REFLECTIVE SUBCATEGORIES, LOCALIZATIONS AND FACTORIZATION SYSTEMS

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Abstract

This work is a detailed analysis of the relationship between reflective subcategories of a category and factorization systems supported by the category.

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1. Introduction

We use "subcategory" to mean "full replete subcategory". Although some authors call any reflexion of a category $\mathcal{C}$ onto a subcategory $\mathcal{B}$ a *localization* of $\mathcal{C}$, we follow the more common practice of reserving "localization" for "*left-exact reflexion*".

Localizations have been extensively studied for certain classes of categories $\mathcal{C}$. When $\mathcal{C}$ is a presheaf category $[\mathcal{K}, \mathbf{Set}]$ they correspond to the Grothendieck topologies on $\mathcal{K}$; and more generally, when $\mathcal{C}$ is any topos, they correspond to the Grothendieck-Lawvere-Tierney topologies on $\mathcal{C}$; see [9]. When $\mathcal{C}$ is an additive functor category $[\mathcal{K}, \mathbf{Ab}]$, such as a category of modules, they correspond to the Gabriel topologies, which are an additive analogue of the Grothendieck ones; see [13] and [14]. This has been further generalized by Borceux [3], replacing **Set** and **Ab** by the symmetric monoidal closed category of algebras for any commutative Lawvere theory.

Our present concern is much wider, and must accordingly be less deep in particulars. We consider reflexions in general, localizations being only a special case; and we ask of $\mathcal{C}$ only that it be *finitely well-complete* (f.w.c.). By this we mean that $\mathcal{C}$ admits finite limits and all intersections (even large ones, if need be) of strong subobjects in the sense of Kelly [10]. Clearly a complete $\mathcal{C}$ is f.w.c. if each object has only a small set of strong subobjects; but so too is the incomplete category of finite sets.

Any factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ determines a reflective subcategory $\Psi\mathcal{F} = \mathcal{M}/1$, consisting of those objects A for which the map $A \to 1$ (into the terminal object) lies in $\mathcal{M}$. Any reflective subcategory $\mathcal{B}$ of $\mathcal{C}$ determines at least a *pre*factorization system $\Phi\mathcal{B}$ on $\mathcal{C}$, whose $\mathcal{E}$-part is the set of maps inverted by the reflexion $\mathcal{C} \to \mathcal{B}$; and $\Phi\mathcal{B}$ is actually a factorization system when $\mathcal{C}$ is f.w.c. What we have here is an adjunction $\Phi \dashv \Psi$ between the ordered set of factorization systems on $\mathcal{C}$ and the ordered set of reflective subcategories, for which $\Psi\Phi = 1$. Accordingly there is a bijection between the reflective subcategories and those factorization systems—we call them the *reflective* ones—which lie in the image of Φ; and there is an *interior operation* $\Phi\Psi$ which sends a general factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ to its *reflective interior* $\mathring{\mathcal{F}} = (\mathring{\mathcal{E}}, \mathring{\mathcal{M}})$. It turns out that $g \in \mathring{\mathcal{E}}$ precisely when $fg \in \mathcal{E}$ for some $f \in \mathcal{E}$; so that the factorization system $\mathcal{F}$ is reflective exactly when $fg \in \mathcal{E}$ and $f \in \mathcal{E}$ imply $g \in \mathcal{E}$. We deal with this in Sections 2 and 3. In doing so we establish our result that $\Phi\mathcal{B}$ is a factorization system in still greater generality, replacing a reflexion by a general adjunction; and thereby give a new proof of Day's Theorem [6] on the factorization of a left adjoint into a reflexion followed by a conservative left adjoint.

We consider in Section 4 those reflexions $r \colon \mathcal{C} \to \mathcal{B}$ for which the process of forming the $\Phi\mathcal{B}$-factorization of a map in $\mathcal{C}$ ends after the first step, so that mere *finite completeness* of $\mathcal{C}$ suffices for $\Phi\mathcal{B}$ to be a factorization system; we call these the *simple* reflexions. We show that every localization is simple, and that for a finitely-complete $\mathcal{C}$ the localizations correspond in the bijection above to those reflective factorization systems $(\mathcal{E}, \mathcal{M})$ for which $\mathcal{E}$ is *stable under pullbacks*.

The connexion with the classical results on localization—which we do not pursue below—is as follows. In any reasonable $\mathcal{C}$, every map f has the form ip where i is a monomorphism and p is a strong epimorphism. Write j for the equalizer of the kernel-pair of f. Then f is inverted by a localization r if and only if i and j are inverted; so that the corresponding $\mathcal{E}$ is fully determined by its intersection $\mathcal{E}'$ with the monomorphisms. In the classical cases, the existence of good generators or a subobject-classifier enables us to describe $\mathcal{E}'$ by something quite small and easy to handle; and this is the *topology* of the appropriate kind.

When $\mathcal{B}$ is reflective, so is its closure under subobjects $\mathcal{B}^{\#}$, for any reasonable $\mathcal{C}$, and certainly when $\mathcal{C}^{\mathrm{op}}$ is f.w.c.; we recall this in Section 5, and calculate $\Phi\mathcal{B}^{\#}$ in terms of $\Phi\mathcal{B}$.

Henceforth we usually suppose that $\mathcal{Q}^{op}$ as well as $\mathcal{Q}$ if f.w.c., and consider coreflective subcategories as well as reflective ones. A factorization system $\mathcal{F}$ now has a *coreflective closure* $\overline{\mathcal{F}}$ as well as a reflective interior $\overset{\circ}{\mathcal{F}}$. These two operations, applied successively to any $\mathcal{F}$, lead to at most seven factorization systems, which in general are distinct. These seven reduce to three if we begin with a reflective $\mathcal{F}$, corresponding to a reflective subcategory $\mathcal{B}$ say. The coreflective subcategory $\mathcal{B}^{\leftarrow} = 0/\mathcal{E}$ corresponds to the coreflective factorization system $\overline{\mathcal{F}}$, which then gives the reflective subcategory $\mathcal{B}^{\leftarrow\rightarrow} = \overline{\mathcal{M}}/1$, containing $\mathcal{B}$ and corresponding to $\overset{\circ}{\overline{\mathcal{F}}}$. Here the process stops; $\mathcal{C} = \mathcal{B}^{\leftarrow}$ and $\mathcal{D} = \mathcal{B}^{\leftarrow\rightarrow}$ satisfy $\mathcal{C}^{\rightarrow} = \mathcal{D}$ and $\mathcal{D}^{\leftarrow} = \mathcal{C}$. Generalizing from the case of a module-category $\mathcal{Q}$, we may call such a pair $(\mathcal{C}, \mathcal{D})$ a *torsion theory*. The above is the content of Section 6.

When $\mathcal{Q}$ is *pointed*, the coreflexion onto $\mathcal{B}^{\leftarrow}$ may be obtained by iterating the kernel of the reflexion onto $\mathcal{B}$; sometimes no iteration is needed, and we then call the reflexion *normal*; every simple reflexion is normal. We treat this in Section 7. In Section 8 we turn to the study of torsion theories for a pointed $\mathcal{Q}$, and derive relations between normality, simplicity, and the property $\mathcal{B}^{\leftarrow\rightarrow} = \mathcal{B}^{\#}$, under various "exactness" conditions on $\mathcal{Q}$. Finally, in Section 9, we return to reflexions which preserve some or all finite limits, and make connexions with those classical results on localizations expressed (see [13] and [14]) in terms of *hereditary torsion theories*.

2. Prefactorization systems and reflexions

We recall the notion of a factorization system, in the sense of [7], on a category $\mathcal{Q}$. If p and i are maps in $\mathcal{Q}$, we write $p \downarrow i$ if, for every commutative square $vp = iu$, there is a unique "diagonal" w with $wp = u$ and $iw = v$. If $\mathcal{N}$ is any class of maps in $\mathcal{Q}$ we write $\mathcal{N}^{\uparrow} = \{p \mid p \downarrow n$ for all $n \in \mathcal{N}\}$ and $\mathcal{N}^{\downarrow} = \{i \mid n \downarrow i$ for all $n \in \mathcal{N}\}$. A *factorization system* $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ on $\mathcal{Q}$ consists of two classes $\mathcal{E}$ and $\mathcal{M}$ of maps, each containing the isomorphisms and closed under composition, such that every map has a factorization $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$, and such that $\mathcal{M}^{\uparrow} \subset \mathcal{E}$ (or equivalently $\mathcal{E}^{\downarrow} \subset \mathcal{M}$). This last condition is in effect the assertion that $(\mathcal{E}, \mathcal{M})$ factorizations are functorial; see [7].

A *prefactorization system* $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ just consists of classes $\mathcal{E}$ and $\mathcal{M}$ with $\mathcal{E}^{\downarrow} = \mathcal{M}$ and $\mathcal{M}^{\uparrow} = \mathcal{E}$. Every factorization system is a prefactorization system, and a prefactorization system is a factorization system just when every map f does admit an $(\mathcal{E}, \mathcal{M})$ factorization as above. For every class $\mathcal{N}$ of maps, $(\mathcal{N}^{\uparrow}, \mathcal{N}^{\uparrow\downarrow})$ is a prefactorization system. Accordingly the prefactorization systems form a complete lattice, if we order them by setting $(\mathcal{E}, \mathcal{M}) \leq (\mathcal{E}', \mathcal{M}')$ when $\mathcal{M} \subset \mathcal{M}'$ (or equivalently $\mathcal{E} \supset \mathcal{E}'$). The greatest element $\mathbf{1} = (\mathcal{I}so, \mathcal{Q}ll)$ of this lattice is the

factorization system in which $\mathscr{E}$ is the isomorphisms and $\mathfrak{M}$ is all maps; and the least element $\mathbf{0}$ is $(\mathcal{Q}ll, \mathscr{I}so)$.

In any prefactorization system, $\mathscr{E} \cap \mathfrak{M}$ consists of the isomorphisms, and $\mathfrak{M}$ is closed under composition, pullbacks, products, and fibred products; moreover we have the cancellation property

$$(\Delta) \qquad\qquad if\ fg \in \mathfrak{M}\ and\ f \in \mathfrak{M}\ then\ g \in \mathfrak{M},$$

with the dual properties for $\mathscr{E}$; see [7] Proposition 2.1.1.

For a map n in $\mathcal{Q}$ and an object B of $\mathcal{Q}$, we write $n \perp B$ if $\mathcal{Q}(n, B)$ is invertible. For any class $\mathfrak{N}$ of maps we set $\mathfrak{N}^{\perp} = \{B \mid n \perp B \text{ for all } n \in \mathfrak{N}\}$, identifying this set of objects with the corresponding subcategory of $\mathcal{Q}$; and for any subcategory $\mathscr{B}$ of $\mathcal{Q}$ we set $\mathscr{B}^{\top} = \{n \mid n \perp B \text{ for all } B \in \mathscr{B}\}$. If $S: \mathcal{Q} \to \mathcal{C}$ is any functor we write Σ_S for the class of maps in $\mathcal{Q}$ inverted by S. We order subcategories of $\mathcal{Q}$ by inclusion.

Now suppose that $\mathcal{Q}$ contains a terminal object 1. We get an order-preserving map Ψ from the set of prefactorization systems satisfying

$$(*) \qquad\qquad every\ A \to 1\ admits\ an\ (\mathscr{E}, \mathfrak{M})\ factorication$$

to the set of reflective subcategories of $\mathcal{Q}$, by setting

$$(2.1) \qquad\qquad \Psi(\mathscr{E}, \mathfrak{M}) = \mathfrak{M}/1,$$

the subcategory of those $B \in \mathcal{Q}$ with $B \to 1$ in $\mathfrak{M}$. The reflexion $\rho_A: A \to rA$ of A onto $\mathfrak{M}/1$ is obtained by taking the $(\mathscr{E}, \mathfrak{M})$ factorization of $A \to 1$; observe that

$$(2.2) \qquad\qquad \rho_A \in \mathscr{E} \quad and \quad \mathscr{E} \subset \Sigma_r.$$

On the other hand, we get an order-preserving map Φ from the set of reflective subcategories to the set of prefactorization systems by setting

$$(2.3) \qquad\qquad \Phi\mathscr{B} = \big((\text{mor } \mathscr{B})^{\top}, (\text{mor } \mathscr{B})^{\top\downarrow}\big).$$

Note that, if $\rho_A: A \to rA$ is the reflexion, we have

$$(2.4) \qquad (\text{mor } \mathscr{B})^{\top} = \{B \to 1 \mid B \in \mathscr{B}\}^{\top} = \mathscr{B}^{\top} = \Sigma_r,$$

the first equality here arising from (Δ) and the last from the observation that $\mathscr{B}(r(f), B)$, and so $\mathcal{Q}(f, B)$, is invertible for all $B \in \mathscr{B}$ precisely when $r(f)$ is invertible. In particular

$$(2.5) \qquad\qquad \rho_A \in \Sigma_r = (\text{mor } \mathscr{B})^{\top},$$

whence $\Phi\mathscr{B}$ satisfies $(*)$, the factorization of $A \to 1$ being $A \to rA \to 1$.

LEMMA 2.1. *Writing $\{\rho\}$ for the class of all the reflexions $\rho_A: A \to rA$, let $\mathfrak{N}$ be any class of maps with $\{\rho\} \subset \mathfrak{N} \subset \Sigma_r$. Then $\mathfrak{N}^{\downarrow}/1 = \mathfrak{N}^{\perp} = \mathscr{B}$.*

PROOF. That $\mathfrak{N}^{\perp}/1 = \mathfrak{N}^{\perp}$ is immediate. We have $\Sigma_r^{\perp} \subset \mathfrak{N}^{\perp} \subset \{\rho\}^{\perp}$. By the last equality in (2.4), we have $\Sigma_r^{\perp} = \mathfrak{B}^{\perp\top} \supset \mathfrak{B}$. However $\{\rho\}^{\perp} \subset \mathfrak{B}$, since every retract of an object of a reflective $\mathfrak{B}$ lies in $\mathfrak{B}$.

The case $\mathfrak{N} = \Sigma_r$ of this gives $\Psi\Phi = 1$, by (2.4). On the other hand $\Phi\Psi \leqslant 1$ by (2.2) and (2.4); thus
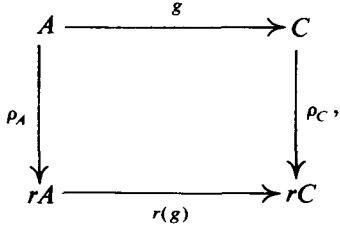
PROPOSITION 2.2. *We have an adjunction* $\Phi \dashv \Psi$ *between the prefactorization systems satisfying* $(*)$ *and the reflective subcategories; and we have* $\Psi\Phi = 1$.

It follows that $\Psi\Phi$ is an interior operation on the set of prefactorization systems satisfying $(*)$, sending $\mathfrak{F} = (\mathcal{E}, \mathfrak{N})$ to what we shall call its *reflective interior* $\mathring{\mathfrak{F}} = (\mathring{\mathcal{E}}, \mathring{\mathfrak{N}})$; and that Φ and Ψ restrict to a bijection between the reflective subcategories and those prefactorization systems satisfying $(*)$ for which $\mathring{\mathfrak{F}} = \mathfrak{F}$. We call these the *reflective* prefactorization systems.

THEOREM 2.3. *Let* $\mathring{\mathfrak{F}}$ *be the reflective interior of* $\mathfrak{F}$. *Then* $g \in \mathring{\mathcal{E}}$ *precisely when* $fg \in \mathcal{E}$ *for some* $f \in \mathcal{E}$. *Thus* $\mathfrak{F}$ *is reflective precisely when it satisfies*

$(\dagger)$ *if* $fg \in \mathcal{E}$ *and* $f \in \mathcal{E}$ *then* $g \in \mathcal{E}$.

PROOF. Let $\rho_A: A \to rA$ be the reflexion onto $\mathfrak{N}/1$, so that $\mathring{\mathcal{E}} = \Sigma_r$ by (2.4). Since $\mathcal{E} \subset \mathring{\mathcal{E}}$, we conclude from $fg \in \mathcal{E}$ and $f \in \mathcal{E}$ that $g \in \Sigma_r = \mathring{\mathcal{E}}$. For the converse, let $g: A \to C$ lie in $\mathring{\mathcal{E}}$. In the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ g\ \ } & C \\
{\scriptstyle \rho_A}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_C} \\
rA & \xrightarrow{\ r(g)\ } & rC
\end{array}
$$

ρ_C and ρ_A lie in $\mathcal{E}$ by (2.2), and $r(g)\rho_A \in \mathcal{E}$ since $r(g)$ is invertible.

REMARK 2.4. Contrast $(\dagger)$ with the dual of (Δ).

REMARK 2.5. There is a connexion with categories of fractions. Let $\mathfrak{F} = (\mathcal{E}, \mathfrak{N})$ be a prefactorization system satisfying $(*)$. By Theorem 2.3, any functor with domain $\mathcal{C}$ that inverts every element of $\mathcal{E}$ also inverts every element of $\mathring{\mathcal{E}}$; but the reflexion of $\mathcal{C}$ onto $\mathfrak{N}/1$ inverts precisely the elements of $\mathring{\mathcal{E}}$. Thus $\mathring{\mathcal{E}}$ is the saturation of $\mathcal{E}$ in the category-of-fractions sense, and $\mathcal{E}$ is saturated precisely when $\mathfrak{F}$ is reflective. It is easy to see that the category of fractions $\mathcal{C}[\mathcal{E}^{-1}]$ is equivalent to $\mathfrak{N}/1$.

3. The existence of $\Phi\mathfrak{B}$-factorizations for f.w.c. $\mathcal{C}$

It is by no means true that every prefactorization system on a complete and cocomplete, wellpowered and cowellpowered category $\mathcal{C}$ is a factorization system. Adámek [1] gives a pointed endofunctor T on the category **Gph** of graphs such that the category $\mathfrak{B}$ of T-algebras is not cocomplete. Hence the full subcategory $\mathfrak{B}$ of $\mathcal{C} = T/\mathbf{Gph}$ is not reflective. Yet by Theorem 14.4 and Proposition 5.2 of Kelly [11], $\mathfrak{B}$ is of the form $\mathfrak{N}^\perp$ in $\mathcal{C}$, and is therefore $\mathfrak{M}/1$ where $(\mathcal{E}, \mathfrak{M})$ is the prefactorization system $(\mathfrak{N}^{\downarrow\uparrow}, \mathfrak{N}^\downarrow)$. Since $\mathfrak{B}$ is not reflective, this is not a factorization system.

