Canad. Math. Bull. Vol. 33 (1), 1990

EXTENSIONS THAT ARE SUBMODULES OF THEIR QUOTIENTS

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ABSTRACT. Let $0 \longrightarrow N \longrightarrow E \longrightarrow F \longrightarrow 0$ be a short exact sequence of torsion-free Kronecker modules. Suppose that N and F have rank one. The module F is classified by a height function h defined on the projective line. If N is finite-dimensional, h is supported on a set of cardinality less than that of its domain and h takes on the value ∞, then E embeds into F. The converse holds if all such E embed into F. This embeddability is in contrast to the situation with other rings such as commutative domains, where it never occurs.

Suppose N, E, F are non-zero, torsion-free modules over a commutative integral domain, sitting in a short exact sequence $0 \longrightarrow N \longrightarrow E \longrightarrow F \longrightarrow 0$. If E is of finite rank, then the extension E cannot be embedded into F because the field of fractions of the domain is flat. However, in many other situations embedding of E into F is possible. In this note we examine how this can happen for the category of Kronecker modules. In this case the ground ring is $\begin{bmatrix} K & K^2 \\ 0 & K \end{bmatrix}$, where K is an algebraically closed field. As in much of the literature, see for example [2], Kronecker modules are more conveniently viewed as quadruples $(V, W; a, b)$, where V, W are vector spaces over K and a, b are K-linear maps from V to W.

We shall deal only with the case where F has rank one, because such are the only torsion-free modules which are completely understood; see [2, 3.4]. The modules N could have any rank, but for simplicity of notations N will be assumed to have rank one also. For various terminologies about Kronecker modules, [1] and [2] are useful references. For a more general viewpoint [7] may be checked.

In order to avoid undue technicalities we merely state our first result. It serves to motivate the subsequent constructions, but it will not be used otherwise.

THEOREM 1. *Suppose that $0 \longrightarrow N \longrightarrow E \longrightarrow F \longrightarrow 0$ is a short exact sequence of torsion-free Kronecker modules with N and F of rank one. If E can be embedded into F, then N must be finite-dimensional.* □

The torsion-free modules of rank one have an explicit construction depending only on a so called height function $h : K \cup \{\infty\} \longrightarrow \{\infty, 0, 1, 2, \ldots\}$, see [2, Sec. 3].

Received by the editors May 25, 1988 and, in revised form, October 7, 1988.

AMS 1980 Subject Classification: 16A64, 16A46.

https://doi.org/10.4153/CMB-1990-016-0 Published online by Cambridge University Press

Indeed, let $K(X)$ be the space of all rational functions in the indeterminate X. For each height function h, let R_h be the subspace consisting of those r in $K(X)$ such that r has a pole at each θ in $K \cup \{\infty\}$ of order not more than $h(\theta)$. Let R_h^- be the subspace of R_h consisting of those r in R_h such that r has a pole at ∞ of order strictly less than $h(\infty)$.

With these notations, the modules $(R_h^-, R_h; a, b)$, where $a : R_h^- \longrightarrow R_h$ is the inclusion map, and $b : R_h^- \longrightarrow R_h$ is the multiplication operator $r \longmapsto Xr$, account for all the torsion-free modules of rank one. A simple equivalence relation among the height functions distinguishes the isomorphism types, see [2, 3.4].

For example, if $k(\theta) = \infty$ for all θ in $K \cup \{\infty\}$, then $R_k^- = R_k = K(X)$. The resulting F_k will be denoted by $\mathcal{R}$. This module is the only indecomposable, torsion-free, divisible module, see [1, 9.8] and [7, 5.3]. Also, every other F_h is a submodule of $\mathcal{R}$.

The torsion-free modules N of rank one, that are finite-dimensional, can be completely classified up to isomorphism by a single positive integer m. Indeed, let P_m denote the space of polynomials of degree less than m. As in [1, 2.1], let III^m denote the module $(P_{m-1}, P_m; a, b)$, where $a, b : P_{m-1} \longrightarrow P_m$ denote the inclusion map and multiplication by X, respectively. This is nothing but F_h when $h(\theta) = 0$ for θ in K and $h(\infty) = m - 1$. The III^m's provide the isomorphism types for the finite-dimensional, torsion-free, rank one modules; see [1, 4.3].

