinct) in Ω_n, we define $A * B = \sum_{j=1}^n 2^j \varepsilon_j$, where for each $j = 1, 2, \cdots, n$, $\varepsilon_j = 1$ or 0 (according to whether $(b_1, b_2, \cdots, b_j) = (a_{n-j+1}, \cdots, a_n)$ or not.

Lemma 1. If A and B are two distinct sequence patterns in Ω_n, then $P(T_A < T_B)/P(T_B < T_A) = (B * B - B * A)/(A * A - A * B)$.

Lemma 2. If $1 \leqq k \leqq n-1$, $k \neq n/2$, $n \geqq 4$, and $A = (a_1, a_2, \cdots, a_n)$ is in Ω_n^k, then $B \ll A$, where $B = (a_n, a_1, a_2, \cdots, a_{n-1})$.

Theorem 1. If $n \geqq 4$, then there is an arrangement $\{B_1, B_2, \cdots, B_n\}$ of Ω_n^1 (or Ω_n^{n-1}) such that $B_1 \ll B_2 \ll \cdots \ll B_n \ll B_1$.

For each $A = (a_1, a_2, \cdots, a_n)$ in Ω_n, let $A^1 = A$ and $A^{j+1} = (a_{n-j+1}, \cdots, a_n, a_1, \cdots, a_{n-j})$ for all $j = 1, 2, \cdots, n-1$. For each $i = 0, 1, \cdots, n-2$, let $A_i = (1, 0, \overbrace{\cdots}^{i}, 0, 1, 0, \cdots, 0)$.

Lemma 3. If $n = 2m + 1$ and $i \leqq n - 3$, then $A_{i+1}^1 \ll A_i^n$ if $m \geqq 3$ or $m = 2$ but $i \neq 1$.

Lemma 4. If $n = 2m + 1 \geqq 5$ and $A = (1, 1, 0, \cdots, 0)$ is in Ω_n^2, then $A \ll A_i^n$ for all $i = 0, 1, 2, \cdots, n-2$.

Lemma 5. If $n = 2m + 1 \geqq 7$, then there is an arrangement $\{B_1, B_2, \cdots, B_{\binom{n}{2}}\}$ of Ω_n^2 (or Ω_n^{n-2}) such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{2}} \ll B_1$.

Proof. Let A_i and A_i^j be as defined just above Lemma 3 for all $j = 1, 2, \cdots, n$ and $i = 0, 1, \cdots, n-2$. By Lemma 2, $A_i^{j+1} \ll A_i^j$ for all $j = 1, 2, \cdots, n-1$ and all $i = 0, 1, \cdots, m-1$. By Lemma 3, $A_{i+1}^1 \ll A_i^n$ for all $i = 0, 1, \cdots, m-1$ (since $m \geqq 3$). By Lemma 4, $A_0^1 \ll A_{m-1}^n$. Therefore, $A_0^1 \ll A_{m-1}^n \ll \cdots \ll A_{m-1}^1 \ll A_{m-2}^n \ll \cdots \ll A_1^1 \ll A_0^n \ll \cdots \ll A_0^2 \ll A_0^1$. Since $n = 2m + 1 \geqq 7$ and $\{A_i^j \mid 1 \leqq j \leqq n \text{ and } 0 \leqq i \leqq m-1\} = \Omega_n^2$, the proof of Lemma 5 is now complete.

Received 17 January 1989; revision received 12 July 1989.

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https://doi.org/10.2307/1427778 Published online by Cambridge University Press

Lemma 6. If $n=2m\geqq 6$, then $A_{i+1}^1 \ll A_i^n$ for all $i=0, 1, \cdots, m-1$.

Lemma 7. If $n=2m\geqq 6$, then there is an arrangement $\{B_1, B_2, \cdots B_{\binom{n}{2}}\}$ of Ω_n^2 (or Ω_n^{n-2}) such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{2}} \ll B_1$.

Proof. By Lemma 2, $A_i^{j+1} \ll A_i^j$ for all $j=1, 2, \cdots, n-1$ and all $i=0, 1, \cdots, m-1$. By Lemma 6, $A_{i+1}^1 \ll A_i^n$ for all $i=0, 1, \cdots, m-2$. Now by a direct computation, $A_0^1 * A_0^1 = 2^{2m}$, $A_0^1 * A_{m-1}^m = 2^{m-1}$, $A_{m-1}^m * A_{m-1}^m = 2^{2m}+2^m$, and $A_{m-1}^m * A_0^1 = 2$. Hence $A_0^1 \ll A_{m-1}^m$. Therefore, $A_0^1 \ll A_{m-1}^m \ll \cdots \ll A_{m-1}^1 \ll A_{m-2}^n \ll \cdots \ll A_{m-2}^1 \ll \cdots \ll A_1^n \ll \cdots \ll A_1^1 \ll A_0^n \ll \cdots \ll A_0^2 \ll A_0^1$. Since $n=2m\geqq 6$ and $\{A_i^j \mid 0\leqq I\leqq m-2 \text{ and } 1\leqq j\leqq n\} \cup \{A_{m-1}^1, \cdots, A_{m-1}^m\} = \Omega_n^2$, the proof of Lemma 7 is now complete.

Theorem 2. If $n\geqq 5$, then there is an arrangement $\{B_1, B_2, \cdots, B_{\binom{n}{2}}\}$ of Ω_n^2 (or Ω_n^{n-2}) such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{2}} \ll B_1$.

Proof. If $n=5$, Theorem 2 can be proved by a direct computation and Lemma 1. If $n\geqq 6$, Theorem 2 is proved by Lemmas 5 and 7.