Write $\mathcal{E}pi$ and $\mathfrak{M}on$ for the classes of epimorphisms and monomorphisms in $\mathcal{C}$. Recall from [10] that the class $\mathcal{S}\mathfrak{M}on$ of *strong monomorphisms* is $\mathfrak{M}on \cap (\mathcal{E}pi)^\downarrow$; strong epimorphisms are defined dually. If either $\mathcal{C}$ or $\mathcal{C}^{op}$ is finitely complete (f.c.), it follows by two applications of Proposition 2.1.4 of [7] that $\mathcal{S}\mathfrak{M}on = (\mathcal{E}pi)^\downarrow$ and that $(\mathcal{S}\mathfrak{M}on)^\uparrow = \mathcal{E}pi$; so in these cases we have prefactorization systems $(\mathcal{E}pi, \mathcal{S}\mathfrak{M}on)$ and $(\mathcal{S}\mathcal{E}pi, \mathfrak{M}on)$. In any case $\mathcal{S}\mathfrak{M}on$ has the closure properties mentioned above for the $\mathfrak{M}$ of a prefactorization system, since $\mathfrak{M}on$ does. Moreover, f is a strong monomorphism if gf is. (A strong monomorphism is an extremal one, and the converse is true if $\mathcal{C}$ admits pushouts or if $(\mathcal{E}pi, \mathcal{S}\mathfrak{M}on)$ is a factorization system: Lemma 2.3.3 of [7] is inexact on this point.) As we said in the introduction, we call $\mathcal{C}$ *finitely well-complete* (f.w.c.) if it admits, besides finite limits, all intersections of strong monomorphisms.

A monomorphism is *regular* if it is a joint equalizer of a family of parallel pairs of maps; then it is the equalizer of its cokernel-pair if the latter exists. Every regular monomorphism is a strong one, and every coretraction is a regular monomorphism.

Any map f factorizes as $f = jq$ through the joint equalizer j of all the pairs u, v with $uf = vf$, if this joint equalizer exists; in which case we call j the *regular image* of f. By Lemma 4.1 and Proposition 4.2 of [10], regular images exist if either $\mathcal{C}$ is f.w.c. or both $\mathcal{C}$ and $\mathcal{C}^{op}$ are f.c. When regular images exist, the following are equivalent: the q in $f = jq$ above is always epimorphic; the regular monomorphisms coincide with the strong ones; the regular monomorphisms are closed under composition. If regular images exist and $\mathcal{C}^{op}$ is f.c., these conditions certainly hold if every pushout of a regular monomorphism is a monomorphism; the converse is false in general, but if $\mathcal{C}^{op}$ is f.c. and additive, every pushout of a regular monomorphism is a monomorphism if every strong monomorphism is regular. For all this, see [10]. A related result, which we express in the dual form, is the following unpublished observation of A. Joyal. The category $\mathcal{C}$ is said to be *regular* if it is f.c., if $(\mathcal{S}\mathcal{E}pi, \mathfrak{M}on)$ is a factorization system, and if $\mathcal{S}\mathcal{E}pi$ is stable under pullbacks; *in a regular $\mathcal{C}$, the strong and the regular epimorphisms coincide*.

Of course $(\mathcal{E}pi, \mathcal{SM}on)$ is a factorization system, by the above, when $\mathcal{Q}$ and $\mathcal{Q}^{op}$ are f.c. and the regular monomorphisms coincide with the strong ones; but such categories are very special, and we need a much wider result. The following lemma generalizes a result of Kennison [12]:

LEMMA 3.1. *Let $\mathfrak{N}$ be a class of monomorphisms in $\mathcal{Q}$, closed under composition; let $\mathcal{Q}$ admit all pullbacks (along any map) of maps in $\mathfrak{N}$, and all intersections of maps in $\mathfrak{N}$; and let these pullbacks and intersections again belong to $\mathfrak{N}$. Then $(\mathfrak{N}^{\uparrow}, \mathfrak{N})$ is a factorization system.*
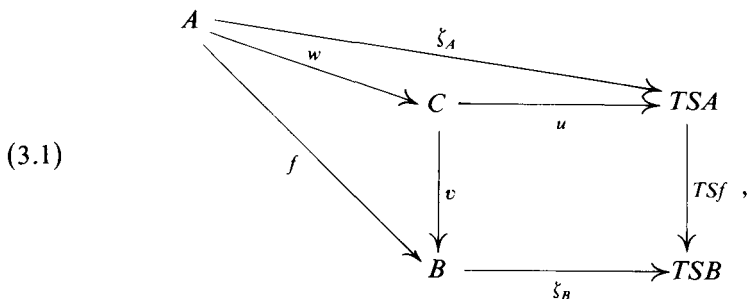
PROOF. $f: A \to B$ factorizes as me, where m is the intersection of the $\mathfrak{N}$-subobjects of B through which f factorizes; it is easy to see that $e \in \mathfrak{N}^{\uparrow}$, by pulling-back the "test-element" of $\mathfrak{N}$.

COROLLARY 3.2. *$(\mathcal{E}pi, \mathcal{SM}on)$ is a factorization system if (i) $\mathcal{Q}$ is f.w.c., or (ii) $\mathcal{Q}$ is finitely cocomplete and admits all cointersections of epimorphisms, or (iii) $\mathcal{Q}$ and $\mathcal{Q}^{op}$ are f.c. and all strong monomorphisms are regular.*

Our central result on the existence of factorizations applies to prefactorization systems formed like the $\Phi\mathcal{B}$ of (2.3), but for a *general* adjunction:

THEOREM 3.3. *Consider an adjunction $(\zeta, \varepsilon): S \dashv T: \mathcal{C} \to \mathcal{Q}$ where $\mathcal{Q}$ is f.w.c., and let $(\mathcal{E}, \mathfrak{M})$ be the prefactorization system $((T(\mathrm{mor}\,\mathcal{C}))^{\uparrow}, (T(\mathrm{mor}\,\mathcal{C}))^{\uparrow\downarrow})$ on $\mathcal{Q}$. Then $\mathcal{E} = \Sigma_S$ and $(\mathcal{E}, \mathfrak{M})$ is a factorization system.*

PROOF. It is clear—see Lemma 4.2.1 of [7]—that $\mathcal{E} = \Sigma_S$. Given $f: A \to B$ in $\mathcal{Q}$, consider the diagram

(3.1)

where the square is a pullback. Since $TSf \in T(\mathrm{mor}\,\mathcal{C}) \subset \mathfrak{M}$, its pullback v is in $\mathfrak{M}$; this it suffices to give an $(\mathcal{E}, \mathfrak{M})$ factorization of w.

Let $\mathfrak{N} = \mathfrak{M} \cap \mathcal{SM}on$; then $(\mathfrak{N}^{\uparrow}, \mathfrak{N})$ is a factorization system by Lemma 3.1. Let the $(\mathfrak{N}^{\uparrow}, \mathfrak{N})$ factorization of w be nf'. Since $n \in \mathfrak{N} \subset \mathfrak{M}$, it suffices to show that $f' \in \mathcal{E}$.

The equality $uw = \zeta_A$ in the diagram (3.1) above translates under the adjunction into $\bar{u} \cdot Sw = 1$ where $\bar{u}: SC \to SA$ is the image of u. Thus Sw is a coretraction, whence Sf' is a coretraction. It thus suffices to show that $f \in \mathcal{E}$ whenever $f \in \mathcal{N}^\uparrow$ and Sf is a coretraction.

Consider the diagram (3.1) again, but for such an f. Since Sf is a coretraction, TSf is a coretraction and hence a strong monomorphism, whence its pullback v is a strong monomorphism. Thus $v \in \mathcal{N}$ (since $v \in \mathcal{M}$). Since $f \in \mathcal{N}^\uparrow$ it follows that the monomorphism v is invertible; we may as well suppose that $v = 1$, so that $f = w$.

Now the diagram (3.1) gives $TSf.u = \zeta_B$, which transforms under the adjunction to $Sf.\bar{u} = 1$. Since we already have $\bar{u}.Sf = \bar{u}.Sw = 1$, it follows that Sf is invertible and $f \in \mathcal{E}$.

COROLLARY 3.4. *When $\mathcal{C}$ is f.w.c. the prefactorization system $\Phi\mathcal{B}$ of (2.3) corresponding to a reflective subcategory $\mathcal{B}$ is a factorization system; the adjunction $\Phi \dashv \Psi$ of Proposition 2.2 restricts to one between factorization systems and reflective subcategories; and the reflective interior $\overset{\circ}{\mathcal{F}}$ of a factorization system $\mathcal{F}$ is a factorization system.*

From Theorem 3.3 we also get a simple proof of the following result of Day [6], which itself generalizes and simplifies an argument of Applegate and Tierney:

PROPOSITION 3.5 (Day). *If $\mathcal{C}$ is f.w.c., every left adjoint $S: \mathcal{C} \to \mathcal{C}$ factorizes, to within isomorphism, as a reflexion $r: \mathcal{C} \to \mathcal{B}$ followed by a conservative (= isomorphism-reflecting) left adjoint $Q: \mathcal{B} \to \mathcal{C}$.*

PROOF. With $S \dashv T$, let $(\mathcal{E}, \mathcal{M})$ be the factorization system on $\mathcal{C}$ given by Theorem 3.3. It is clearly reflective since $\mathcal{E} = \Sigma_S$ satisfies (†) of Theorem 2.3; let $\mathcal{B}$ be the corresponding reflective subcategory $\mathcal{M}/1$ of $\mathcal{C}$, with inclusion $j: \mathcal{B} \to \mathcal{C}$ and reflexion $r: \mathcal{C} \to \mathcal{B}$. Then $T: \mathcal{C} \to \mathcal{C}$ factorizes as $T = jP$ for some $P: \mathcal{C} \to \mathcal{B}$, since $TC \in \mathcal{E}^\perp = \mathcal{B}$. Hence P has the left adjoint $Q = Sj$, so that $S \cong Qr$. The functor Q is conservative, since $\Sigma_Q = \Sigma_S \cap \text{mor } \mathcal{B} \subset \mathcal{E} \cap \mathcal{M}$, which consists of the isomorphisms.

REMARK 3.6. It follows easily from Theorem 4.7 below (with $\mathcal{C}$ now only f.c.) that if the S above is left exact, so are r and Q. Thus Day's result also generalizes that of Lawvere-Tierney (see Theorem 4.14 of [9]) on the factorization of geometric morphisms between topoi.

4. Cases where finite limits suffice; localizations

We transcribe the factorization process of Theorem 3.3. when ζ_A is the reflexion $\rho_A: A \to rA$ onto a reflective subcategory $\mathcal{B}$ of $\mathcal{C}$, and $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ is the reflective prefactorization system $\Phi\mathcal{B}$. We suppose for the moment only that $\mathcal{C}$ admits finite limits. For each $K \in \mathcal{C}$ and each $g: B \to rK$ with $B \in \mathcal{B}$, let

(4.1)

$$
\begin{array}{ccc}
J(g) & \xrightarrow{\;g_0\;} & B \\
{\scriptstyle g_1}\Big\downarrow & & \Big\downarrow{\scriptstyle g} \\
K & \xrightarrow[\;\rho_K\;]{} & rK
\end{array}
$$

denote the pullback. For any map $f: A \to K$ in $\mathcal{C}$ the diagram (3.1) now becomes

(4.2)

we have $r(f)_1 \in \mathcal{M}$, so that we obtain an $(\mathcal{E}, \mathcal{M})$ factorization of f by finding one of $\bar{f}$. In certain cases, however, $\bar{f}$ *already* lies in $\mathcal{E}$; then (4.2) provides an $(\mathcal{E}, \mathcal{M})$ factorization of f, under no hypotheses but our blanket one (for this section) that $\mathcal{C}$ admits finite limits. We say that the reflexion r onto $\mathcal{B}$ is *simple* if $\bar{f} \in \mathcal{E}$ for every map f in $\mathcal{C}$.

THEOREM 4.1. *Let* $\rho: 1 \to r$ *be a reflexion of the f.c.* $\mathcal{C}$ *onto* $\mathcal{B}$. *Then the following are equivalent, and imply that* $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$ *is a factorization system:*
   (i) $\bar{f} \in \mathcal{E}$ *in* (4.2) *for all* f; *that is, the reflexion* r *is simple.*
   (ii) *A map* f *lies in* $\mathcal{M}$ *if and only if*

(4.3)

$$
\begin{array}{ccc}
A & \xrightarrow{\;\rho_A\;} & rA \\
{\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle r(f)} \\
K & \xrightarrow[\;\rho_K\;]{} & rK
\end{array}
$$

is a pullback.

(iii) *For each map* $g: B \to rK$ *in* $\mathfrak{B}$ *we have a pullback*

(4.4)

$$
\begin{array}{ccc}
J(g) & \xrightarrow{\ \rho_{J(g)}\ } & rJ(g) \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow {\scriptstyle r(g_0)} \\
J(g) & \xrightarrow[\ g_0\ ]{} & B
\end{array}
$$

,

where $J(g)$ *and* g_0 *are as in* (4.1).

(iv) *For each map* $g: B \to rK$ *in* $\mathfrak{B}$, *with* g_0 *as in* (4.1), *the map* $r(g_0)$ *is invertible if it is a retraction.*

When $\mathcal{Q}$ *is additive, the conditions are further equivalent to*

(v) *For each* $g: B \to rK$ *in* $\mathfrak{B}$, *the map* $r(g_0)$ *is a monomorphism.*

PROOF. To say that (4.3) is a pullback is to say that $\bar{f}$ is invertible; hence (i) implies (ii) since $r(f)_1 \in \mathfrak{M}$. To see that (ii) implies (iii), consider the diagram (4.3) when f is the g_1 of (4.1), which lies in $\mathfrak{M}$. Since we may always suppose that $r(\rho_K) = 1$, we have $r(g_1) = gr(g_0)$. Thus, in the commutative diagram

$$
\begin{array}{ccc}
J(g) & \xrightarrow{\ \rho_{J(g)}\ } & rJ(g) \\
{\scriptstyle 1}\big\downarrow & & \big\downarrow {\scriptstyle r(g_0)} \\
J(g) & \xrightarrow[\ g_0\ ]{} & B \\
{\scriptstyle g_1}\big\downarrow & & \big\downarrow {\scriptstyle g} \\
K & \xrightarrow[\ \rho_K\ ]{} & rK
\end{array}
$$

,

the bottom square is a pullback by the definition (4.1) and the exterior is a pullback by (ii). Hence the top square is a pullback, as desired. To see that (iii) implies (iv), let $t: B \to rJ(g)$ satisfy $r(g_0)t = 1$; then $r(g_0)tg_0 = g_0$, so (iii) gives $tg_0 = \rho_{J(g)}$, and applying r gives $tr(g_0) = 1$. To see that (iv) implies (i), write g for the $r(f)$ of (4.2); applying r to the top triangle of (4.2) gives $r(g_0)r(\bar{f}) = 1$, so that $r(\bar{f})$ is invertible by (iv) and $\bar{f} \in \mathcal{E}$. Clearly, (v) always implies (iv);
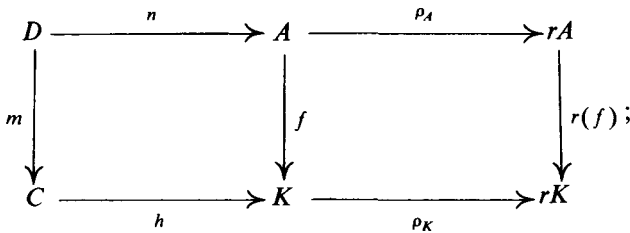
while (iii) implies (v) for additive $\mathcal{C}$, since then $r(g_0)$ is monomorphic because its pullback 1 is so.

EXAMPLE 4.2. Reflexions are not simple in general, even for abelian $\mathcal{C}$, and even when $\mathcal{B}$ is closed under subobjects. Let $\mathcal{C}$ be the category **Ab** of abelian groups, and $\mathcal{B}$ the subcategory of groups of exponent 2, so that $\rho_A \colon A \to rA$ is $A \to A/2A$. Take $K = \mathbf{Z}$ and $B = 0$ in (4.1); then $J(g) \cong \mathbf{Z}$, and $r(g_0)$ is the retraction $\mathbf{Z}/2\mathbf{Z} \to 0$; since this is not invertible, it follows from (iv) of the theorem above that r is not simple.
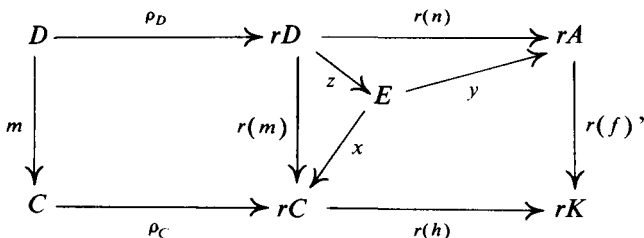
THEOREM 4.3. *For a reflexion* $\rho\colon 1 \to r$ *of the f.c.* $\mathcal{C}$ *onto* $\mathcal{B}$, *with* $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$, *the following are equivalent, and imply that* r *is simple:*
(i) *For each* $g\colon B \to rK$ *in* $\mathcal{B}$ *the pullback* g_0 *of* ρ_K *along* g, *as in* (4.1), *lies in* $\mathcal{E}$; *that is,* $r(g_0)$ *is invertible.*
(ii) r *preserves the pullback of* $f\colon A \to K$ *and* $h\colon C \to K$ *if* $f \in \mathcal{M}$.
(iii) *Every pullback of an* $\mathcal{E}$ *by an* $\mathcal{M}$ *is an* $\mathcal{E}$.

PROOF. (ii) implies (iii) since $\mathcal{E} = \Sigma_r$, and (iii) implies (i) as a special case by (2.5). It remains to show that (i) implies (ii); for it clearly implies condition (iv) of Theorem 4.1. Let the pullback of f and h be the left square in

since $f \in \mathcal{M}$ the right square is a pullback by (ii) of Theorem 4.1, so that the exterior is a pullback. By naturally, this exterior is also the exterior of the diagram

in which (E, x, y) is the pullback of $r(h)$ and $r(f)$. Note that $E \in \mathcal{B}$ since $\mathcal{B}$ is closed under limits in $\mathcal{C}$; so that $z \in \mathfrak{M}$ since mor $\mathcal{B} \subset \mathfrak{M}$. Since the exterior of this diagram is a pullback and the square with vertex E is a pullback, so too is the diagram

(4.5)

$$
\begin{array}{ccc}
D \xrightarrow{\ \ \rho_D\ \ } rD \xrightarrow{\ \ z\ \ } E \\
\downarrow m \qquad\qquad\qquad\qquad \downarrow x \ . \\
C \xrightarrow{\qquad\qquad \rho_C \qquad\qquad} rC
\end{array}
$$

If we look upon (4.5) as an instance of (4.1), we have $r(x_0) = z$, so that z is invertible by (i). Thus r preserves the pullback of f and h.