We are therefore looking at extensions $0 \longrightarrow \mathrm{III}^m \longrightarrow E \longrightarrow F_h \longrightarrow 0$. According to [5, Thm. A], such extensions may be realized by a K-linear functional $\alpha : K(X) \longrightarrow K$. This goes as follows. Let $a : P_{m-1} \oplus R_h^- \longrightarrow P_m \oplus R_h$ denote the obvious inclusion map. With the value of a linear functional α at a rational function r denoted by $\langle \alpha, r \rangle$, let $b : P_{m-1} \oplus R_h^- \longrightarrow P_m \oplus R_h$ stand for the mapping $(p, r) \longmapsto (Xp + \langle \alpha, r \rangle, Xr)$. Then

$$E = E_\alpha = (P_{m-1} \oplus R_h^-, P_m \oplus R_h; a, b). \tag{1}$$

The pair of exact sequences of vector spaces

$$0 \longrightarrow P_{m-1} \longrightarrow P_{m-1} \oplus R_h^- \longrightarrow R_h^- \longrightarrow 0, \quad 0 \longrightarrow P_m \longrightarrow P_m \oplus R_h \longrightarrow R_h \longrightarrow 0,$$

defined in the obvious way, constitutes the desired short exact sequence $0 \longrightarrow \mathrm{III}^m \longrightarrow E_\alpha \longrightarrow F_h \longrightarrow 0$ of modules.

The class of such E_α's is also interesting because it contains all purely simple Kronecker modules of rank two, see [5, Prop. B].

Given such an extension E_α, any attempt to embed E_α into its quotient F_h will require the construction of a homomorphism into $\mathcal{R}$, because F_h is a submodule of $\mathcal{R}$. All of the homomorphisms of E into $\mathcal{R}$ can explicitly be defined with the aid of a kind of derived function which we shall now discuss.

Let $K(X)^*$ denote the space of K-linear functionals on $K(X)$. Given α in $K(X)^*$ and t in $K(X)$, let $\alpha * t$ denote the functional $r \longmapsto \langle \alpha, tr \rangle$. With this we have a unique K-bilinear map

$$\partial : K(X)^* \times K(X) \longrightarrow K(X); \quad (\beta, t) \longmapsto \partial_\beta(t)$$

such that

$$\partial_\alpha(rs) = r\partial_\alpha(s) + \partial_{\alpha * s}(r), \tag{2}$$

$$\partial_\alpha(X) = \langle \alpha, 1 \rangle \tag{3}$$

for all α in $K(X)^*$ and all r, s in $K(X)$. For a definition of ∂ see [3, Sec. 3], where it is referred to as the derived function.

Since the field K is algebraically closed, a basis for the space $K(X)$ consists of the functions $(X-\theta)^{-n}$, $n = 1, 2, \ldots$ and X^n, $n = 0, 1, 2, \ldots$. On this basis, each ∂_α is computed as follows:

$$\partial_\alpha((X-\theta)^{-1}) = -\sum_{j=1}^{n} \langle \alpha, (X-\theta)^{-j} \rangle (X-\theta)^{j-n-1} \tag{4}$$

$$\partial_\alpha(X^n) = \sum_{j=0}^{n-1} \langle \alpha, X^{n-j-1} \rangle X^j, \quad n \geqq 1; \ \partial_\alpha(1) = 0. \tag{5}$$

By decomposing rational functions into their partial fraction expansion it can then be seen from (4) and (5) that, if a rational function r has a pole at θ in $K \cup \infty$ of non-negative order, then $\partial_\alpha(r)$ has a pole at θ of order not greater than that of r. Furthermore, this order is strictly less at $\theta = \infty$. That is, $\partial_\alpha(R_h) \subset R_h^-$ for each height function h.

Now any homomorphism from E_α to $\mathcal{R}$ is a pair of linear maps

$$\varphi : P_{m-1} \oplus R_h^- \longrightarrow K(X), \qquad \psi : P_m \oplus R_h \longrightarrow K(X)$$

such that

$$a(\varphi(p,r)) = \psi(a(p,r)), \quad b(\varphi(p,r)) = \psi(b(p,r)) \tag{6}$$

for (p, r) in $P_{m-1} \oplus R_h^-$. By the definitions of E_α and $\mathcal{R}$, the first equation of (6) means that $\varphi = \psi$ restricted to $P_{m-1} \oplus R_h^-$. Then the second equation of (6) boils down to

$$X\psi(p,r) = \psi(Xp + \langle \alpha, r \rangle, Xr) \tag{7}$$

for (p, r) in $P_{m-1} \oplus R_h^-$.