For each $i=0, 1, \cdots, n-3$, and $j=0, 1, \cdots, n-3-i$, let $A_{ij}=(1, 0, \overbrace{\cdots}^{i}, 0, 1, 0, \overbrace{\cdots}^{j}, 0, 1, 0, \cdots, 0)$ be a sequence pattern in Ω_n^3.

The following lemmas are essential to Theorem 3 below. However, the proofs of these lemmas are omitted.

Lemma 8. In $n\geqq 8$, then $A_{i+1,0}^1 \ll A_{0,n-i-3}^{i+1}$ for all $i=0, 1, \cdots, n-4$.

Lemma 9. If $n\geqq 8$, then $A_{11}^1 \ll A_{01}^{n-3}$.

Lemma 10. If $n=3m+3$ and $m\geqq 2$, then $A_{m,m}^m \ll A_{m,m-1}^{m+1}$.

Lemma 11. If $n=3m+3$ and $m\geqq 1$, then $A_{m,m-1}^{m+2} \ll A_{m,m}^1$.

Lemma 12. If $n=3m+3$ and $m\geqq 2$, then $A_{00}^1 \ll A_{m+1,m}^m$.

Lemma 13. If $1\leqq i\leqq n-2i-7$, then $A_{i+1,i+1}^1 \ll A_{n-2i-4,i+1}^{i+1}$.

Lemma 14. If $1\leqq i\leqq j$ and $i+1\leqq n-i-j-3$, then $A_{00}^1 \ll A_{j,n-i-j-3}^{i+1}$.

Lemma 15. If $1\leqq i\leqq j$ and $i+2\leqq n-i-j-3$, then $A_{i,j+1}^1 \ll A_{j,n-i-j-3}^{i+1}$.

Theorem 3. If $n\geqq 7$, then there is an arrangement $\{B_1, B_2, \cdots, B_{\binom{n}{3}}\}$ of Ω_n^3 (or Ω_n^{n-3}) such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{3}} \ll B_1$.

Proof. Since the case that $\{B_1, B_2, \cdots, B_{\binom{n}{3}}\} = \Omega_n^{n-3}$ can be proved by interchanging 0 and 1, we prove only the case that $\{B_1, B_2, \cdots, B_{\binom{n}{3}}\} = \Omega_n^3$. If $n=7$, then by a direct computation and Lemma 2, the sequence patterns in Ω_7^3 can be arranged as follows:

$$\begin{aligned}(1110000) &\ll (0101001) \ll (1010010) \ll (0100101) \ll (1001010) \ll (0010101)\\ &\ll (0101010) \ll (1010100) \ll (0001011) \ll (0010110) \ll (0101100) \ll (1011000)\\ &\ll (0110001) \ll (1100010) \ll (1000101) \ll (1001001) \ll (0010011) \ll (0100110)\\ &\ll (1001100) \ll (0011001) \ll (0110010) \ll (1100100) \ll (0001101) \ll (0011010)\\ &\ll (0110100) \ll (1101000) \ll (1010001) \ll (0100011) \ll (1000110) \ll (1100001)\\ &\ll (1000011) \ll (0000111) \ll (0001110) \ll (0011100) \ll (0111000) \ll (1110000)\end{aligned}$$

Now we assume that $k=3$ and $n\geqq 8$. When $n\neq 3m+3$ for some positive integer m, Theorem 3 is proved by combining Lemmas 2, 8, 9, 13, 14, and 15. When $n=3m+3$ for some positive integer m, Theorem 3 is proved by combining Lemmas 2, 8, 13, 14, 15, and at certain steps, Lemmas 10, 11, 12 will be used to avoid the cyclic behavior. Since the detailed proof is very lengthy, we omit it.

From our constructive proofs of Theorems 2 and 3, we strongly believe that Theorem 3 holds in general, i.e., if $n \geqq 4$, $k = 1, 2, \cdots, n-1$, and $k \neq n/2$, then there is an arrangement $\{B_1, B_2, \cdots, B_{\binom{n}{k}}\}$ of Ω_n^k such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{n}{k}} \ll B_1$.

Chen and Lin (1984) proved that if $n \geqq 4$ and $n = 2k$, then $P(T_A < T_{A_i}) = \frac{1}{2}$ for $i = 1, 2$ and $P(T_A < T_D) < \frac{1}{2}$ for all $D \in \Omega_{2k}^k - \{A, A_1, A_2\}$; here $A = (0, 1, 0, 1, \cdots, 0, 1)$, $A_1 = (1, 0, 1, 0, \cdots, 1, 0)$, and $A_2 = (0, 1, 0, 1, \cdots, 0, 1, 1, 0)$. Hence there does not exist an arrangement $\{B_1, B_2, \cdots, B_{\binom{2k}{k}}\}$ of Ω_{2k}^k such that $B_1 \ll B_2 \ll \cdots \ll B_{\binom{2k}{k}} \ll B_1$.

Under some mild conditions, the conjecture and the results in this paper can presumably be extended to the situation in which $\Omega = \{1, 2, \cdots, r\}$ and $P(X_1 = i) = 1/r$ for all $i = 1, 2, \cdots, r$; here r is a positive integer $\geqq 3$. Chen and Zame (1979) also briefly discussed this case.

Acknowledgement

We should like to thank Dr Frank K. Hwang for some useful conversation about the circular ordering among elements in a set. We also sincerely thank the referee for his invaluable comments.

References

Chen, R. and Lin, H. E. (1984) On the fair coin tossing process. *J. Multivariate Anal.* **15,** 222–227.

Chen, R. and Zame, A. (1979) On fair coin-tossing games. *J. Multivariate Anal.* **9,** 150–157.

https://doi.org/10.2307/1427778 Published online by Cambridge University Press