We shall call the reflexion r of $\mathcal{C}$ onto $\mathcal{B}$ *semi-left-exact* if it satisfies the conditions of Theorem 4.3.

EXAMPLE 4.4. A simple reflexion need not be semi-left-exact, even when $\mathcal{C}$ is additive, regular, and coregular. Let $\mathcal{C}$ be the category of normed vector spaces and continuous linear maps, and $\mathcal{B}$ the subcategory of Banach spaces, so that ρ_A: $A \to rA$ is the embedding into the completion of A. Then the $J(g)$ of (4.1) is the subspace $J = g^{-1}(K)$ of B, so that $r(g_0)$ is the inclusion $\bar{J} \to B$, and r is simple by (v) of Theorem 4.1. Yet $r(g_0)$ is not invertible in general; we have only to take the complex numbers for B, and take for g an injection not landing in the dense K, so that $J = \{0\}$. It is otherwise for an additive regular $\mathcal{C}$ when $\mathcal{B}$ is closed under subobjects—see Theorem 8.18 below.

We may say that the reflexion $\rho: 1 \to r$ of $\mathcal{C}$ onto $\mathcal{B}$ *has stable units* if *every* pullback of each ρ_K lies in $\mathcal{S}$. Clearly

THEOREM 4.5. *For a finitely-complete $\mathcal{C}$, a reflexion with stable units is semi-left-exact and hence simple.*

EXAMPLE 4.6. Not every reflexion that is semi-left-exact has stable units, even when $\mathcal{B}$ is closed under subobjects. Let $\mathcal{C}$ be the category of M-sets for a group M (sets with a left action of M), and let $\mathcal{B}$ consist of those M-sets A (the *discrete* ones) having $ma = a$ for all $m \in M$ and all $a \in A$; so that ρ_A: $A \to rA$ is the canonical map from A onto the set rA of orbits (connected components) of A. It is easy to verify that (i) of Theorem 4.3 is satisfied. Yet the reflexion has stable units only when $M = 1$; as we see on considering the pullback by itself of ρ_M. Once again, Theorem 8.18 below shows it to be otherwise when $\mathcal{C}$ is additive and regular, with $\mathcal{B}$ closed under subobjects.

THEOREM 4.7. *For a reflexion r of the f.c. $\mathcal{C}$ onto $\mathcal{B}$, with $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$, the following are equivalent, and imply that r has stable units and is therefore simple:*

(i) *The reflexion r is left exact; that is, a localization.*

(ii) *Every pullback of an $\mathcal{E}$ is an $\mathcal{E}$.*

PROOF. It is clear that (i) implies (ii) since $\mathcal{E} = \Sigma_r$; and it remains to show that (ii) implies (i). Since r trivially preserves the terminal object because this lies in $\mathcal{B}$, it suffices to show that r preserves pullbacks. $(\mathcal{E}, \mathcal{M})$ is certainly, by Theorems 4.5 and 4.1, a factorization system, so that any h has the $(\mathcal{E}, \mathcal{M})$ factorization $h = me$. The pullback of $f \in \mathcal{E}$ along h is the pullback along m of the pullback of f along e. Given (ii), the latter is trivially preserved by r, while the former is so by Theorem 4.3.

COROLLARY 4.8. *The localizations of a finitely-complete $\mathcal{C}$ are in bijection, via Φ and Ψ, with those reflective factorization systems $(\mathcal{E}, \mathcal{M})$ for which $\mathcal{E}$ is stable under pullback.*

EXAMPLE 4.9. Even for abelian $\mathcal{C}$ and for $\mathcal{B}$ closed under subobjects, a reflexion with stable units need not be left exact. Take $\mathcal{C} = \mathbf{Ab}$ and $\mathcal{B}$ the subcategory of torsion-free groups, so that rA is the quotient of A by its torsion subgroup. It follows from Theorem 8.18 below (and can easily be verified directly) that r has stable units; yet r does not preserve the kernel of the non-zero map $\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z}$.

REMARK 4.10. There is a close relation between Theorem 4.7 and Exercise 3.2 of [9]. For an extension of part of Theorem 4.7, see Theorem 9.12 below.

5. Subobject-hulls

A factorization system $(\mathcal{E}_0, \mathcal{M}_0)$ on $\mathcal{C}$ is said to be *proper* if $\mathcal{E}_0 \subset \mathcal{E}pi$ and $\mathcal{M}_0 \subset \mathcal{M}on$. The results of this section apply to any such; but we state them for the case of greatest importance for us, namely $(\mathcal{E}_0, \mathcal{M}_0) = (\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$, which we suppose for this section to be a *factorization system*: recall the sufficient conditions for this to be so, given in Corollary 3.2. Some of the following results are well known.

By the *subobject-hull*, or the *closure under subobjects*, of a subcategory $\mathcal{B}$ of $\mathcal{C}$, we mean the subcategory $\mathcal{B}^{\#}$ given by those $A \in \mathcal{C}$ which admits a monomorphism $A \to B$ into some $B \in \mathcal{B}$.

PROPOSITION 5.1. *Let $\rho\colon 1 \to r$ be a reflexion of $\mathcal{Q}$ onto the subcategory $\mathcal{B}$, and let*

$$(5.1) \qquad\qquad A \underset{\rho_A^\#}{\to} r^\# A \underset{\phi_A}{\to} rA$$

be the $(\mathcal{S}\mathcal{E}_{pi}, \mathcal{M}_{on})$ factorization of ρ_A. Then

(i) *$\rho^\#\colon 1 \to r^\#$ is a reflexion of $\mathcal{Q}$ onto the subobject-hull $\mathcal{B}^\#$ of $\mathcal{B}$;*

(ii) *$A \in \mathcal{B}^\#$ if and only if ρ_A is monomorphic;*

(iii) *$\mathcal{B}$ is closed under subobjects, in the sense that $\mathcal{B}^\# = \mathcal{B}$, precisely when each ρ_A is a strong epimorphism.*

PROOF. For (i), let $f\colon A \to C$ where $C \in \mathcal{B}^\#$, and let $i\colon C \to B$ be a monomorphism with $B \in \mathcal{B}$. Then we have $if = g\rho_A = g\phi_A \rho_A^\#$ for some g, whence $f = h\rho_A^\#$ for some h since $\rho_A^\# \in \mathcal{M}$ on†. Moreover h is unique since $\rho_A^\#$ is epimorphic. Now (ii) and (iii) are immediate.

REMARK 5.2. The conclusion (iii) above does not need our hypothesis that $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ be a factorization system; it suffices that $\mathcal{Q}$ be finitely complete, so that extremal epimorphisms are strong ones; then a simple direct argument applies.

REMARK 5.3. It is possible that $\mathcal{B}^\#$ be all of $\mathcal{Q}$, when $\mathcal{B}$ is not; see Example 4.4 above. When this happens, each ρ_A is epimorphic as well as monomorphic; for if $u, v\colon rA \to C$ satisfy $u\rho_A = v\rho_A$, we have $\rho_C u\rho_A = \rho_C v\rho_A$, given $\rho_C u = \rho_C v$ by the universal property of the reflexion, so that $u = v$ since ρ_C is monomorphic.

LEMMA 5.4. *If the reflexion r of $\mathcal{Q}$ onto $\mathcal{B}$ inverts the composite ip, and if p is epimorphic, then r inverts i and p.*

PROOF. $r(p)$ is epimorphic in $\mathcal{B}$, since left adjoints preserve epimorphisms; it is also a coretraction in $\mathcal{B}$, since $r(i)r(p)$ is invertible; so $r(p)$ is invertible.

PROPOSITION 5.5. *In the situation of Proposition 5.1, suppose that $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$ is a factorization system. Write $\mathcal{E}_*$ for $\mathcal{E} \cap \mathcal{S}\mathcal{E}pi$ and $\mathcal{M}_*$ for the set of maps mi with $m \in \mathcal{M}$ and $i \in \mathcal{E} \cap \mathcal{M}on$. Then $(\mathcal{E}_*, \mathcal{M}_*)$ is a factorization system $\mathcal{F}_*$, and $\mathcal{B}^\# = \mathcal{M}_*/1$. Thus $\Phi\mathcal{B}^\#$ is $\mathcal{F}^\# = (\mathcal{F}_*)^\circ$.*

PROOF. Since $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ are prefactorization systems, so is $(\mathcal{E}_*, \mathcal{E}_*^\downarrow)$ by a simple general argument, and $\mathcal{E}_*^\downarrow$ contains $\mathcal{M}_*$ since it contains $\mathcal{M}$ and $\mathcal{M}on$. An arbitrary map f in $\mathcal{Q}$ has an $(\mathcal{E}, \mathcal{M})$ factorization $f = me$, and e has an $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ factorization $e = ip$. Then i and p are in $\mathcal{E}$ by Lemma 5.4,

so that $f = (mi)p$ with $mi \in \mathfrak{M}_*$ and $p \in \mathcal{E}_*$. It follows that $(\mathcal{E}_*, \mathcal{E}_*^{\downarrow})$ is a factorization system with $\mathcal{E}_*^{\downarrow} = \mathfrak{M}_*$. If $A \in \mathfrak{B}^{\#}$ then $\rho_A \in \mathfrak{M}on$ by Proposition 5.1(ii), while $\rho_A \in \mathcal{E}$ by (2.5); so that $\rho_A \in \mathcal{E} \cap \mathfrak{M}on$. Since $rA \to 1$ lies in $\mathfrak{M}$, we have $A \in \mathfrak{M}_*/1$. Conversely, if $A \in \mathfrak{M}_*/1$, the map $A \to 1$ is mi with $m \in \mathfrak{M}$ and $i \in \mathfrak{M}on$, so that $\mathcal{C} \in \mathfrak{B}^{\#}$.

EXAMPLE 5.6. In general $\mathfrak{F}^{\#}$ is not $\mathfrak{F}_*$; the latter need not be a reflective factorization system (or a coreflective one). The lattice of prefactorization systems on the category **Set** has the four elements $\mathbf{0} \leqslant \mathfrak{F}_0 \leqslant \mathfrak{F}_1 \leqslant \mathbf{1}$, all of which are factorization systems. Here $\mathbf{0}$ and $\mathbf{1}$ are the extreme systems ($\mathcal{All}, \mathcal{Iso}$) and ($\mathcal{Iso}, \mathcal{All}$) as in Section 2, and $\mathfrak{F}_1 = (\mathcal{E}pi, \mathfrak{M}on)$; while $\mathfrak{F}_0 = (\mathcal{E}_0, \mathfrak{M}_0)$, where $\mathfrak{M}_0$ consists of the isomorphisms together with all maps whose domain is the empty set 0, and $\mathcal{E}_0$ consists of all maps with non-empty domain together with $1: 0 \to 0$. The reflective ones are $\mathbf{0}$, $\mathfrak{F}_0$, and $\mathbf{1}$, corresponding to the reflective subcategories $\{1\}$, $\{0,1\}$, and **Set**; while the only coreflective ones are $\mathbf{0}$ and $\mathbf{1}$, corresponding to the coreflective subcategories **Set** and $\{0\}$. When $\mathfrak{B} = \{1\}$ so that $\Phi\mathfrak{B} = \mathfrak{F} = \mathbf{0}$, we have $\mathfrak{B}^{\#} = \{0,1\}$ and $\mathfrak{F}^{\#} = \mathfrak{F}_0$; however $\mathfrak{F}_* = \mathfrak{F}_1$.

A *family* $(f_\lambda: A \to B_\lambda)_{\lambda \in \Lambda}$ of maps in $\mathcal{C}$ is (jointly) *monomorphic* if $f_\lambda u = f_\lambda v$ for all λ implies $u = v$, where $u, v: C \to A$. If the product $B = \prod_\lambda B_\lambda$ exists, this is just to say that the corresponding map $f: A \to B$ is monomorphic; but the general concept is needed even for a complete $\mathcal{C}$, since Λ may be large. If the $(\mathcal{S}\mathcal{E}pi, \mathfrak{M}on)$ factorization of f_λ is

(5.2)
$$A \xrightarrow[p_\lambda]{} C_\lambda \xrightarrow[i_\lambda]{} B_\lambda \ ,$$

the family (f_λ) is monomorphic exactly when the family (p_λ) is.

By the *closure under monomorphic families* of a subcategory $\mathfrak{B}$ we mean the subcategory $\mathfrak{B}^{\dagger}$ given by those A which admit a monomorphic family $(f_\lambda: A \to B_\lambda)$ with each $B_\lambda \in \mathfrak{B}$. It clearly comes to the same thing to ask of A that the family of *all* maps $f: A \to B_f$ with codomain in $\mathfrak{B}$ be monomorphic. When $\mathfrak{B}$ is reflective in $\mathcal{C}$, this is equally to ask that the reflexion $\rho_A: A \to rA$ be monomorphic; so that then, by Proposition 5.1, $\mathfrak{B}^{\dagger}$ coincides with the subobject-hull $\mathfrak{B}^{\#}$ of $\mathfrak{B}$. For a general $\mathfrak{B}$, it is clear that $\mathfrak{B}^{\dagger}$ contains all subobjects of (small) products of elements of $\mathfrak{B}$; and it consists of these alone if $\mathcal{C}$ admits small products and is *weakly cowellpowered*, in the sense that each $A \in \mathcal{C}$ has but a small set of $(\mathcal{S}\mathcal{E}pi)$-quotients. For then, if $(f_\lambda: A \to B_\lambda)$ is a monomorphic family with $B_\lambda \in \mathfrak{B}$, the set $\{p_\mu\}_{\mu \in M}$ of *distinct* p_λ in (5.2) is small; and clearly A is a subobject of $\prod_{\mu \in M} C_\mu$ and hence of $\prod_{\mu \in M} B_\mu$.

The subcategory $\mathfrak{B}$ is *closed under monomorphic families* when $\mathfrak{B}^{\dagger} = \mathfrak{B}$. Such a $\mathfrak{B}$ is of course closed under subobjects and under small products; and the converse is true if $\mathcal{C}$ admits small products and is weakly cowellpowered.

THEOREM 5.7. *If $\mathcal{Q}^{op}$ is f.w.c., the following properties of a subcategory $\mathscr{B}$ are equivalent*:

(i) *$\mathscr{B}$ is reflective and the unit $\rho_A: A \to rA$ of the reflexion is a strong epimorphism.*

(ii) *$\mathscr{B}$ is reflective and closed under subobjects.*

(iii) *$\mathscr{B}$ is closed under monomorphic families.*

PROOF. By Corollary 3.2, $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ is a factorization system. So (i) and (ii) are equivalent by Proposition 5.1, and we have seen above that (ii) implies (iii). It remains to show that (iii) implies (i). For $A \in \mathcal{Q}$ consider the family of all maps $g: A \to B_g$ with codomain in $\mathscr{B}$, and let $\rho_A: A \to rA$ be the cointersection of all the strong epimorphisms with domain A through which each such g factorizes; then each such g factorizes through ρ_A, say as $g = f_g\rho_A$. The family (f_g) is monomorphic; for $f_g u = f_g v$ for each g implies that each f_g factorizes through the coequalizer w of u and v, so that each g factorizes through the strong epimorphism $w\rho_A$, and w is invertible by the definition of ρ_A. Therefore $rA \in \mathscr{B}$ by (iii). It follows that ρ_A is a reflexion: each g factorizes through ρ_A, and uniquely so because ρ_A is epimorphic.

REMARK 5.8. There is a better-known form of this theorem with the f.w.c. hypothesis on $\mathcal{Q}^{op}$ replaced by the following: *$\mathcal{Q}$ admits small products and is weakly cowellpowered, and $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ is a factorization system.* To see that (iii) implies (i) in these circumstances, we consider the small set of strong epimorphisms $p: A \to B_p$ with $B_p \in \mathscr{B}$, form the corresponding map $A \to \Pi B_p$ into the product, and find ρ_A as the first factor in the $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ factorization of this. However this form of the result is weaker, once $\mathcal{Q}^{op}$ is f.c.; for then $\mathcal{Q}^{op}$ is f.w.c., the cointersection of any (necessarily small) family $A \to B_p$ of strong epimorphisms being obtained by factorizing $A \to \Pi B_p$ as above.

6. Relations between reflexions and coreflexions

Suppose now that $\mathcal{Q}$ has an initial object 0 as well as a terminal object 1, so that a factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ gives rise, not only to a reflexion $\rho_A: A \to rA$ onto $\mathscr{B} = \mathcal{M}/1$, but also a coreflexion $\sigma_A: sA \to A$ onto $\mathcal{C} = 0/\mathcal{E}$. Then the factorization system has not only a *reflective interior* $\overset{\circ}{\mathcal{F}} = (\overset{\circ}{\mathcal{E}}, \overset{\circ}{\mathcal{M}})$ given by Theorem 2.3, but also a *coreflective closure* $\overline{\mathcal{F}} = (\overline{\mathcal{E}}, \overline{\mathcal{M}})$ given dually; and these are again factorization systems if both $\mathcal{Q}$ and $\mathcal{Q}^{op}$ are f.w.c., as we now suppose for the time being.