For example, let $\psi : P_m \oplus R_h \longrightarrow K(X)$ be defined by $\psi(p, r) = p - \partial_\alpha(r)$. Using properties (2) and (3) of ∂ it is easy to check that ψ satisfies (7) and thereby gives rise to a homomorphism $E_\alpha \longrightarrow \mathcal{R}$. Indeed, for (p, r) in $P_{m-1} \oplus R_h^-$,

$$\begin{aligned} \psi(Xp + \langle \alpha, r \rangle, Xr) &= Xp + \langle \alpha, r \rangle - \partial_\alpha(Xr) \\ &= Xp + \langle \alpha, r \rangle - X\partial_\alpha(r) - \partial_{\alpha * r}(X) \\ &= Xp + \langle \alpha, r \rangle - X\partial_\alpha(r) - \langle \alpha * r, 1 \rangle \\ &= X(p - \partial_\alpha(r)) = X\psi(p, r). \end{aligned}$$

A more trivial example is given by $\psi(p,r) = r$. Also, if ψ_1, ψ_2 satisfy (7) and thereby give homomorphisms $E_\alpha \longrightarrow \mathcal{R}$, then so does $s_1\psi_1 + s_2\psi_2$ for any rational functions s_1, s_2. It follows that for any rational functions s, t the mapping $\psi : P_m \oplus R_h \longrightarrow K(X)$ given by

$$\psi(p,r) = s(p - \partial_\alpha(r)) + tr \tag{8}$$

gives rise to a homomorphism $E_\alpha \longrightarrow \mathcal{R}$.

THEOREM 2. *Every homomorphism $E_\alpha \longrightarrow \mathcal{R}$ arises from a linear map $\psi : P_m \oplus R_h \longrightarrow K(X)$ defined according to* (8) *for some s, t in $K(X)$.*

PROOF. For the following argument the notion of torsion closure as given in [2, Sec. 2] will be used.

Let $\psi : P_m \oplus R_h \longrightarrow K(X)$ satisfy (7) and thereby give a homomorphism $E_\alpha \longrightarrow \mathcal{R}$. Let $s = \psi(1,0)$, $t = \psi(0,1)$ and let $\psi_1 : P_m \oplus R_h \longrightarrow K(X)$ be given by formula (8), using these s, t. The mapping $\psi - \psi_1$ will satisfy (7), and give a homomorphism from E_α into the torsion-free module $\mathcal{R}$, which kills the torsion generators (1, 0) and (0, 1) of E_α. Hence $\psi - \psi_1$ must vanish. So $\psi = \psi_1$ is given according to (8). □

In light of Theorem 2, the extension E_α will be embeddable into $\mathcal{R}$ whenever rational functions s, t can be found making ψ, as defined by (8), injective. Furthermore E_α will be embedded in its quotient F_h whenever such an injection ψ maps $P_{m-1} \oplus R_h^-$ into R_h^- and $P_m \oplus R_h$ into R_h.

The set $\{\nu \in K : h(\nu) > 0\}$ is called the *support* of a height function h. Note that the function $(X - \nu)^{-1} \in R_h$ if and only if ν is in the support of h.

THEOREM 3. *Let $0 \longrightarrow \mathrm{III}^m \longrightarrow E_\alpha \longrightarrow F_h \longrightarrow 0$ be an extension as prescribed by* (1). *If the support of h has less cardinality than K, then E_α is embeddable in $\mathcal{R}$.*

PROOF. By the cardinality assumption, the totality of roots of all polynomials $(\nu - X)^m + \langle \alpha, (X - \nu)^{-1} \rangle$, with ν in the support of h, does not exhaust K. So choose θ in K such that, for every ν in the support of h, $(\nu - \theta)^m + \langle \alpha, (X - \nu)^{-1} \rangle \neq 0$. The mapping $\psi : P_m \oplus R_h \longrightarrow K(X)$ given by $\psi(p,r) = p - \partial_\alpha(r) + (X - \theta)^m r$ will provide the desired embedding of E_α into $\mathcal{R}$.