We have of course $\overset{\circ}{\underline{\mathcal{F}}} \leqslant \mathcal{F} \leqslant \overline{\overset{\circ}{\mathcal{F}}}$, and $\overset{\circ}{\underline{\mathcal{F}}} = \underline{\overset{\circ}{\mathcal{F}}}$ and $\overline{\overline{\mathcal{F}}} = \overline{\mathcal{F}}$. Moreover, because (°) and (¯) are functors, we have $\overline{\mathcal{F}} \leqslant \overline{\mathcal{G}}$ and $\overset{\circ}{\mathcal{F}} \leqslant \overset{\circ}{\mathcal{G}}$ if $\mathcal{F} \leqslant \mathcal{G}$. From these properties we get for each $\mathcal{F}$ a diagram

(6.1)
$$\underline{\mathcal{F}} \leqslant \underset{\nwarrow \overset{\circ}{\underline{\mathcal{F}}} \swarrow}{\overset{\swarrow \overline{\overset{\circ}{\mathcal{F}}} \nwarrow}{\overset{\circ}{\overline{\mathcal{F}}} \qquad \overline{\underline{\mathcal{F}}}}} \leqslant \overline{\mathcal{F}};$$

for instance, the last inclusion follows by applying (¯) to $\overset{\circ}{\mathcal{F}} \leqslant \overline{\mathcal{F}}$. If we now apply (°) to the last two inclusions of the bottom edge of (6.1), we get—including the dual result as well—

(6.2)
$$\overset{\circ}{\underline{\overset{\circ}{\mathcal{F}}}} = \overset{\circ}{\underline{\mathcal{F}}}; \qquad \overset{\circ}{\overline{\underline{\mathcal{F}}}} = \overset{\circ}{\overline{\mathcal{F}}}.$$

Thus the number of different factorization systems we get from a given $\mathcal{F}$ by applying the two operations is at most seven; $\mathcal{F}$ itself and the six in (6.1). The number of different reflective subcategories is at most three, namely $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{B}''$ corresponding to the reflecitve factorization systems

(6.3)
$$\overset{\circ}{\underline{\mathcal{F}}} \leqslant \overset{\circ}{\mathcal{F}} \leqslant \overset{\circ}{\overline{\mathcal{F}}};$$

while the number of different coreflective subcategories is again at most three, namely $\mathcal{C}'' \supset \mathcal{C}' \supset \mathcal{C}$ corresponding to the coreflective factorization systems

(6.4)
$$\overline{\underline{\mathcal{F}}} \leqslant \overline{\overset{\circ}{\mathcal{F}}} \leqslant \overline{\mathcal{F}}.$$

These seven factorization systems are distinct in general. It suffices to find, for each pair of them, an example $\mathcal{C}_i$ in which they are distinct, whereupon all seven are distinct in the product $\prod_i \mathcal{C}_i$. Certainly when $\mathcal{F}$, like $(\mathcal{E}pi, \mathcal{M}on)$ in the case $\mathcal{C} = \mathbf{Set}$, is neither reflective nor coreflective, it differs from all the terms in (6.1). For the rest, it suffices to show for each pair of immediate successors in (6.1) that they may be distinct; and by duality three examples will suffice. We in fact include a fourth, for its general interest.

EXAMPLE 6.1. When $\mathcal{C} = \mathbf{Set}$ and $\mathcal{F}$ is the reflective $\mathcal{F}_0$ of Example 5.5, we have a strict inequality

$$\overset{\circ}{\underline{\mathcal{F}}} < \overset{\circ}{\mathcal{F}} = \mathbf{1}.$$

EXAMPLE 6.2. Let $\mathcal{C}$ be the category $\mathbf{Grp}$ of groups and let $\mathcal{F} = (\mathcal{E}pi, \mathcal{M}on)$. It is clear that $\overset{\circ}{\mathcal{F}} = \mathbf{0}$ and $\overline{\mathcal{F}} = \mathbf{1}$, so that we have strict inequalities

$$\overset{\circ}{\underline{\mathcal{F}}} < \overset{\circ}{\mathcal{F}}, \qquad \overline{\overset{\circ}{\mathcal{F}}} < \overline{\mathcal{F}}.$$

EXAMPLE 6.3. Let $\mathcal{C}$ be the five-element complete lattice given by $0 < a < b < 1$ and $0 < c < 1$, with no further relations. Let $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ be the reflective

factorization system corresponding to the reflective subcategory $\mathcal{B} = \{c, 1\}$. It is clear that $\mathcal{C} = 0/\mathcal{E}$ is the coreflective subcategory $\{c, 0\}$. Thus, by duality, $\mathcal{B} = \mathcal{B}' = \mathcal{B}''$ and $\mathcal{C} = \mathcal{C}' = \mathcal{C}''$; the three terms in (6.3) coincide, as do those in (6.4). Yet $\overset{\circ}{\mathcal{F}} = \mathcal{F} < \overline{\mathcal{F}}$, since $a \to b$ belongs to $\mathcal{E}$ (and hence not to $\mathcal{M}$) and belongs to $\overline{\mathcal{M}}$. So we have a strict inequality

$$\overset{\circ}{\mathcal{F}} < \overline{\mathcal{F}}.$$

EXAMPLE 6.4. Bousfield, in Example 5.5 of [4], exhibits a factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ on **Grp** in which $f: A \to B$ lies in $\mathcal{E}$ if and only if $f_*: H_i(A; G) \to H_i(B; G)$ is invertible for $i = 1$ and epimorphic for $i = 2$; here G is a fixed abelian group. Note that the corresponding coreflective $\mathcal{C} = 0/\mathcal{E}$ consists of those A with $H_1(A; G) = H_2(A; G) = 0$; but that the coreflective $\mathcal{C}'' = 0/\overset{\circ}{\mathcal{E}}$ consists of those A with $H_1(A; G) = 0$. (When $G = \mathbf{Z}$, therefore, $\mathcal{C}''$ consists of the perfect groups.) So we have a strict inequality

$$\overset{\circ}{\mathcal{F}} < \overline{\mathcal{F}}.$$

Some relations or coincidences between these seven factorization systems imply others; thus (6.1) and (6.2) give:

$$(6.5) \qquad \overline{\overset{\circ}{\mathcal{F}}} = \overset{\overline{\circ}}{\mathcal{F}} \quad \Leftrightarrow \quad \overline{\overset{\circ}{\mathcal{F}}} = \overset{\circ}{\mathcal{F}};$$

$$(6.6) \qquad \mathcal{F} \leqslant \overset{\overline{\circ}}{\mathcal{F}} \quad \Leftrightarrow \quad \overset{\overline{\circ}}{\mathcal{F}} = \overline{\mathcal{F}};$$

$$(6.7) \qquad \mathcal{F} \leqslant \overline{\overset{\circ}{\mathcal{F}}} \quad \Leftrightarrow \quad \overline{\overset{\circ}{\mathcal{F}}} = \overset{\overline{\circ}}{\mathcal{F}} = \overline{\mathcal{F}} \quad \Rightarrow \quad \overline{\overset{\circ}{\mathcal{F}}} = \overset{\circ}{\mathcal{F}};$$

$$(6.8) \qquad \overset{\circ}{\mathcal{F}} \leqslant \mathcal{F} \leqslant \overline{\mathcal{F}} \quad \Leftrightarrow \quad \left(\overline{\overset{\circ}{\mathcal{F}}} = \overset{\overline{\circ}}{\mathcal{F}} = \overline{\mathcal{F}} \right) \text{ and } \left(\overset{\overline{\circ}}{\mathcal{F}} = \overline{\overset{\circ}{\mathcal{F}}} = \overset{\circ}{\mathcal{F}} \right).$$

When $\mathcal{F}$ is both reflective and coreflective, all seven factorization systems coincide. This happens for any $\mathcal{A}$ when $\mathcal{F} = \mathbf{0}$ or $\mathbf{1}$; but it can happen for other $\mathcal{F}$ as well.
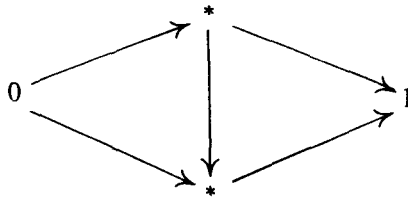
EXAMPLE 6.5. We return to Example 4.9, where $\mathcal{A} = \mathbf{Ab}$ and $\mathcal{B}$ is the reflective subcategory of torsion-free groups; and we take for $\mathcal{F}$ the corresponding reflective factorization system. Since r has stable units, $\mathcal{M}$ is determined by (ii) of Theorem 4.1; it is easily seen to consist of those $f: A \to K$ which induce an isomorphism of the torsion subgroups. On the other hand, $\mathcal{C} = 0/\mathcal{E}$ clearly consists of the torsion groups, and the coreflexion $\sigma_A: sA \to A$ onto $\mathcal{C}$ is the inclusion of the torsion subgroup sA of A. It follows that $\overline{\mathcal{M}} = \mathcal{M}$, so that $\mathcal{F} = \overset{\circ}{\mathcal{F}} = \overline{\mathcal{F}}$. See Theorem 8.20 below for a general result on such coincidences in abelian categories.

EXAMPLE 6.6. For another example, let $\mathcal{A}$ be again the lattice of Example 6.3, but now let $\mathcal{F}$ be the reflective factorization system corresponding to the reflective
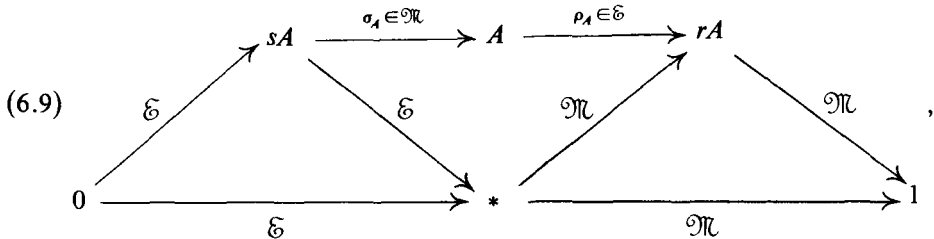
subcategory $\mathcal{B} = \{a, b, 1\}$. It is easy to see that $\mathcal{C} = 0/\mathcal{E}$ is $\{0, a\}$, and that once again $\mathcal{F} = \overset{\circ}{\mathcal{F}} = \overline{\mathcal{F}}$.

The hypothesis at the beginning of this section, that $\mathcal{A}$ and $\mathcal{A}^{op}$ are f.w.c., was needed only to ensure that all the prefactorization systems in (6.1) were factorization systems, which in turn gives the existence of the reflective $\mathcal{B}'$ and $\mathcal{B}''$ and the coreflective $\mathcal{C}'$ and $\mathcal{C}''$. However it is clear from Section 4 that in certain cases weaker hypotheses may suffice for this; which is relevant because, mild though the f.w.c. hypotheses are in practice, we should like to state various results below for, say, arbitrary abelian categories, where only finite limits and colimits are guaranteed. Accordingly we henceforth drop all blanket assumptions on $\mathcal{A}$ except the existence of 0 and 1; observe that the order-relations (6.1)–(6.8) hold equally for prefactorization systems.

Consider now, for any factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$, the reflexion $\rho: 1 \to r$ onto $\mathcal{B} = \mathcal{M}/1$ and the coreflexion $\sigma: s \to 1$ onto $\mathcal{C} = 0/\mathcal{E}$. Let the $(\mathcal{E}, \mathcal{M})$ factorization of $0 \to 1$ be $0 \to * \to 1$; consideration of

shows that every epimorphism of $*$ is the identity. We have for each A the diagram

(6.9)

which we now proceed to explain. Recall the description of r from Section 2. Clearly we have

(6.10) $r0 = s1 = *;$

the map $* \to rA$ in (6.9) is intended to be $r(0 \to A)$, and it lies in $\mathcal{M}$ by (Δ) of Section 2 since $rA \to 1$ and $* \to 1$ lie in $\mathcal{M}$; similarly $sA \to *$ is $s(A \to 1)$ and lies in $\mathcal{E}$. To see that the top region of (6.9) commutes, it suffices by the "uniqueness of the diagonal" in Section 2 above to verify that its composite with $0 \to sA$ commutes, and its composite with $rA \to 1$; and these do so by naturality.

PROPOSITION 6.7.

(i) $\mathcal{B} \cap \mathcal{C}$ is the subcategory $\{*\}$ of objects isomorphic to $*$;

(ii) $*$ is terminal in $\mathcal{C}$ and initial in $\mathcal{B}$;

(iii) for each $A \in \mathcal{Q}$ we have $srA \cong rsA \cong *$.

PROOF. (i) is immediate since $\mathcal{C} = 0/\mathcal{E}$ and $\mathcal{B} = \mathcal{M}/1$; (ii) follows from (6.10) since a right adjoint s preserves terminal objects; and (iii) is clear from (6.9).

It follows from Proposition 6.7(iii) that $\mathcal{C} \subset \{A \,|\, rA \cong *\}$; but we can give more precise results. Write $\mathcal{Q}(D, E) = 1$ to mean that there is exactly one map in $\mathcal{Q}$ from D to E, and for any subcategory $\mathcal{D}$ set

(6.11) $\mathcal{D}^{\rightarrow} = \{A \in \mathcal{Q} \,|\, \mathcal{Q}(D, A) = 1 \text{ for all } D \in \mathcal{D}\},$

with $\mathcal{D}^{\leftarrow}$ defined dually. These operations $\leftarrow$ and $\rightarrow$ clearly constitute a Galois connexion, with $\mathcal{D} \subset \mathcal{D}^{\leftarrow\rightarrow}$ and $\mathcal{D}^{\leftarrow\rightarrow\leftarrow} = \mathcal{D}^{\leftarrow}$.

THEOREM 6.8. For any factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$, setting $\mathcal{C} = 0/\mathcal{E}$ and $\mathcal{B} = \mathcal{M}/1$ with reflexion r, we have

(6.12) $\mathcal{C} \subset \{A \,|\, rA \cong *\} = \mathcal{B}^{\leftarrow},$

with

(6.13) $\mathcal{C} = \{A \,|\, rA \cong *\} = \mathcal{B}^{\leftarrow} \quad \text{if } \mathcal{F} \text{ is reflective}.$

For any $\mathcal{F}$ such that $\overset{\circ}{\mathcal{F}}$ is a factorization system, we have

(6.14) $\mathcal{C}'' = \{A \,|\, rA \cong *\} = \mathcal{B}^{\leftarrow}.$

In fact we still have $\mathcal{C} \subset \mathcal{B}^{\leftarrow}$ even for a prefactorization system $\mathcal{F}$, with the rest of (6.12) if $\mathcal{B}$ is reflective, and with (6.13) if $\mathcal{F} = \Phi\mathcal{B}$ for a reflective $\mathcal{B}$.

PROOF. For $C \in \mathcal{C}$ and $B \in \mathcal{B}$ we have $0 \to C$ in $\mathcal{E}$ and $B \to 1$ in $\mathcal{M}$, and now the existence of a unique diagonal in the evident square gives $\mathcal{Q}(C, B) = 1$ or $\mathcal{C} \subset \mathcal{B}^{\leftarrow}$. When $\mathcal{F} = \Phi\mathcal{B}$ and $C \in \mathcal{B}^{\leftarrow}$, it is clear that $0 \to C$ lies in $(\text{mor } \mathcal{B})^{\dagger} = \mathcal{E}$, so that $\mathcal{B}^{\leftarrow} \subset \mathcal{C}$. When $\mathcal{B}$ is reflective, $* = r0$ is the initial object of $\mathcal{B}$, and thus to say that $rA \cong *$ is to say that $\mathcal{B}(rA, B) = 1$ for all $B \in \mathcal{B}$, or equivalently that $\mathcal{Q}(A, B) = 1$ for all $B \in \mathcal{B}$, or that $A \in \mathcal{B}^{\leftarrow}$. We get (6.14) by applying (6.13) to the reflective $\overset{\circ}{\mathcal{F}}$, which has the same $\mathcal{B}$ as $\mathcal{F}$ but has $\mathcal{C}''$ in place of $\mathcal{C}$.

REMARK 6.9. When all the systems in (6.1) are factorization systems, the remark above that $\overset{\circ}{\mathcal{F}}$ has the same $\mathcal{B}$ as $\mathcal{F}$ gives by Proposition 6.7 that $*$ is also terminal in $\mathcal{C}''$ and hence in $\mathcal{C}'$, so that each system in (6.1) has the same $*$. It also gives $\mathcal{B} \cap \mathcal{C}'' = \{*\}$ and then, by duality, $\mathcal{B}' \cap \mathcal{C}'' = \{*\}$. However Example 6.2 shows that $\mathcal{B}'' \cap \mathcal{C}''$ may be all of $\mathcal{Q}$.

From now on we leave aside the study of general factorization systems $\mathscr{F}$; since our primary interest is in those that are reflective or coreflective, and in the corresponding reflective and coreflective subcategories. Our standard situation of interest, and our standard notation, will henceforth be as follows. We consider a reflective subcategory $\mathscr{B}$, with reflexion $\rho\colon 1 \to r$, and the corresponding reflective $\mathscr{F} = \Phi\mathscr{B}$. Now (6.1) collapses to the extent given by (6.7). Whether $\mathscr{F}$ is a factorization system or not, we can write $\mathcal{C}$ for $0/\mathscr{E}$, which is $\mathscr{B}^{\leftarrow}$ by Theorem 6.8; when $\mathscr{F}$ is a factorization system, $\mathcal{C}$ is coreflective, and we write $\sigma\colon s \to 1$ for the coreflexion. By (6.7), $\mathcal{C}'$ and $\mathcal{C}''$ coincide with $\mathcal{C}$, while $\mathscr{B}''$ coincides with $\mathscr{B}' \supset \mathscr{B}$. Once $\mathcal{C}$ is coreflective, Theorem 6.8 gives

$$(6.15) \qquad\qquad \mathcal{C}^{\rightarrow} = \{A \,|\, sA \cong *\} = \mathscr{B}' = \overline{\mathfrak{M}}/1,$$

and $\mathscr{B}'$ is reflective if $\overline{\mathscr{F}}$ is a factorization system; we write $\rho'\colon 1 \to r'$ for the reflexion onto $\mathscr{B}'$. We then have (see Remark 6.9)

$$(6.16) \qquad\qquad \mathscr{B}'^{\leftarrow} = \{A \,|\, r'A \cong *\} = \mathcal{C}.$$

The analogy of the classical nomenclature in module-categories might suggest that we call a pair $(\mathcal{C}, \mathscr{B})$ of subcategories a *torsion theory* if $\mathscr{B}^{\leftarrow} = \mathcal{C}$ and $\mathcal{C}^{\rightarrow} = \mathscr{B}$. Even when $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are f.w.c., however, there is no guarantee that $\mathscr{B}^{\leftarrow}$ is coreflective for an arbitrary subcategory $\mathscr{B}$; although it is then so by Theorem 6.8 if $\mathscr{B}$ is reflective, and for any $\mathscr{B}$ by Theorem 8.1 below if the category $\mathcal{C}$ is *pointed*.

Accordingly we agree instead to call $(\mathcal{C}, \mathscr{B})$ a *torsion theory* when there is a reflective factorization system $\mathscr{F}$, with $\overline{\mathscr{F}}$ also a factorization system, such that $\mathscr{B} = \mathfrak{M}/1 = \overline{\mathfrak{M}}/1$ and $\mathcal{C} = 0/\mathscr{E} = 0/\overline{\mathscr{E}}$. Then of course we do have $\mathscr{B}^{\leftarrow} = \mathcal{C}$ and $\mathcal{C}^{\rightarrow} = \mathscr{B}$ by the above. Clearly a general reflective $\mathscr{B}$ gives rise, as above, to a torsion-theory $(\mathcal{C}, \mathscr{B}')$ if $\overline{\mathscr{F}}$ and $\overset{\circ}{\mathscr{F}}$ are factorization systems. For a torsion theory $(\mathcal{C}, \mathscr{B})$, (6.1) of course collapses even further, the only two distinct elements being $\mathscr{F} = \overset{\circ}{\mathscr{F}}$ and $\overline{\mathscr{F}}$. It may collapse further still, when $\overset{\circ}{\mathscr{F}} = \overline{\mathscr{F}}$, as in the trivial cases $\mathscr{F} = \mathbf{0}$ or $\mathbf{1}$, and as in Examples 6.5 and 6.6; but more typical is Example 6.3, where $(\mathcal{C}, \mathscr{B})$ is a torsion theory with $\mathscr{F} = \overset{\circ}{\mathscr{F}} < \overline{\mathscr{F}}$.
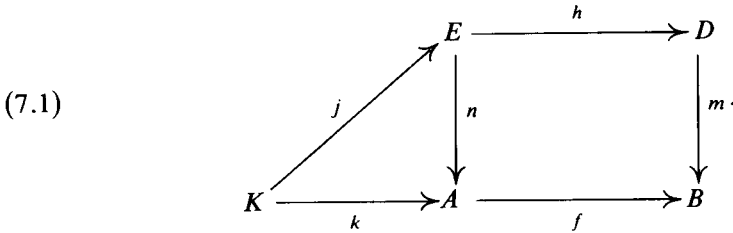
REMARK 6.10. The reflexion r onto $\mathscr{B}$ in the torsion theory of Example 6.6 is easily seen to be simple, but not semi-left-exact in the sense of Section 4; the pullback of $c \to 1$ in $\mathscr{E}$ along $b \to 1$ in $\mathfrak{M}$ is $0 \to b$, which is not in $\mathscr{E}$. Compare this with Example 4.4, where $(\mathcal{C}, \mathscr{B})$ was not a torsion theory but $\mathcal{C}$ was more special; and see Theorem 8.18 below.

REMARK 6.11. We have no example of a torsion theory $(\mathcal{C}, \mathscr{B})$ in which the reflexion r onto $\mathscr{B}$ is not simple.

7. The case of a pointed category

We suppose now that $\mathcal{C}$ is *pointed*, so that $0 \cong 1$; we write 0 for their common value, and also write $0\colon A \to B$ for the map $A \to 0 \to B$. If $f\colon A \to B$ has a kernel $k\colon K \to A$, we write as usual $k = \ker f$ or $K = \ker f$, according to the context. We sometimes also use 0 for the functor $\mathcal{C} \to \mathcal{C}$ constant at 0.

LEMMA 7.1. *In the following diagram, let the square be a pullback, let $fk = 0$, and let j be the unique map with $nj = k$ and $hj = 0$:*
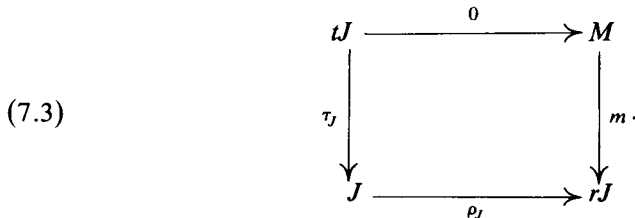
(7.1)

$$
\begin{array}{ccc}
E & \xrightarrow{\quad h \quad} & D \\
\Big\uparrow{\scriptstyle j} \quad \Big\downarrow{\scriptstyle n} & & \Big\downarrow{\scriptstyle m\,\cdot} \\
K \xrightarrow{\quad k \quad} A & \xrightarrow{\quad f \quad} & B
\end{array}
$$

Then $k = \ker f$ if and only if $j = \ker h$.

PROPOSITION 7.2. *Let $\mathcal{B}$ be a reflective subcategory of the f.c. pointed $\mathcal{C}$, and let $\tau\colon t \to 1$ be the kernel of the reflexion $\rho\colon 1 \to r$. Then*

(7.2) $$tr \cong 0,$$

and the following conditions are equivalent:
   (i) $t^2 = t$ *as subfunctors of 1; that is, t is idempotent.*
   (ii) $rt \cong 0$.
   (iii) *For every map $g\colon B \to rK$ in $\mathcal{B}$, the kernel of the g_0 of (4.1) is $\tau_J\colon tJ \to J$, where J denotes $J(g)$.*
   (iv) *In the situation of* (iii), *the diagram*

(7.3)

$$
\begin{array}{ccc}
tJ & \xrightarrow{\quad 0 \quad} & M \\
\Big\downarrow{\scriptstyle \tau_J} & & \Big\downarrow{\scriptstyle m\,\cdot} \\
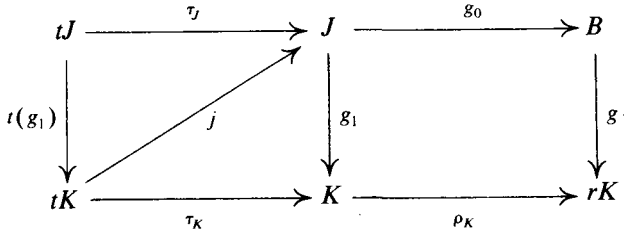J & \xrightarrow{\quad \rho_J \quad} & rJ
\end{array}
$$

where m is the kernel of $r(g_0)$, is a pullback.
   These conditions surely obtain whenever the reflexion r onto $\mathcal{B}$ is simple.

PROOF. $tr \cong 0$ since $\tau r\colon tr \to r$ is the kernel of $\rho r\colon r \to r^2$, which is the identity; and (i) is equivalent to (ii) because $\tau t\colon t^2 \to t$ is the kernel of $\rho t\colon t \to rt$. Given (i),

consider the diagram

where j is determined by $g_1 j = \tau_K$ and $g_0 j = 0$; so that $j = \ker g_0$ by Lemma 7.1. The upper triangle on the left commutes too; for $g_1 jt(g_1) = \tau_K t(g_1) = g_1\tau_J$, while $g_0 jt(g_1) = 0 = g_0\tau_J$ (since g_0 factorizes through ρ_J). Applying t to the two triangles on the left now exhibits $t(j)$ as inverse to $t(g_1)$, so that τ_J like j is a kernel of g_0, giving (iii). Now given (iii), apply it with $B = 0$; then $g_1 = \ker\rho_K$, so that g_0 is $tK \to 0$, whose kernel is tK. Since this kernel is $t^2 K$ by (iii), we have $t^2 = t$, giving (i). As for (iv), it merely re-states (iii); the diagram (7.3) commutes trivially, and to say that it is a pullback is to say that τ_J is the kernel of $r(g_0)\rho_J = g_0$. Finally, if r is simple, we obtain (iii) by applying Lemma 7.1 to the pullback (4.4), with τ_J for j.

REMARK 7.3. For a pointed $\mathcal{C}$, the $\mathcal{D}^\rightarrow$ of (6.11) consists of those A such that every map $D \to A$ with $D \in \mathcal{D}$ is 0; and is therefore closed under monomorphic families and *a fortiori* under subobjects.

Recall our standard notation from the latter part of Section 6.

THEOREM 7.4. *Let* $\mathcal{F} = \Phi\mathcal{B}$ *where* $\mathcal{B}$ *is reflective in the pointed* $\mathcal{C}$. *Now* (6.13) *becomes*

$$(7.4) \qquad \mathcal{C} = \{A \mid rA \cong 0\} = \{A \mid \rho_A = 0\} = \mathcal{B}^\leftarrow,$$

and $\mathcal{C}$ *is closed under epimorphic images. When* $\mathcal{C}$ *is coreflective (and so in particular when* $\mathcal{F}$ *is a factorization system), we have*

$$(7.5) \qquad \rho_A\sigma_A = 0,$$

and $\sigma_A\colon sA \to A$ *is a strong monomorphism if either* $(\mathcal{E}pi, \mathcal{SM}on)$ *factorizations exist or* $\mathcal{C}^{\mathrm{op}}$ *is f.c. When this is so, for any subobject* D *of* A *with* $sA \leqslant D \leqslant A$ *we have* $sD = sA$ *as subobjects of* A; *and* sA *is the largest subobject of* A *that lies in* $\mathcal{C}$.

PROOF. Clearly $* = 0$ now; and $\rho_A = 0$ gives $1_{rA}\rho_A = 0_{rA}\rho_A$, so that $1_{rA} = 0_{rA}$ and $rA \cong 0$. By Remark 7.3, $\mathcal{C} = \mathcal{B}^\leftarrow$ is closed under epimorphic images; so that σ_A is a strong monomorphism under the given hypotheses by Proposition 5.1 or

Remark 5.2. The next statement follows from $s^2 = s$ and implies the final statement.

EXAMPLE 7.5. When $\mathcal{C}$ is not pointed, σ_A need not be monomorphic. Let $\mathcal{C}$ be the category of rings and $\mathcal{B}$ the subcategory of those rings for which $2 = 0$; then $\rho_A: A \to rA$ is $A \to A/2A$. Since the initial ring 0 is $\mathbf{Z}$, while the terminal ring is $1 = \{0\}$, we have $* = r0 = \mathbf{Z}/2\mathbf{Z}$, so that $\sigma_1: * = s1 \to 1$ is not monomorphic.

PROPOSITION 7.6. *Let $\mathcal{B}$ be reflective in the pointed $\mathcal{C}$, and let the reflexion ρ: $1 \to r$ have a kernel $\tau: t \to 1$. Then*

$$(7.6) \qquad\qquad A \in \mathcal{C} \quad \text{if and only if} \quad tA = A;$$

and whenever $tA \in \mathcal{C}$, the map $\tau_A: tA \to A$ is the coreflexion of A in $\mathcal{C}$. If $\mathcal{C}$ is coreflective, $\sigma_A: sA \to A$ factorizes uniquely through τ_A, so that $sA \leq tA$ if (as is usual) σ is monomorphic. In any case we have $s\tau: st \cong s$.

PROOF. (7.6) comes from (7.4), $tA = A$ being equivalent to $\rho_A = 0$. When $tA \in \mathcal{C}$, the monomorphism τ_A is the coreflexion since, if $f: C \to A$ with $C \in \mathcal{C}$, we have $f\tau_C = \tau_A t(f)$, and τ_C is invertible by (7.6). The next statement follows from (7.5). If this factorization is $\sigma = \tau\psi$, we have $s\sigma = s\tau \cdot s\psi$ in $\mathcal{C}$; since $s\sigma = 1$, and $s\tau$ is monomorphic in $\mathcal{C}$ because $s: \mathcal{C} \to \mathcal{C}$ is a right adjoint, $s\tau$ is invertible.

THEOREM 7.7. *For a reflexion $\rho: 1 \to r$ onto the subcategory $\mathcal{B}$ of the pointed $\mathcal{C}$, the following are equivalent:*

(i) *The kernel t of ρ exists and is idempotent.*

(ii) *$\mathcal{C}$ is coreflective and $\sigma: s \to 1$ is $\ker \rho$; that is, $s = t$.*

These conditions surely obtain when $\mathcal{C}$ is finitely complete and the reflexion r is simple; and then $\mathcal{F}$ is a factorization system.

PROOF. If $t^2 = t$ we have $tA \in \mathcal{C}$ by (7.6), so that $\tau: t \to 1$ is the coreflexion onto $\mathcal{C}$ by Proposition 7.6. Thus (i) implies (ii), while (ii) trivially implies (i). Propostion 7.2 and Theorem 4.1 give the final assertions.

REMARK 7.8. We call the reflexion r in a pointed $\mathcal{C}$ *normal* if it satisfies the equivalent conditions of Theorem 7.7. We have no example in a pointed $\mathcal{C}$ of a normal reflexion that is not simple. For positive results in this direction, see Theorems 8.10, 8.14, 8.17, and 8.18 below.

In the non-normal case where s is strictly less than t, we can still describe s explicitly in terms of t for an f.w.c. $\mathcal{C}$. Define $t^\alpha A \leq A$ inductively for all ordinals

α, setting $t^0 A = A$, setting $t^{\alpha+1} A = t t^\alpha A$, and setting $t^\alpha A = \bigcap_{\beta<\alpha} t^\beta A$ for a limit-ordinal α; observing that each $t^\alpha A$ is a strong subobject of A. For some α—not necessarily small—we have $t^{\alpha+1} A = t^\alpha A$; write $t^\infty A$ for this stationary value. Clearly t^∞ is an endofunctor of $\mathcal{C}$.

THEOREM 7.9. *When the pointed $\mathcal{C}$ is f.w.c., we have $s = t^\infty$.*

PROOF. Since $(\mathcal{E}pi, \mathcal{SM}on)$ is now a factorization system by Corollary 3.2, σ: $s \to 1$ is monomorphic by Theorem 7.4. Since $tt^\infty A = t^\infty A$, we have $t^\infty A \in \mathcal{C}$ by (7.6). If $C \leqslant A$ with $C \in \mathcal{C}$, we have $t^\infty C \leqslant t^\infty A$; giving $C \leqslant t^\infty A$ since $t^\infty C = C$ by (7.6). It follows from Theorem 7.4 that $t^\infty A = sA$.

Still with $\mathcal{B}$ reflective, suppose now that $\mathcal{C} = \mathcal{B}^\leftarrow$ is coreflective, and consider the subcategory $\mathcal{C}^\rightarrow = \mathcal{B}' \supset \mathcal{B}$ of (6.15). Since $\mathcal{B}'$ is closed under subobjects by Remark 7.3, we have

(7.7) $$\mathcal{B} \subset \mathcal{B}^\# \subset \mathcal{B}',$$

where $\mathcal{B}^\#$ is the subobject-hull of $\mathcal{B}$ as in Section 5. Recall from Section 5 that we have a reflexion $\rho^\#: 1 \to r^\#$ onto $\mathcal{B}^\#$ when $\mathcal{C}$ admits $(\mathcal{S}\mathcal{E}pi, \mathcal{M}on)$ factorizations, and from Section 6 that we have a reflexion $\rho': 1 \to r'$ onto $\mathcal{B}'$ when $\overline{\mathcal{F}}$ is a factorization system.

PROPOSITION 7.10. *Let $\ker \rho$ exist. Then if $\rho^\#$ exists we have $\ker \rho^\# = \ker \rho$. If ρ' exists and $\ker \rho'$ exists, we have $\sigma \leqslant \ker \rho' \leqslant \ker \rho$ if (as is usual) σ is monomorphic. Thus $\ker \rho' = \sigma$ if $\ker \rho = \sigma$; so that r' is normal whenever r is normal.*

PROOF. $\ker \rho^\# = \ker \rho$ since the ϕ_A of (5.1) is monomorphic. We have $\rho'\sigma = 0$ by the dual of (7.5), while ρ factorizes through ρ' since $\mathcal{B} \subset \mathcal{B}'$; the statements about $\ker \rho'$ follow.

REMARK 7.11. Since, as we said in Remark 7.8, we have no example where r is normal but not simple, we have *a fortiori* no example where r is normal and r' is not simple. In fact, by Remark 6.11, we have no example *at all* in which r' is not simple.

REMARK 7.12. We saw in (6.16) that $\mathcal{B}'^\leftarrow = \mathcal{B}^\leftarrow = \mathcal{C}$; it follows from Remark 7.3 that we also have $\mathcal{B}^{\#\leftarrow} = \mathcal{C}$ when $\mathcal{C}$ is pointed.

REMARK 7.13. When $\mathcal{C}$ is not pointed we need not have $\mathcal{B}^\# \subset \mathcal{B}'$; when $\mathcal{C} = \mathbf{Set}$ and $\mathcal{B} = \{1\}$, we have $\mathcal{B}' = \mathcal{B}$ but $\mathcal{B}^\# = \{0,1\}$. Here $\mathcal{B}'$ is not closed in $\mathcal{C}$ even under *strong* subobjects; and the $\mathcal{C}$ in Example 7.5 is not closed under

strong epimorphic images, since the σ there is not monomophic. Again, in both the Examples 6.3 and 6.6, we have $\mathfrak{B}' = \mathfrak{B}$ but $\mathfrak{B}^\# = \mathcal{C}$. These examples also show that $\mathfrak{B}^{\#\,\frown}$ need not be $\mathcal{C}$ in the non-pointed case.

We turn now to some illustrative examples for the results of this section.

REMARK 7.14. Example 4.4 is one in which r is simple. There ρ is monomorphic as well as epimorphic, and we have $\mathcal{C} = \{0\}$, while $\mathfrak{B}^\# = \mathfrak{B}' = \mathcal{C}$. Another example with a simple r is that of Examples 4.9 and 6.5; as we have observed, $\mathfrak{F}$ is both reflective and coreflective, so that here $\mathfrak{B}' = \mathfrak{B}$ and $(\mathcal{C}, \mathfrak{B})$ is a torsion theory. Of course $\mathfrak{B}^\# = \mathfrak{B}$ in any such case, by (7.7).

EXAMPLE 7.15. The category $\mathcal{C} = \textbf{Nil}$ of nilplotent groups is finitely complete, but not complete and not even finitely cocomplete; although it is the union of the well-behaved categories $\textbf{Nil}_c$ of groups nilpotent of class c. So the usual existence theorems do not apply directly; one can deal instead with $\textbf{Nil}_c$ and pass to the union. For a set P of primes with complementary set P', there is a reflexion (see [8]) onto the subcategory $\mathfrak{B}$ of P-local groups. This is a localization, and hence $\mathfrak{F} = \Phi\mathfrak{B}$ is a factorization system by Theorem 4.7. In fact $\mathcal{E}$ consists of the P-bijections and $\mathfrak{M}$ of the P'-isolated homomorphisms; see [5]. The coreflexion s onto $\mathcal{C}$ has for sA the P'-torsion subgroup of $\mathcal{C}$. Moreover $\overline{\mathfrak{F}}$ can be shown to be a factorization system, and $A \to A/sA$ is the reflexion r' onto $\mathfrak{B}'$.

We pass to some non-normal and hence non-simple examples.