To see this let $(p,r) \in P_m \oplus R_h$. If r in R_h is not a polynomial, then at some ν in the support of h, r has a pole of positive order n. Write $r = u + c(X - \nu)^{-n}$ with $c \in K, c \neq 0$ and u in R_h having a pole at ν of order less than n. By formula (4) and the fact ∂_α does not increase the order of a pole, it follows that $\partial_\alpha(r) = w - c\langle \alpha, (X - \nu)^{-1} \rangle (X - \nu)^{-n}$ where w has a pole at ν of order less than n. Hence

$$\psi(p,r) = [p - w + (X - \theta)^m u] + c[(X - \theta)^m + \langle \alpha, (X - \nu)^{-1} \rangle](X - \nu)^{-n}.$$

The first bracket above has a pole at ν of order less than n. The second term contributes a pole of order n at ν because the polynomial $(X - \theta)^m + \langle \alpha, (X - \nu)^{-1} \rangle$ does not vanish

https://doi.org/10.4153/CMB-1990-016-0 Published online by Cambridge University Press

at ν, and $c \neq 0$. So these two terms don't add up to 0. It follows that if $\psi(p,r) = 0$, then r is a polynomial.

When r is a non-zero polynomial, then $\psi(p,r) = p - \partial_\alpha(r) + (X-\theta)^m r$ is a non-zero polynomial as well of degree $m + \deg r$. This is because ∂_α drops degrees and because $m > \deg p$. Thus $\psi(p,r) = 0$ implies $r = 0$, and then trivially $p = 0$ too. □

If the support of a height function h has cardinality the same as K, then there may exist E_α's constructed as in (1) which don't embed in $\mathcal{R}$, and others which do. We now construct examples of each sort.

Given a height function h, the construction of an E_α which does embed in $\mathcal{R}$ can be discovered from the proof of Theorem 3. There we see that all that are needed are a positive integer m, some θ in K and a functional $\alpha : K(X) \longrightarrow K$ such that $(\nu-\theta)^m + \langle \alpha, (X-\nu)^{-1} \rangle \neq 0$ when ν is in the support of h. For instance, $m = 1$, $\theta = 0$ and any α such that $\nu + \langle \alpha, (X-\nu)^{-1} \rangle \neq 0$ will do. The cardinality of the support of h here is quite irrelevant.

To build an E_α that does not embed in $\mathcal{R}$ let h have support S, and assume card S = card K. With this assumption one can find a function $f : S \longrightarrow K$ such that for any rational function t the set $\{\theta \in S : f(\theta) = t(\theta)\}$ is infinite, see e.g. [3, 2.1]. Let α be any functional such that $\langle \alpha, (X-\theta)^{-1} \rangle = -f(\theta)$ for all θ in S. Using any value of m, let E_α be constructed as in (1). By Theorem 2 any embedding $E_\alpha \longrightarrow \mathcal{R}$ would arise from an injection $\psi : P_m \oplus R_h \longrightarrow R_h$ given by $\psi(p,r) = s(p - \partial_\alpha(r)) + tr$ for suitable rational functions s, t. Since $s = \psi(1,0)$ and ψ is injective, $s \neq 0$. By replacing ψ by $s^{-1}\psi$, there is no loss in supposing $s = 1$, and $\psi(p,r) = p - \partial_\alpha(r) + tr$. For each θ in S, $(0, (X-\theta)^{-1}) \in P_m \oplus R_h$ and from (4)

$$\begin{aligned}\psi(0,(X-\theta)^{-1}) &= -\partial_\alpha((X-\theta)^{-1}) + t(X-\theta)^{-1} \\ &= (\langle \alpha, (X-\theta)^{-1} \rangle + t)(X-\theta)^{-1} \\ &= (t - f(\theta))(X-\theta)^{-1}.\end{aligned}$$

Now for infinitely many θ in S, $t(\theta) - f(\theta) = 0$. Thus, for these θ, the functions $(t - f(\theta))(X-\theta)^{-1}$ lie in the finite-dimensional space of functions having poles of order no more than the orders of the poles of t. Hence ψ maps an infinity of linearly independent elements of the form $(0, (X-\theta)^{-1})$ into a finite-dimensional space, contrary to its injective property. There are no embeddings $E_\alpha \longrightarrow \mathcal{R}$.

THEOREM 4. *Let* $0 \longrightarrow \mathrm{III}^m \longrightarrow E_\alpha \longrightarrow F_h \longrightarrow 0$ *be an extension as prescribed by* (1). *If* E_α *embeds in* $\mathcal{R}$ *and* h *assumes the value* ∞, *then* E_α *embeds in* F_h.