EXAMPLE 7.16. It is quite possible to have $\mathfrak{B} = \mathfrak{B}^\# \neq \mathfrak{B}'$, even when $\mathcal{C} = \textbf{Ab}$; although by Theorem 8.18 below this cannot happen if r is simple. Consider Example 4.2, where ρ_A is $A \to A/2A$ and tA is $2A \leqslant A$. Clearly t is not idempotent; which by Proposition 7.2 gives another proof that r is not simple. For $sA = t^\infty A$ we may write $2^\infty A$; this is the coreflexion onto the subcategory $\mathring{c}$ of groups divisible by 2. By Theorem 7.4, $\mathfrak{B}'$ consists of the A with $2^\infty A = 0$, which is strictly larger than $\mathfrak{B}$. It follows from Theorem 8.15 below, and can be easily verified directly, that the reflexion ρ'_A onto $\mathfrak{B}'$ is $A \to A/2^\infty A$.

EXAMPLE 7.17. We give another example with $\mathfrak{B} = \mathfrak{B}^\# \neq \mathfrak{B}'$, this time for $\mathcal{C} = \textbf{Ab}^{\text{op}}$. The subcategory of $\textbf{Ab}$ given by the groups of exponent 2 is not only reflective, but also coreflective. We called it $\mathfrak{B}$ in Examples 4.2 and 7.16; but we now call it $\mathcal{C}$. The coreflexion σ_A: $sA \to A$ onto $\mathcal{C}$ is the inclusion $_2A \to A$, where $_2A = \{a \in A \,|\, 2a = 0\}$. By Theorem 7.4, $\mathfrak{B} = \mathcal{C}^\frown$ consists of those A with

$_2A = 0$; that is, the groups with no 2-torsion; and the reflexion r onto $\mathcal{B}$ is $A \to A/_{(2)}A$, where $_{(2)}A$ is the 2-torsion subgroup of A. Since r is not coker σ, the coreflexion s is by Theorem 7.7 non-normal and hence non-simple. The coreflective $\mathcal{C}' = \mathcal{B}^\leftharpoonup$ consists of the 2-torsion groups, and is strictly bigger than $\mathcal{C}$, which is of course closed under epimorphic images. We can easily determine the *coreflective* factorization system $(\mathcal{E}, \mathcal{M})$ corresponding to $\mathcal{C}$; a map $f\colon X \to Y$ lies of course in $\mathcal{M}$ exactly when it induces an isomorphism $_2X \to {}_2Y$; one verifies that it lies in $\mathcal{E}$ just when coker $f \in \mathcal{C}$ and ker $f \in \mathcal{C}'$.

EXAMPLE 7.18. Let $\mathcal{A}$ be the category of groups and $\mathcal{B}$ the subcategory of abelian groups; so that $rA = A/tA$ where tA is the derived subgroup A' of A. Once again $\mathcal{B}^{\#} = \mathcal{B}$. The kernel t is not idempotent, and the coreflexion sA onto the subcategory $\mathcal{C} = \mathcal{B}^\leftharpoonup$ of perfect groups is $t^\infty A = A^\infty$, the infinitely (and not just countably) derived group of A. By Theorem 8.15 below (recalling that A^∞ is a normal subgroup of A), or directly, we see that the reflexion onto $\mathcal{B}' = \mathcal{C}^\rightharpoonup$ is $A \to A/A^\infty$; and again $\mathcal{B}'$ is strictly larger than $\mathcal{B}$. Note that $\mathcal{C}$ remains unchanged if we replace $\mathcal{B}$ by the reflective subcategory $\mathbf{Nil}_c$ for some c.

REMARK 7.19. For an abelian example of the other extreme, where $\mathcal{B} \neq \mathcal{B}^{\#} = \mathcal{B}'$, see Example 9.18 below, in which r is a localization, and hence simple and normal.

REMARK 7.20. We now take stock of some aspects, in this pointed case, of the "standard situation" $\mathcal{B}$, $\mathcal{C}$, $\mathcal{B}'$, $\mathcal{B}^{\#}$ arising from a reflexion r onto $\mathcal{B}$. We have various examples, even for $\mathcal{A} = \mathbf{Ab}$, where r is not simple; however we have no examples (Remark 7.11) where r' and s are not simple. (By "examples" we mean ones in which $\mathcal{A}$ is decently complete and cocomplete, and not artificial ones.) We have seen (Theorem 7.7) that the simplicity of r implies its normality, but we are ignorant (Remark 7.8) about the converse in general. In the examples we have given, $\mathcal{B}' = \mathcal{B}^{\#}$ was true precisely when r was normal; but we have no general proof that either of these implies the other, not any counterexample. In the next section, in which we study torsion theories for pointed $\mathcal{A}$, we give positive answers to all the questions above (with one exception) when $\mathcal{A}$ has suitable "exactness" properties, along the lines of regularity and coregularity. The exception is that, even for abelian $\mathcal{A}$, we establish that normality of r implies simplicity only when $\mathcal{B}$ is closed under subobjects. We further show that, for $\mathcal{A}$ with suitable properties, r' has stable units whenever it is simple; although by Example 4.9 it need not be left exact, even for abelian $\mathcal{A}$.

8. Torsion theories for pointed and additive categories

Recall the precise definition of *torsion theory* from the end of Section 6. We remarked there that a pair $(\mathcal{C}, \mathcal{B})$ of subcategories with $\mathcal{B}^{\leftarrow} = \mathcal{C}$ and $\mathcal{C}^{\rightarrow} = \mathcal{B}$ need not be a torsion theory even when $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ are f.w.c.; for one cannot show in this generality that such a $\mathcal{B}$ is reflective. It is otherwise when $\mathcal{C}$ is pointed; for Remark 7.3 and Theorem 5.7 give:

THEOREM 8.1. *If $\mathcal{C}$ is pointed and $\mathcal{C}^{\mathrm{op}}$ is f.w.c., the subcategory $\mathcal{D}^{\rightarrow}$ is reflective for any $\mathcal{D}$. If $\mathcal{C}$ too is f.w.c., $(\mathcal{C}, \mathcal{B})$ is a torsion theory if and only if $\mathcal{C} = \mathcal{B}^{\leftarrow}$ and $\mathcal{B} = \mathcal{C}^{\rightarrow}$; and any subcategory $\mathcal{D}$ generates a torsion theory $(\mathcal{D}^{\rightarrow\leftarrow}, \mathcal{D}^{\rightarrow})$.*

REMARK 8.2. We shall call a torsion theory $(\mathcal{C}, \mathcal{B})$ *left normal* if the coreflexion $\sigma\colon s \to 1$ onto $\mathcal{C}$ is the kernel of the reflexion $\rho\colon 1 \to r$ onto $\mathcal{B}$; that is, if the reflexion r is normal in the sense of Remark 7.8. We call $(\mathcal{C}, \mathcal{B})$ *right normal* if $\rho = \mathrm{coker}\,\sigma$; that is, if the coreflexion s is normal; and we call $(\mathcal{C}, \mathcal{B})$ *normal* when it is both left and right normal. If it is left normal and ρ is a cokernel, it is normal; for then ρ is the cokernel of its kernel. If $(\mathcal{C}, \mathcal{B}')$ is a torsion theory of the form $(\mathcal{B}^{\leftarrow}, \mathcal{B}^{\leftarrow\rightarrow})$ for a reflexion r onto $\mathcal{B}$, then $(\mathcal{C}, \mathcal{B}')$ is left normal if r is normal, by Proposition 7.10. By Remark 7.20, we know of no non-normal torsion theories; for certain $\mathcal{C}$ we show in Theorems 8.15 and 8.18 below that all torsion theories are normal.

We say that the subcategory $\mathcal{B}$ of the pointed $\mathcal{C}$ is *closed under extensions* if, whenever $f\colon A \to B$ is a strong epimorphism with $B \in \mathcal{B}$, and $\ker f$ exists and lies in $\mathcal{B}$, we have $A \in \mathcal{B}$.

LEMMA 8.3. *Whenever $(\mathcal{S}\&pi, \mathcal{M}on)$ is a factorization system, the following are equivalent*:
   (i) *The subcategory $\mathcal{B}$ is closed under subobjects and extensions.*
   (ii) *For any map $f\colon A \to B$ with $B \in \mathcal{B}$ and $\ker f \in \mathcal{B}$ we have $A \in \mathcal{B}$.*

PROOF. (i) gives (ii) on factorizing f, and (ii) gives (i) on taking f monomorphic.

REMARK 8.4. The subcategory $\mathcal{B}$ in Example 4.2, although closed under subobjects, is not closed under extensions; it does not contain the extension $\mathbf{Z}/4\mathbf{Z}$ of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}/2\mathbf{Z}$.

We observed in Section 7 that, by Remark 7.3, the $\mathcal{B}$ of a torsion theory $(\mathcal{C}, \mathcal{B})$ is closed under subobjects; but more is true:

THEOREM 8.5. *For any subcategory $\mathcal{D}$, the subcategory $\mathcal{D}^{\rightarrow}$ is closed under subobjects and extensions; so the $\mathcal{B}$ of any torsion theory $(\mathcal{C}, \mathcal{B})$ is so. On the other hand, if $\mathcal{B}$ is any reflective subcategory whose reflexion r is normal, and if $\mathcal{B}$ is closed under subobjects and extensions, we have $\mathcal{B}' = \mathcal{B}$; so that $(\mathcal{C}, \mathcal{B})$ is a left-normal torsion theory whenever $\mathcal{F} = \Phi\mathcal{B}$ and $\overline{\mathcal{F}}$ are factorization systems, and a normal one if ρ is a cokernel.*

PROOF. We use Lemma 8.3. If $k\colon E \to A$ is the kernel of $f\colon A \to B$ with E, $B \in \mathcal{D}^{\rightarrow}$, any $g\colon D \to A$ with $D \in \mathcal{D}$ has $fg = 0$, so that g factorizes through some $D \to E$ and is therefore 0, giving $A \in \mathcal{D}^{\rightarrow}$. For the second assertion, recall from the definition of normality in Remark 7.8 that the kernel $\sigma_A\colon sA \to A$ of $\rho_A\colon A \to rA$ is the coreflexion onto $\mathcal{C} = \mathcal{B}^{\leftarrow}$. When $A \in \mathcal{B}' = \mathcal{C}^{\rightarrow}$, we have $sA \cong 0$ by (7.4), so that $sA \in \mathcal{B}$. Since $rA \in \mathcal{B}$, we conclude from the closure hypothesis on $\mathcal{B}$ that $A \in \mathcal{B}$.

REMARK 8.6. When all torsion theories on $\mathcal{A}$ are normal, and when $\mathcal{A}$ is complete and cocomplete enough for the $\mathcal{F}$ and $\overline{\mathcal{F}}$ above to be factorization systems, Theorem 8.5 gives a *necessary and sufficient* condition for $\mathcal{B}$ to be part of a torsion theory $(\mathcal{C}, \mathcal{B})$; namely that it be normally reflective, and closed under subobjects and extensions.

PROPOSITION 8.7. *Let $\mathcal{B}$ be reflective with reflexion $\rho\colon 1 \to r$, and let $\mathcal{C} = \mathcal{B}^{\leftarrow}$ be coreflective with coreflexion $\sigma\colon s \to 1$. Then the following are equivalent:*
(i) $\rho = \mathrm{coker}\,\sigma$.
(ii) *Each ρ_A is the cokernel of some $f\colon C \to A$ with $C \in \mathcal{C}$.*
(iii) *For each $A \in \mathcal{A}$ there is some $C \in \mathcal{C}$ and some $f\colon C \to A$ with $\mathrm{coker}\,f \in \mathcal{B}$.*
These conditions imply that $\mathcal{B}' = \mathcal{C}^{\rightarrow}$ is $\mathcal{B}$, and hence that $(\mathcal{C}, \mathcal{B})$ is a right-normal torsion theory if $\mathcal{F} = \Phi\mathcal{B}$ and $\overline{\mathcal{F}}$ are factorization systems. Conversely, of course, any right-normal torsion theory satisfies (i). *Moreover* (i) *certainly holds if the reflexion r is normal and ρ is a cokernel; and then $(\mathcal{C}, \mathcal{B}')$, if it is a torsion theory, is normal.*

PROOF. Trivially (i) implies (ii) implies (iii). Given (iii), let the cokernel of the f there be $g\colon A \to B$. Since the left adjoint $r\colon \mathcal{A} \to \mathcal{B}$ preserves cokernels, $r(g)\colon rA \to rB$ is the cokernel of $r(f)\colon rC \to rA$; but $rC \cong 0$ by (7.4), so that $r(g)$ is invertible. Hence $\mathrm{coker}\,f = \rho_A$, giving (ii). Because f factorizes through σ_A, any x with $x\sigma_A = 0$ has $xf = 0$, and so factorizes uniquely through ρ_A; giving (i), since $\rho_A\sigma_A = 0$ by (7.5). If $A \in \mathcal{B}' = \mathcal{C}^{\rightarrow}$ we have $\sigma_A = 0$, so that ρ_A is invertible by (i), and $A \in \mathcal{B}$. If r is normal we have $\sigma = \ker\rho$, so that $\rho = \mathrm{coker}\,\sigma$ if ρ is a cokernel.

PROPOSITION 8.8. *If $(\mathcal{C}, \mathcal{B})$ is a right-normal torsion theory for which (as is usual by Theorem* 7.4) σ *is monomorphic,* $\mathcal{B}$ *can be characterized in terms of $\mathcal{C}$ as the unique subcategory of $\mathcal{Q}$ which is closed under subobjects, has $\mathcal{B} \cap \mathcal{C} = \{0\}$, and satisfies* (iii) *of Proposition* 8.7.

PROOF. Suppose $\mathcal{D}$ is such a subcategory. For $A \in \mathcal{D}$ we have $sA \in \mathcal{D}$ since σ_A is monomorphic; so that $sA \in \mathcal{C} \cap \mathcal{D} = \{0\}$, giving $A \in \mathcal{B}$ by (7.4). For $A \in \mathcal{B}$ we have by hypothesis some $f: C \to A$ with $C \in \mathcal{C}$ and coker $f \in \mathcal{D}$; but then $f = 0$ by (7.4), so that coker $f = A$ and $A \in \mathcal{D}$.

We now consider normal reflexions $\rho: 1 \to r$ in which the unit is an epimorphism of some kind, stable under pullbacks, and show that in certain cases r is then simple; we use our standard notation. First, however, we extend the scope of Theorem 4.1(v). Call the pointed $\mathcal{Q}$ *quasi-additive* if a map f in $\mathcal{Q}$ is monomorphic whenever ker $f = 0$; thus additive categories are quasi-additive, but so too are **Grp**, **Grp**$^{\mathrm{op}}$, and the duals of pointed sets and pointed topological spaces. Now when (4.4) is a pullback, Lemma 7.1 easily gives that ker $r(g_0) = 0$; whence

PROPOSITION 8.9. *We can replace "additive" by "quasi-additive" in* (v) *of Theorem* 4.1.

THEOREM 8.10. *Let $\rho: 1 \to r$ be a normal reflexion onto $\mathcal{B}$ in the f.c. pointed $\mathcal{Q}$.*

(i) *If every pullback of each ρ_A by any kernel is an epimorphism and $\mathcal{Q}$ is quasi-additive, the reflexion r is simple and $\mathcal{F} = \Phi\mathcal{B}$ is a factorization system.*

(ii) *If every pullback of each ρ_A is a strong epimorphism and $\mathcal{Q}$ is quasi-additive, the reflexion r is semi-left-exact.*

(iii) *If every pullback of each ρ_A is a cokernel, and $\mathcal{Q}$ is arbitrary, the reflexion r has stable units, $\mathcal{F} = \Phi\mathcal{B}$ is a factorization system, and $\mathcal{B}' = \mathcal{B}$. Thus $(\mathcal{C}, \mathcal{B})$ is a normal torsion theory if $\overline{\mathcal{F}}$ is a factorization system.*

PROOF. (i) By Theorem 7.7 and Proposition 7.2, the diagram (7.3) is a pullback. The map $0: tJ \to M$ therein being epimorphic by hypothesis, we have $M = 0$ and so ker $r(g_0) = m = 0$; thus $r(g_0)$ is monomorphic and the result follows from Proposition 8.9.

(ii) The g_0 of (4.1) is a strong epimorphism by hypothesis; but $g_0 = r(g_0)\rho_J$, and $r(g_0)$ is monomorphic by (i). Hence $r(g_0)$ is invertible; which is condition (i) of Theorem 4.3.

(iii) Let the pullback of $\rho_K: K \to rK$ by $f: A \to K$ be D, with projections $u: D \to K$ and $v: D \to A$. By Lemma 7.1, ker v is the map $j: sK \to D$ with $uj = \sigma_K$ and $vj = 0$. Since v is by hypothesis a cokernel, it is coker j. Because $r: \mathcal{Q} \to \mathcal{B}$ is

a left adjoint, $r(v)$ is the cokernel in $\mathscr{B}$ of $r(j)$: $rsK \to rD$. But $rsK \cong 0$ by (7.4), so that $r(v)$ is invertible and $v \in \mathscr{E}$. Thus r has stable units; whence $\mathscr{F}$ is a factorization system by Theorems 4.5 and 4.1. Since ρ is itself a cokernel and $\sigma = \ker \rho$, the rest follows from Proposition 8.7.

REMARK 8.11. If s too is normal in Theorem 8.10 and $\mathscr{C}^{\mathrm{op}}$ too is finitely complete, we can apply the theorem in its dual form; concluding that s is simple and that $\overline{\mathscr{F}}$ is a factorization system if every pushout of σ_A by a cokernel is a monomorphism and $\mathscr{C}^{\mathrm{op}}$ is quasi-additive, or if every pushout of σ_A is a kernel. Certainly s is normal if ρ is a cokernel, by Proposition 8.7; and then $\mathscr{B}' = \mathscr{B}$.

We are now in a position to study the effect of various "exactness" conditions on $\mathscr{C}$ (using "exactness" in the very broad sense of relations between limits and colimits), and to give positive answers for suitable pointed $\mathscr{C}$ to the various questions mentioned in Remark 7.20. First, however, we extend to pointed categories the additive-category versions (with cokernels in place of coequalizers) of some of the results of [10] that were recalled in Section 3 above.

PROPOSITION 8.12. *Let the pointed $\mathscr{C}$ be finitely complete and finitely cocomplete. Then the following are equivalent:*
    (i) *Every map f factorizes as $f = jq$ with j monomorphic and q a cokernel.*
    (ii) *All strong epimorphisms are cokernels.*
    (iii) *Composites of cokernels are cokernels and $\mathscr{C}$ is quasi-additive.*
Moreover the condition
    (iv) *Any pullback of a cokernel by a kernel is epimorphic*
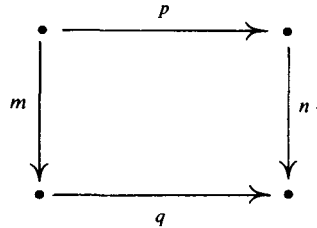*implies the condition*
    (v) *Composites of cokernels are cokernels;*
and (iv) is in turn implied by any one of the conditions
    (vi) *All pullbacks of cokernels by kernels are cokernels;*
    (vii) *All pullbacks of cokernels are epimorphic;*
    (viii) *All pullbacks of cokernels are cokernels;*
    (ix) *$\mathscr{C}$ is regular.*
When $\mathscr{C}$ is quasi-additive, (viii) and (ix) are equivalent, and imply (i)–(vii).

PROOF. (i) implies (ii) trivially. Since (ii) certainly implies that all strong epimorphisms are regular, it implies by Corollary 3.2 the existence of $(\mathscr{E}pi, \mathscr{M}on)$ factorizations, and hence implies (i). Since strong epimorphisms are closed under composition, (ii) implies (v). On the other hand, (i) implies that $\mathscr{C}$ is quasi-additive. For if $f = jq$ as in (i), we have $\ker f = \ker q$, so that $q = \mathrm{coker}(\ker f)$; and if $\ker f = 0$ we have q invertible and so f monomorphic. Thus (i) implies (iii). To see

that (iii) implies (i), let $k = \ker f$ and let $f = jq$ where $q = \operatorname{coker} k$. Then let $n = \ker j$ and let $j = mr$ where $r = \operatorname{coker} n$. Since $f = mrq$ and $\ker f = \ker q$, we also have $\ker f = \ker(rq)$. However rq is a cokernel by hypothesis, so that it is the cokernel of its kernel k. Since q is $\ker k$, it follows that r is invertible, so that $n = 0$. Since $\mathcal{C}$ is quasi-additive, this implies that j is monomorphic.

To see that (iv) implies (v), let rq be a composite of cokernels, let $n = \ker r$, let $k = \ker q$, and consider the pullback

Then $m = \ker(rq)$, and since $rqk = 0$ we have $k = mv$ for some v. We claim that $rq = \operatorname{coker} m$. If $fm = 0$ we have $fk = 0$, so that $f = gq$ for some g. Now $gnp = gqm = 0$, so that $gn = 0$ since p is epimorphic by hypothesis. Hence $g = hr$ for some h, and $f = hrq$ factorizes through rq; uniquely, since rq is epimorphic.

It is trivial that (vi)–(ix) each imply (iv) and hence (v); so that when $\mathcal{C}$ is quasi-additive they imply (i)–(iii). Since strong epimorphisms then coincide with cokernels, (viii) and (ix) are then equivalent.

REMARK 8.13. Recall that **Grp** is the category of groups; write **Set**$_*$ and **Top**$_*$ for the categories of pointed sets and pointed topological spaces. Each of **Grp**, **Set**$_*^{\mathrm{op}}$, and **Top**$_*^{\mathrm{op}}$ is regular and quasi-additive, and thus has all of the properties above. The quasi-additive category **Grp**$^{\mathrm{op}}$ has none of them. The category **Set**$_*$, being regular, satisfies (iv), (v), and (vii); in fact it also satisfies (vi); but it does not satisfy (viii) and is not quasi-additive. The category **Top**$_*$ is neither regular nor quasi-additive, but it does satisfy (vii) and hence (iv) and (v). Recall from Section 3 that *when the $\mathcal{C}$ of Proposition 8.12 is additive, the conditions (i)–(iii) are equivalent to (vii) and hence to (iv).*

THEOREM 8.14. (i) *Suppose that all strong epimorphisms are cokernels in the finitely complete and finitely cocomplete pointed $\mathcal{C}$, and let $\rho: 1 \to r$ be a normal reflexion onto a subcategory $\mathcal{B}$. Then $\mathcal{C}^{\to} = \mathcal{B}'$ is the subobject-hull $\mathcal{B}^{\#}$ of $\mathcal{B}$, which is reflective with $\sigma = \ker \rho^{\#}$ and $\rho^{\#} = \operatorname{coker} \sigma$; so that $(\mathcal{C}, \mathcal{B}^{\#})$ is a normal torsion theory if $\overline{\mathcal{F}}$ and $\overset{\circ}{\mathcal{F}} = \mathcal{F}^{\#}$ are factorization systems, where $\mathcal{F} = \Phi \mathcal{B}$. In particular, every left-normal torsion theory on $\mathcal{C}$ is normal.*

(ii) *Under the stronger hypothesis on $\mathcal{C}$ that all pullbacks of cokernels by kernels are epimorphic (which is not in fact stronger if $\mathcal{C}$ is additive), the reflexion $r^{\#} = r'$ is simple and $\mathcal{F}^{\#}$ is certainly a factorization system.*
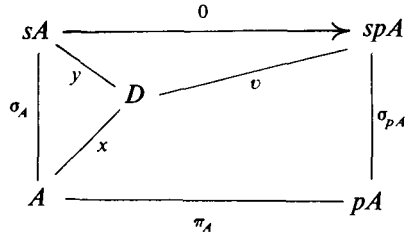
(iii) *Add to the hypothesis of* (i) *either the hypotheses that all pushouts of kernels by cokernels are monomorphic and that* $\mathcal{Q}^{\mathrm{op}}$ *is quasi-additive, or else the hypothesis that all pushouts of kernels are kernels. Then s is simple and* $\overline{\mathfrak{F}}$ *is certainly a factorization system.*

PROOF. As for (i), by the very definition of normality in Remark 7.8, the category $\mathcal{C} = \mathfrak{B}^{\leftarrow}$ is coreflective, and the coreflexion $\sigma: s \to 1$ is $\ker \rho$. Moreover, since (ii) implies (i) in Proposition 8.12, it follows from Proposition 5.1 that $\mathfrak{B}^{\#}$ is reflective and $\rho^{\#}$ is a cokernel. By Proposition 7.10 we have $\sigma = \ker \rho^{\#}$ and hence $\rho^{\#} = \operatorname{coker} \sigma$. The result now follows from Remark 7.12 and Proposition 8.7. Then (ii) is immediate from Proposition 8.12 and Theorem 8.10, and (iii) follows from Remark 8.11.

THEOREM 8.15. *Suppose that all pullbacks of cokernels by kernels are cokernels in the finitely-complete and finitely-cocomplete pointed* $\mathcal{Q}$. *Then a torsion theory* $(\mathcal{C}, \mathfrak{B})$ *is normal whenever* $\sigma: s \to 1$ *is a kernel. If, in addition, all strong monomorphisms in* $\mathcal{Q}$ *are kernels, every torsion theory is normal.*

PROOF. Let $\pi: 1 \to p$ be the cokernel of $\sigma: s \to 1$. Consider the diagram

in which the inner square is a pullback; the outside commutes because $\pi_A \sigma_A = 0$. Because $\sigma_A = \ker \pi_A$ we have $y = \ker v$ by Lemma 7.1; whence $v = \operatorname{coker} y$, since the pullback v is a cokernel by hypothesis. By the dual of Theorem 8.5, we have $D \in \mathcal{C}$. Since the pullback x of the monomorphism σ_{pA} is monomorphic, we have $sA \leqslant D \leqslant A$; whence y is invertible by Theorem 7.4. Hence $0: sA \to spA$ like v is epimorphic, giving $sp \cong 0$. By the dual of Proposition 7.2, therefore, p is idempotent; and by Theorem 7.7 this gives $\rho = \pi = \operatorname{coker} \sigma$, whence also $\sigma = \ker \rho$. We have the last statement since σ is a strong monomorphism by Theorem 7.4.

There is a partial converse to part of Theorem 8.14:

THEOREM 8.16. *If* $\rho: 1 \to r$ *is a reflexion of the pointed* $\mathcal{Q}$ *onto* $\mathfrak{B}$, *and if* $(\mathcal{C}, \mathfrak{B}') = (\mathfrak{B}^{\leftarrow}, \mathfrak{B}^{\leftrightarrow})$ *is a normal torsion theory, then* $\mathfrak{B}' = \mathfrak{B}^{\#}$ *implies that* r *is normal.*

PROOF. We have $\sigma = \ker \rho' = \ker \rho^{\#} = \ker \rho$.

THEOREM 8.17. *In the category* **Grp** *of groups, all torsion theories are normal. For any reflexion r onto a subcategory $\mathcal{B}$, we have a normal torsion theory $(\mathcal{C}, \mathcal{B}')$, and the reflexion r' has stable units. The reflexion r is normal precisely when $\mathcal{B}' = \mathcal{B}^\#$; so a normal reflexion r has stable units if $\mathcal{B}^\# = \mathcal{B}$.*

PROOF. The first statement follows from Theorem 8.15: all pullbacks of cokernels are cokernels; and although not all strong monomorphisms are kernels, the natural transformation $\sigma: s \to 1$ surely is one. For every endormorphism f of A restricts to an endomorphism $s(f)$ of sA, so that sA is a fully-invariant and hence normal subgroup of A. Since **Grp** and **Grp**$^{\mathrm{op}}$ are f.w.c., $(\mathcal{C}, \mathcal{B}')$ is always a torsion theory; and r' has stable units by Theorem 8.10(iii). If r is normal we have $\mathcal{B}' = \mathcal{B}^\#$ by Theorem 8.14(i), while Theorem 8.16 gives the converse.

THEOREM 8.18. *Let the finitely complete and finitely cocomplete $\mathcal{C}$ be additive.*

(i) *If all strong epimorphism are cokernels, any normal reflexion onto $\mathcal{B}$ (and a fortiori any simple one) gives rise to a normal torsion theory $(\mathcal{C}, \mathcal{B}')$ with $\mathcal{B}' = \mathcal{B}^\#$, provided that $\overline{\mathcal{F}}$ is a factorization system; and r' is simple, so that r is simple if $\mathcal{B}^\# = \mathcal{B}$.*

(ii) *If $\mathcal{C}$ is regular, any torsion theory $(\mathcal{C}, \mathcal{B}')$ is normal if σ is a kernel; and then the reflexion r' has stable units, so that r does too if $\mathcal{B}^\# = \mathcal{B}$.*

(iii) *If all strong epimorphisms are cokernels and all strong monomorphisms are kernels, the $\overline{\mathcal{F}}$ of (i) is always a factorization system, and s is simple.*

(iv) *If $\mathcal{C}$ is regular and all strong monomorphisms are kernels, every torsion theory is normal.*

(v) *If $\mathcal{C}$ satisfies either the conditions of (iv) or their dual, we have $\mathcal{B}' = \mathcal{B}^\#$ if and only if r is normal.*

PROOF. (i) follows from Theorem 8.14 and the final sentence of Remark 8.13; (ii) follows from Theorems 8.15 and 8.10(iii); again, (iii) follows from Theorem 8.14(iii), and (iv) from Theorem 8.15; while (v) follows from (i), (iv), and Theorem 8.16.

For abelian categories we can make a further observation.

LEMMA 8.19. *Consider a map of short exact sequences in an abelian category:*

The right square is a pullback if and only if g is invertible.

THEOREM 8.20. *If* $(\mathcal{C}, \mathcal{B})$ *is a torsion theory in an abelian* $\mathcal{C}$, *the reflective factorization system* $\mathcal{F} = \Phi\mathcal{B}$ *is also coreflective, so that* $\mathcal{F} = \overset{\circ}{\mathcal{F}} = \overline{\mathcal{F}}$.

PROOF. Given $f: A \to K$ in $\mathcal{C}$, consider the map of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & sA & \xrightarrow{\sigma_A} & A & \xrightarrow{\rho_A} & rA & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s(f)} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r(f)} & & \\
0 & \longrightarrow & sK & \xrightarrow[\sigma_K]{} & K & \xrightarrow[\rho_K]{} & rK & \longrightarrow & 0
\end{array}
$$

r is simple by Theorem 8.18; so by Theorem 4.1 we have $f \in \mathcal{M}$ precisely when the right square is a pullback; while by (2.4) we have $f \in \overline{\mathcal{M}}$ precisely when $s(f)$ is invertible. Lemma 8.19 gives $\overline{\mathcal{M}} = \mathcal{M}$, so that $\overline{\mathcal{F}} = \mathcal{F}$.

REMARK 8.21. Compare Example 6.5, where we verified this in a particular case by direct calculation.

9. Further comments on localizations

When a reflexion r of $\mathcal{C}$ onto $\mathcal{B}$ is left exact, the associated torsion theory $(\mathcal{C}, \mathcal{B}')$ has various special properties. In some cases the map $\mathcal{B} \mapsto (\mathcal{C}, \mathcal{B}')$ from localizations to torsion theories is injective, and its image may be determined. Thus, in an additive functor category $[\mathcal{K}, \mathbf{Ab}]$, such as a category of modules, it is classical (see [13] and [14]) that localizations are in bijection, not only with Gabriel topologies as we noted in the Introduction, but also with *hereditary* torsion theories; or again with *hereditary radicals*. We consider such matters briefly in this final section, partly to assist the reader in making the connexions with the classical results; but we have nothing really new to add on the *classification* of localizations, beyond the classical results and those of Corollary 4.8. *For this section we suppose* $\mathcal{C}$ *and* $\mathcal{C}^{\mathrm{op}}$ *to be finitely complete.*

Let us extend, for comparison, a terminology classically used for module categories. Let s be endofunctor of $\mathcal{C}$ with a monomorphism $\sigma: s \to 1$ into the identity. Call s *hereditary* if we have $sD = sA \cap D$ for every *regular* monomorphism $D \to A$, and call s a *preradical* if $\sigma: s \to 1$ is itself a regular monomorphism. Of course we call s *idempotent* if $s^2 = s$ as subobjects of 1, whereupon s is a coreflexion of $\mathcal{C}$ onto a subcategory $\mathcal{C}$. We call a subcategory $\mathcal{C}$ *hereditary* if it is

closed under *regular* subobjects; and we call a torsion theory $(\mathcal{C}, \mathcal{B})$ *hereditary* if $\mathcal{C}$ is hereditary. (In the classical applications of this terminology, *all* monomorphisms in $\mathcal{A}$ were regular.)

LEMMA 9.1. *If s is an endofunctor of $\mathcal{A}$ with a monomorphism $\sigma: s \to 1$, then s: $\mathcal{A} \to \mathcal{A}$ preserves equalizers if and only if s is hereditary. If σ is itself a regular monomorphism, so that s is a preradical, it is idempotent whenever it is hereditary.*

LEMMA 9.2. *Let $\sigma: s \to 1$ be any coreflexion of $\mathcal{A}$ onto a subcategory $\mathcal{C}$. The following are equivalent*:

    (i) $\mathcal{C}$ *is closed in $\mathcal{A}$ under limits of some class.*
    (ii) *The inclusion $\mathcal{C} \to \mathcal{A}$ preserves limits of this class.*
    (iii) s: $\mathcal{A} \to \mathcal{A}$ *preserves limits of this class.*
If the coreflexion $\sigma: s \to 1$ here is monomorphic, the following are equivalent:

    (iv) $\mathcal{C}$ *is closed in $\mathcal{A}$ under equalizers.*
    (v) s: $\mathcal{A} \to \mathcal{A}$ *preserves equalizers.*
    (vi) s *is hereditary.*
    (vii) $\mathcal{C}$ *is hereditary.*

REMARK 9.3. The kernel t of the reflexion r of **Ab** onto the groups of exponent 2 is a preradical that, as we saw in Example 7.16, is not idempotent. There is a torsion theory $(\mathcal{C}, \mathcal{B})$ on **Ab** where $\mathcal{C}$ consists of the divisible groups and $\mathcal{B}$ of the reduced ones; here the preradical s is idempotent but not hereditary, since clearly $\mathcal{C}$ is not hereditary.

THEOREM 9.4. *Let r be a left-exact reflexion of $\mathcal{A}$ onto $\mathcal{B}$; then $(\mathcal{E}, \mathcal{M}) = \Phi\mathcal{B}$ is a factorization system, and $\mathcal{C} = 0/\mathcal{E}$ is hereditary; so that the torsion theory $(\mathcal{C}, \mathcal{B}')$, if it exists, is hereditary.*

PROOF. $\Phi\mathcal{B}$ is a factorization system by Theorems 4.7 and 4.1. Let $k: D \to C$ be the equalizer of u, v: $C \to A$, where $C \in \mathcal{C}$. Then, since r is left exact, $r(k)$: $rD \to rC$ is the equalizer of $r(u)$, $r(v)$: $rC \to rA$. By Theorem 6.8, $rC \cong *$, which by Proposition 6.7 is the initial object of $\mathcal{B}$. Hence $r(u) = r(v)$ and $rD \cong *$, giving $D \in \mathcal{C}$ by Theorem 6.8 again.

REMARK 9.5. We see no reason to suppose that the map $\mathcal{B} \mapsto (\mathcal{C}, \mathcal{B}')$ from localizations to torsion theories is injective in general; we show below that it is so for abelian $\mathcal{A}$. For coregular $\mathcal{A}$, Barr [2] shows that the map sending a localization $\mathcal{B}$ to its closure $\mathcal{B}\check{\ }$ under *strong* subobjects is injective, and determines its image when $\mathcal{A}$ is a topos or a category of modules.

We now turn to the case of a pointed $\mathcal{C}$.

PROPOSITION 9.6. *Let r be an endofunctor of $\mathcal{C}$, let ρ: $1 \to r$ be a natural transformation, and let σ: $s \to 1$ be the kernel of ρ.*

(i) *If r: $\mathcal{C} \to \mathcal{C}$ sends regular monomorphisms to monomorphisms, the preradical s is hereditary and hence idempotent, thus giving a coreflexion onto some hereditary $\mathcal{C}$. If r is the reflexion onto some $\mathcal{B}$, we have $\mathcal{C} = \mathcal{B}^\leftarrow$, so that $\mathcal{C} = 0/\mathcal{E}$ where $(\mathcal{E}, \mathcal{M}) = \Phi\mathcal{B}$.*

(ii) *If r is left exact, we have more; s: $\mathcal{C} \to \mathcal{C}$ is left exact, and $\Phi\mathcal{B}$ is always a factorization system.*

PROOF. (i) For a regular monomorphism i: $D \to A$ consider the diagram

(9.1)

Since $r(i)$ is monomorphic, σ_D is the kernel of $r(i)\rho_D = \rho_A i$. But the kernel of $\rho_A i$ is the pullback by i of the kernel σ_A of ρ_A. Thus $sD = D \cap sA$, so that s is hereditary, and hence idempotent by Lemma 9.1. If r is a reflexion onto $\mathcal{B}$, we have by (7.4) that $A \in \mathcal{B}^\leftarrow$ precisely when $\rho_A = 0$, which is to say that σ_A is invertible or that $A \in \mathcal{C}$. The final remark follows from Theorem 6.8. As for (ii), the kernel s of ρ: $1 \to r$ preserves all limits that 1 and r do, and hence all finite ones; Theorem 9.4 gives the final statement.