PROOF. The embedding of E_α into $\mathcal{R}$ comes from a map $\psi : P_m \oplus R_h \longrightarrow K(X)$ given by $\psi(p,r) = s(p - \partial_\alpha(r)) + tr$ for some s, t in $K(X)$. Suppose h assumes the value ∞ at θ in K. By thinking of rational functions as quotients of polynomials in $(X-\theta)^{-1}$ we obtain rational functions u, v with poles only at θ such that uvs and uvt have poles at most only at θ. Then $\psi_1 : P_m \oplus R_h \longrightarrow K(X)$ where $\psi_1(p,r) = (X-\theta)^{-m}uvs(p - \partial_\alpha(r)) + (X-\theta)^{-m}uvtr$ defines a new embedding into $\mathcal{R}$. Since $h(\theta) = \infty$, it follows that $(X-\theta)^{-m}uvtR_h + (X-\theta)^{-m}uvsR_h \subset R_h$, and $(X-\theta)^{-m}uvsP_m \subset R_h$. Furthermore, $\partial_\alpha(R_h) \subset R_h$. These facts combined yield $\psi_1(P_m \oplus R_h) \subset R_h$. It is just as evident that $\psi_1(P_{m-1} \oplus R_h^-) \subset R_h^-$. So, if $h(\theta) = \infty$, ψ_1 yields an embedding into F_h. If h assumed the value ∞ at ∞ rather than at θ in K, a similar argument would again yield the desired embedding into F_h. □

THEOREM 5. *If* $0 \longrightarrow \mathrm{III}^m \longrightarrow E_\alpha \longrightarrow F_h \longrightarrow 0$ *is an extension as prescribed by* (1), *and* $h(\theta) < \infty$ *for all* θ *in* $K \cup \{\infty\}$, *then no embedding of* E_α *into* F_h *exists.*

PROOF. Let $E_\alpha \longrightarrow \mathcal{R}$ be a homomorphism. This comes from a map $\psi : P_m \oplus R_h \longrightarrow K(X)$ which, by Theorem 2, is given by $\psi(p,r) = s(p - \partial_\alpha(r) + tr$ for suitable rational functions s, t. Suppose that $E_\alpha \longrightarrow \mathcal{R}$ sends E_α into its quotient F_h. In that case $\psi(P_m \oplus R_h) \subset R_h$.

For θ in K let $k(\theta) = h(\theta)$ whenever θ is a finite pole of s or t, and let $k(\theta) = 0$ otherwise. Also let $k(\infty) = h(\infty)$. As the height function k is supported on a finite set and k, like h, is finite valued, it follows that R_k is a finite-dimensional subspace of R_h. Note that $u \in R_k$ if and only if $u \in R_h$ and the finite poles of u, i.e. the poles in K, are among those of s or t.

If $(p,r) \in P_m \oplus R_k$, then $\psi(p,r) \in R_h$. Let θ be a finite pole of $\psi(p,r) = s(p - \partial_\alpha(r)) + tr$. Since p is a polynomial, θ must be a pole of one of s, t, r or $\partial_\alpha(r)$. Since $r \in R_k$, so is $\partial_\alpha(r) \in R_k$. Thus the poles of r and $\partial_\alpha(r)$ are among those of s or t. Therefore θ is a pole of s or t. This means that $\psi(p,r) \in R_k$, i.e. $\psi(P_m \oplus R_k) \subset R_k$. From this, the fact that $\dim P_m \geqq 1$ prevents ψ from being injective. □

In the light of the preceding results and examples, the following holds.

COROLLARY 6. *For a given height function* h, *every extension* $0 \longrightarrow \mathrm{III}^m \longrightarrow E_\alpha \longrightarrow F_h \longrightarrow 0$ *is such that* E_α *embeds in* F_h *if and only if the support of* h *has cardinality less than* K *and* h *takes on the value* ∞. □

In certain ways the extensions E_α which do embed in their quotients F_h are the more complicated modules. For instance such E_α, even if purely simple, may admit endomorphisms other than scalar multiples of the identity; whereas the endomorphism ring of each non-embeddable, purely simple E_α must be K. Consideration of endomorphisms is deferred to [6]. It will also be shown in [6] that the results of this paper are valid for modules over arbitrary finite-dimensional hereditary tame algebras.

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