REMARK 9.7. In the torsion theory $(\mathcal{C}, \mathcal{B})$ on **Ab**, where $\mathcal{C}$ consists of the torsion groups and $\mathcal{B}$ of the torsion-free ones, the coreflexion s is left exact, but the reflexion r is not. More generally, in the situation of Proposition 9.6(ii), the reflexion r' onto $\mathcal{B}'$ is not usually left exact; see Theorem 9.13 and Example 9.18 below.

Again generalizing the classical terminology, we may call a preradical σ: $s \to 1$ a *radical* if its cokernel ρ: $1 \to r$ is such that r is idempotent, and is therefore a reflexion onto some $\mathcal{B}$. So a radical corresponds to a reflexion in which ρ: $1 \to r$ is a cokernel. It follows from Theorem 7.7 that an *idempotent radical* corresponds to a normal torsion theory $(\mathcal{C}, \mathcal{B})$. *Hereditary* radicals are idempotent by Lemma 9.1, and correspond bijectively, by Lemma 9.2, with hereditary normal torsion theories.

REMARK 9.8. The examples in Remark 9.3 of preradicals in **Ab** that are not idempotent, or that are idempotent but not hereditary, are in fact radicals. To see that a hereditary preradical in **Ab** need not be a radical, consider the coreflexion $\sigma: s \to 1$ of Example 7.17 onto the subcategory $\mathcal{C}$ consisting of the groups of exponent 2.

Theorem 9.4, Proposition 9.6, and Theorem 8.14 give

PROPOSITION 9.9 *Let all strong epimorphisms be cokernels in the pointed $\mathcal{C}$, and let r be a left-exact reflexion onto $\mathcal{B}$. Then the hereditary torsion theory $(\mathcal{C}, \mathcal{B}')$ (if it exists) is normal; $\mathcal{B}' = \mathcal{B}^{\#}$; and s is an hereditary radical, in fact a left-exact one.*

We now turn to the case of an additive $\mathcal{C}$.

LEMMA 9.10. *Let $f: A \to B$ and $g: B \to C$ be maps in the f.c. additive $\mathcal{C}$, and let $\ker gf = \ker f$. Then, if every pullback of f is epimorphic, g is monomorphic.*

PROOF. We apply Lemma 7.1 above with $k = \ker f$ and $m = \ker g$; see (7.1). Then the pullback n is $\ker gf$; so that, by hypothesis, j is invertible. Since $j = \ker h$ by Lemma 7.1, we have $h = 0$. But h is epimorphic by hypothesis; so $D = 0$, $m = 0$, and g is monomorphic.

REMARK 9.11. When $\mathcal{C}$ is additive, it is clear that any subfunctor and any quotient functor of an additive endofunctor of $\mathcal{C}$ are additive. So if $\sigma: s \to 1$ is monomorphic, it follows from Lemma 9.1 that s is hereditary precisely when it is left exact. Again, if $\rho: 1 \to r$ is epimorphic, r is left exact precisely when it preserves kernels. We saw in Remark 9.7 that, even when $\mathcal{C} = $ **Ab**, the reflexion r in a torsion theory $(\mathcal{C}, \mathcal{B})$ need not be left exact even when the coreflexion s is so. However we now have a partial converse to Proposition 9.6(i), and at the same time an extension of part of Theorem 4.7:

THEOREM 9.12. *Let r be an endofunctor of the additive $\mathcal{C}$, and $\rho: 1 \to r$ a natural transformation for which every pullback of each ρ_A is epimorphic; let $\sigma: s \to 1$ be the kernel of ρ. Then s is hereditary (or equivalently left exact) if and only if r sends regular monomorphisms to monomorphisms. If a reflexion r onto $\mathcal{B}$ has this property, it is simple, so that $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$ is a factorization system; and then s is the coreflexion onto $\mathcal{C} = 0/\mathcal{E}$.*

PROOF. Consider the diagram (9.1) for a regular monomorphism i. When s is hereditary, the left square is a pullback; so σ_D is the kernel of $\rho_A i$, and hence of

$r(i)\rho_D$; and now $r(i)$ is monomorphic by Lemma 9.10. Since the kernel s of the reflexion ρ: $1 \to r$ is idempotent, the final statement follows from Theorem 8.10(i).

THEOREM 9.13. (i) *Let all strong epimorphisms be regular in the additive $\mathcal{Q}$, which is finitely complete and finitely cocomplete; and let ρ: $1 \to r$ be a left-exact reflexion of $\mathcal{Q}$ onto $\mathcal{B}$. Then r is simple and $\Phi\mathcal{B}$ is a factorization system $\mathcal{F} = (\mathcal{E}, \mathcal{M})$. The coreflexion s: $\sigma \to 1$ onto $\mathcal{C} = \mathcal{B}^{\leftarrow}$ is the kernel of ρ, and s is left exact. Moreover $\mathcal{B}' = \mathcal{C}^{\rightarrow}$ is $\mathcal{B}^{\#}$, which is reflective with $\sigma = \ker \rho^{\#}$ and $\rho^{\#} = \operatorname{coker} \sigma$. The reflexion $r^{\#}$ is simple, and sends regular monomorphisms to monomorphisms; and $\mathcal{F}^{\#}$ is a factorization system. So $(\mathcal{C}, \mathcal{B}')$ is an hereditary normal torsion theory whenever $\overline{\mathcal{F}}$ is a factorization system; this is certainly so whenever all strong monomorphisms in $\mathcal{Q}$ are regular, and then s is simple. When $\mathcal{Q}$ is regular, the reflexion $r' = r^{\#}$ has stable units.*

PROOF. r is simple by Theorem 4.7 and $\mathcal{F}$ is a factorization system by Theorem 4.1. Since the simple r is normal by Theorem 7.7, we can apply Theorem 8.10. For the remaining observations, s is left exact by Proposition 9.6, and $r^{\#}$ sends regular monomorphisms to monomorphisms by Theorem 9.12; while if $\mathcal{Q}$ is regular $r^{\#}$ has stable units by Theorem 8.18.

We turn finally to the case of an abelian $\mathcal{Q}$.

PROPOSITION 9.14. *Let $(\mathcal{C}, \mathcal{B}')$ be any torsion theory in the abelian $\mathcal{Q}$. Then $(\mathcal{C}, \mathcal{B}')$ is hereditary if and only if $\mathcal{C}$ is closed under subobjects. This implies that $\mathcal{B}'$ is closed under essential extensions; and it is implied by the latter if $\mathcal{Q}$ admits injective envelopes.*

PROOF. The first assertion is trivial, since all monomorphism are now regular. Supposing $\mathcal{C}$ hereditary, consider an essential extension i: $B \to A$ where $B \in \mathcal{B}'$. For any map f: $C \to A$ with $C \in \mathcal{C}$, consider the pullback

(9.2)

$$\begin{array}{ccc} D & \xrightarrow{\;\;g\;\;} & B \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ C & \xrightarrow{\;\;f\;\;} & A \end{array}$$

Since the pullback j of the monomorphism i is monomorphic, we have $D \in \mathcal{C} = \mathcal{B}^{\leftarrow}$, so that $g = 0$. In other words, $\operatorname{im} f \cap B = 0$; so that $f = 0$ since i is essential. Thus $A \in \mathcal{C}^{\rightarrow} = \mathcal{B}$. For the converse, let j: $D \to C$ be a monomorphism

with $C \in \mathcal{C}$, and consider any $g: D \to B$ with $B \in \mathcal{B}$. Let $i: B \to A$ be an injective envelope of B in $\mathcal{C}$, so that i is essential and $A \in \mathcal{B}$ by hypothesis. Since A is injective, there is a map $f: C \to A$ rendering (9.2) commutative. However $f = 0$ since $A \in \mathcal{C}^{\to} = \mathcal{B}$, whence $g = 0$ since i is monomorphic. Thus $D \in \mathcal{B}^{\leftarrow} = \mathcal{C}$.

Henceforth we distinguish $\mathcal{E}pi$, the class of epimorphisms in $\mathcal{C}$, called simply "epimorphisms", from the class $\mathcal{E}pi(\mathcal{B}')$ of epimorphisms in $\mathcal{B}'$; and similarly for $\mathcal{M}on$ or $\mathcal{SM}on$.

PROPOSITION 9.15. *Let $(\mathcal{C}, \mathcal{B}')$ be an hereditary torsion theory in the abelian $\mathcal{C}$. Then $\mathcal{M}on(\mathcal{B}')$ is closed under pushouts in $\mathcal{B}'$. Thus strong and regular monomorphisms coincide in $\mathcal{B}'$, and $(\mathcal{E}pi(\mathcal{B}'), \mathcal{SM}on(\mathcal{B}'))$ is a factorization system.*

PROOF. A map $i: D \to E$ in $\mathcal{B}'$ is monomorphic in the reflective $\mathcal{B}'$ if and only if it is monomorphic in $\mathcal{C}$. If $f: D \to F$ is a map in $\mathcal{B}'$, and if $j: F \to A$ is the pushout of i by f in $\mathcal{C}$, then $r'(j)$ is the pushout of i by f in $\mathcal{B}'$. But j is monomorphic since $\mathcal{C}$ is abelian, and then $r'(j)$ is monomorphic by Theorem 9.12. The remaining assertions follow from Remark 8.13 and Corollary 3.2.

THEOREM 9.16. *For any abelian $\mathcal{C}$, the map $\mathcal{B} \mapsto (\mathcal{C}, \mathcal{B}')$ from localizations to (hereditary) torsion theories is injective. In fact $\mathcal{B}$ is recovered as follows:*

(**) *An object A of $\mathcal{B}'$ lies in $\mathcal{B}$ exactly when every monomorphism $i: A \to D$ with $D \in \mathcal{B}'$ is a strong monomorphism in $\mathcal{B}'$.*

PROOF. Let $A \in \mathcal{B}$, and consider a monomorphism $i: A \to D$ with $D \in \mathcal{B}'$. Since i is a regular monomorphism in $\mathcal{C}$ and r is left exact, $r(i)$ is a regular monomorphism in $\mathcal{B}'$. Since $\rho_A = 1$, we have $r(i) = \rho_D i$, and i is a strong monomorphism in $\mathcal{B}'$; see Section 3 above. For the converse, observe that ρ_A is monomorphic if $A \in \mathcal{B}' = \mathcal{B}^{\#}$ by Proposition 5.1, and is thus epimorphic in $\mathcal{B}'$ by Remark 5.3. If it is also a strong monomorphism in $\mathcal{B}'$, it is invertible, and $A \in \mathcal{B}$.

We end by including, for completeness, the following classical result in the other direction:

THEOREM 9.17. *Let the abelian $\mathcal{C}$ admit injective envelopes. Then the map $\mathcal{B} \mapsto (\mathcal{C}, \mathcal{B}')$ is a bijection from localizations to hereditary torsion theories.*

PROOF. Let $(\mathcal{C}, \mathcal{B}')$ be an hereditary torsion theory, and define the subcategory $\mathcal{B} \subset \mathcal{B}'$ by (**) of Theorem 9.16. To prove $\mathcal{B}$ reflective in $\mathcal{C}$, it suffices to prove it reflective in $\mathcal{B}'$. Given $A \in \mathcal{B}'$, choose a monomorphism $i: A \to J$ with J injective,

and let jp be the $(\mathcal{E}pi(\mathcal{B}'), \mathcal{SM}on(\mathcal{B}'))$ factorization of $\rho'_j i$, which exists by Proposition 9.15:

(9.3)

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & J \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'_j} \\
B & \xrightarrow{\ j\ } & r'J
\end{array}
$$

We assert that $B \in \mathcal{B}$. First, $\rho'_j i = r'(i)$ is monomorphic by Theorem 9.12, whence p is monomorphic. Consider a monomorphism $k \colon B \to D$ with $D \in \mathcal{B}'$. Since J is injective, we have $i = hkp$ for some $h \colon D \to J$. So $jp = \rho'_j i = r'(i) = r'(h)kp$, whence $j = r'(h)k$ since $p \in \mathcal{E}pi(\mathcal{B}')$. Because j lies in $\mathcal{SM}on(\mathcal{B}')$, so does k; see Section 3 above. Thus $B \in \mathcal{B}$ by the definition $(\ast\ast)$.

   Suppose that $p \colon A \to B$ is *any* element of $\mathcal{M}on \cap \mathcal{E}pi(\mathcal{B}')$ with $A \in \mathcal{B}'$ and $B \in \mathcal{B}$; such as the p of (9.3). Then p is a reflexion of A into $\mathcal{B}$. To see this, let $f \colon A \to B'$ with $B' \in \mathcal{B}$, and form the pushout in $\mathcal{B}'$

(9.4)

$$
\begin{array}{ccc}
A & \xrightarrow{\ p\ } & B \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle v} \\
B' & \xrightarrow{\ u\ } & E
\end{array}
$$

Since p is monomorphic in $\mathcal{C}$ and hence in $\mathcal{B}'$, so is its pushout u, by Proposition 9.15. Thus $u \in \mathcal{SM}on(\mathcal{B}')$ by $(\ast\ast)$, since $B' \in \mathcal{B}$. Because $p \in \mathcal{E}pi(\mathcal{B}')$, there is a diagonal w in (9.4). Thus $f = wp$ for some w, and such a w is unique since $p \in \mathcal{E}pi(\mathcal{B}')$.

   Thus the p of (9.3) is a reflexion $\rho_A \colon A \to rA$ of A into $\mathcal{B}$ for $A \in \mathcal{B}'$. Since it is epimorphic in $\mathcal{B}'$, it follows from the analogue of Proposition 5.1 that $\mathcal{B}$ is closed in $\mathcal{B}'$ under $\mathcal{SM}on(\mathcal{B}')$-subobjects. Since the p of (9.3) is monomorphic, the subobject-closure $\mathcal{B}^{\#}$ of $\mathcal{B}$ in $\mathcal{C}$ is $\mathcal{B}'$; for $\mathcal{B}'$ is closed in $\mathcal{C}$ under subobjects, the reflexion ρ' being a cokernel by hypothesis.

   Given any monomorphism $\quad i \colon A \to B'$ with $A \in \mathcal{B}'$ and $B' \in \mathcal{B}$, let its $(\mathcal{E}pi(\mathcal{B}'), \mathcal{SM}on(\mathcal{B}'))$ factorization be $i = jp$, where $p \colon A \to B$ and $j \colon B \to B'$. Then $B \in \mathcal{B}$ by the last paragraph, and p is ρ_A by the penultimate paragraph.

   Suppose we have a monomorphism $k \colon A \to D$ with $A, D \in \mathcal{B}'$. Applying the last paragraph with $i = \rho_D k \colon A \to rD$, we have $\rho_D k = j \rho_A$ for some $j \in \mathcal{SM}on(\mathcal{B}')$, which is clearly $r(k)$. Thus $r \colon \mathcal{B}' \to \mathcal{B}$ *sends monomorphisms to monomorphisms which are strong in $\mathcal{B}'$.*

Now write $\rho: 1 \to r$ for the reflexion of $\mathcal{C}$ onto $\mathcal{B}$, so that ρ_A is the composite of $\rho'_A: A \to r'A$ and $\rho_{r'A}: r'A \to rA$. Since r' preserves monomorphisms by Theorem 9.12, it follows from the last paragraph that $r: \mathcal{C} \to \mathcal{C}$ *preserves monomorphisms, sending them to monomorphisms that are regular in $\mathcal{B}'$*.

Since ρ'_A is epimorphic in $\mathcal{C}$ and $\rho_{r'A}$ is epimorphic in $\mathcal{B}'$, it follows by two applications of Remark 9.11 that r is additive. Thus, to prove it left exact, it remains to show that it preserves kernels. If $i = \ker f$ in $\mathcal{C}$, we have $f = kq$ where $q = \operatorname{coker} i$ and k is monomorphic. Since r preserves monomorphisms, to show that $r(i) = \ker r(f)$ we have only to show that $r(i) = \ker r(q)$. Because $r: \mathcal{C} \to \mathcal{B}$ is a left adjoint, $r(q)$ is the cokernel in $\mathcal{B}$ of $r(i)$, and is therefore $\rho_E s$, where $s: D \to E$ say is the cokernel of $r(i)$ in $\mathcal{B}'$. Since $r(i)$ is, by the last paragraph, a regular monomorphism in $\mathcal{B}'$, it is $\ker s$ in $\mathcal{B}'$, and hence in $\mathcal{C}$; and since ρ_E is monomorphic, it is also $\ker(\rho_E s) = \ker r(q)$.

EXAMPLE 9.18. The most classical example of an hereditary torsion theory on **Ab** is $(\mathcal{C}, \mathcal{B}')$, where $\mathcal{C}$ consists of the torsion groups and $\mathcal{B}'$ of the torsion-free groups, as in Example 6.5 above. In this case $\mathcal{B}$ consists of the *torsion-free divisible* groups. As we observed in Example 4.9, the reflexion r' here is not left exact. It is easy in this example to calculate $\Phi\mathcal{B} = (\mathcal{E}, \mathcal{M})$; we find that $f \in \mathcal{E}$ when $\ker f$ and $\operatorname{coker} f$ are both torsion groups, while $f \in \mathcal{M}$ when $\ker f$ is torsion-free and divisible and $\operatorname{coker} f$ is torsion-free. Contrast $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$, which by Example 6.5 is both the reflective factorization system corresponding to $\mathcal{B}'$ and the coreflective one corresponding to $\mathcal{C}$. We have $f: A \to B$ in $\overline{\mathcal{M}}$ when f induces an isomorphism $sA \cong sB$ of the torsion subgroups, and thus when both $\ker f$ and $\operatorname{coker} f$ are torsion-free; while $f \in \overline{\mathcal{E}}$ when it induces an isomorphism $A/sA \to B/sB$, and thus when $\operatorname{coker} f$ is a torsion group and $f^{-1}(sB) \leqslant sA$.

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