# 14
# Eguchi–Kawai model

The large-$N$ reduction was discovered by Eguchi and Kawai [EK82] who showed that the Wilson lattice gauge theory on a $d$-dimensional hypercubic lattice is equivalent at $N = \infty$ to that on a hypercube with periodic boundary conditions. This construction is based on an extra $U(1)^d$ symmetry which is present in the reduced model to each order of the strong-coupling expansion.

Soon after that it was recognized that a phase transition occurs in the reduced model with decreasing coupling constant, so this symmetry is broken in the weak-coupling regime. To cure the construction at weak coupling, a quenching prescription was proposed by Bhanot, Heller and Neuberger [BHN82] and elaborated by many authors. The quenching prescription results in a reduced model which recovers multicolor QCD both on the lattice and in the continuum.

We start this chapter with the simplest example of a matrix-valued scalar theory. The quenched reduced model for this case was advocated by Parisi [Par82] on the lattice and elaborated by Gross and Kitazawa [GK82] in the continuum. Then we consider the Eguchi–Kawai reduction of multicolor QCD both on the lattice and in the continuum.

## 14.1 Reduction of the scalar field (lattice)

Let us begin with the simplest example of a matrix-valued scalar theory on a lattice, the partition function of which is defined by the path integral

$$Z = \int \prod_x \prod_{i \geq j} \mathrm{d}\varphi_x^{ij} \, \mathrm{e}^{-S[\varphi]} \tag{14.1}$$



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with the action

$$S[\varphi] \;=\; \sum_x N\,\mathrm{tr}\Big[-\sum_\mu \varphi_x\varphi_{x+a\hat{\mu}} + V(\varphi_x)\Big]\,. \tag{14.2}$$

Here $\varphi(x)$ is an $N \times N$ Hermitian matrix field and $V(\varphi)$ is a certain interaction potential, say

$$V(\varphi) \;=\; \frac{M}{2}\varphi^2 + \frac{\lambda_3}{3}\varphi^3 + \frac{\lambda_4}{4}\varphi^4\,. \tag{14.3}$$

The prescription of the large-$N$ reduction is formulated as follows.
We substitute

$$\varphi(x) \;\stackrel{\text{red.}}{\to}\; D^\dagger(x)\tilde{\varphi}D(x)\,, \tag{14.4}$$

where

$$D(x) \;=\; \mathrm{e}^{-\mathrm{i}P_\mu x_\mu} \tag{14.5}$$

with

$$P^\mu \;=\; \mathrm{diag}\left(p_1^\mu,\ldots,p_N^\mu\right) \tag{14.6}$$

being a diagonal Hermitian matrix. Explicitly we have

$$\varphi^{kj}(x) \;\stackrel{\text{red.}}{\to}\; \mathrm{e}^{\mathrm{i}(p_k-p_j)^\mu x_\mu}\tilde{\varphi}^{kj}. \tag{14.7}$$

The matrix $D(x)$ in Eq. (14.4) subsumes the coordinate dependence, so that $\tilde{\varphi}$ does *not* depend on $x$.

The averaging of a functional $F[\varphi_x]$ which is defined with the same weight as in Eq. (14.1),

$$\Big\langle F[\varphi_x]\Big\rangle \;=\; Z^{-1}\int\prod_x\prod_{i\geq j}\mathrm{d}\varphi_x^{ij}\,\mathrm{e}^{-S[\varphi]}\,F[\varphi_x]\,, \tag{14.8}$$

can be calculated at $N=\infty$ using

$$\Big\langle F[\varphi_x]\Big\rangle \;\stackrel{\text{red.}}{=}\; a^{Nd}\int\limits_{-\pi/a}^{\pi/a}\prod_{\mu=1}^{d}\prod_{i=1}^{N}\frac{\mathrm{d}p_i^\mu}{2\pi}\Big\langle F[D^\dagger(x)\tilde{\varphi}D(x)]\Big\rangle_{\mathrm{RM}}. \tag{14.9}$$

Here the RHS is calculated [Par82, GK82, DW82] for the *quenched reduced model*, for which the averages are defined by

$$\Big\langle F[\tilde{\varphi}]\Big\rangle_{\mathrm{RM}} \;\equiv\; Z_{\mathrm{RM}}^{-1}\int\mathrm{d}\tilde{\varphi}\,\mathrm{e}^{-S_{\mathrm{R}}[\tilde{\varphi}]}\,F[\tilde{\varphi}] \tag{14.10}$$

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with the reduced action

$$S_{\mathrm{R}}[\tilde{\varphi}] \quad = \quad -N \sum_{ij} |\tilde{\varphi}_{ij}|^2 \sum_{\mu} \cos\left[(p_i^\mu - p_j^\mu)a\right] + N \,\mathrm{tr}\, V(\tilde{\varphi})\,. \qquad (14.11)$$

We have put the symbol "red." on the top of the equality sign in Eq. (14.9) to emphasize that it holds as a result of the large-$N$ reduction.

The partition function of the reduced model is given by

$$Z_{\mathrm{RM}} \quad = \quad \int \mathrm{d}\tilde{\varphi}\, \mathrm{e}^{-S_{\mathrm{R}}[\tilde{\varphi}]} \qquad (14.12)$$

which can be deduced, modulo the volume factor in the action (14.11), from the partition function (14.1) by the substitution (14.4). The measure $\mathrm{d}\tilde{\varphi}$ in Eqs. (14.10) and (14.12) (as well as that in Eqs. (14.1) and (14.8)) is the one for integrating over $N \times N$ Hermitian matrices given by Eq. (13.2).

Similarly to Eq. (14.9), the free energy of the reduced model determines at large $N$ the free energy per unit volume of the $d$-dimensional theory:

$$\frac{1}{N^2}\frac{\ln Z}{V} \quad \stackrel{\text{red.}}{=} \quad a^{d(N-1)} \int_{-\pi/a}^{\pi/a} \prod_{\mu=1}^{d}\prod_{i=1}^{N} \frac{\mathrm{d}p_i^\mu}{2\pi} \frac{1}{N^2} \ln Z_{\mathrm{RM}}\,. \qquad (14.13)$$

Note that the integration over the momenta $p_i^\mu$ on the RHS of Eq. (14.9) is taken *after* the calculation of averages in the reduced model. Analogously, the logarithm of $Z_{\mathrm{RM}}$ is integrated over $p_i^\mu$ on the RHS of Eq. (14.13), rather than $Z_{\mathrm{RM}}$ itself. Such variables are usually called *quenched* in statistical mechanics which clarifies the terminology.

Since $N \to \infty$, it is not necessary to integrate over the quenched momenta in Eq. (14.9) or Eq. (14.13). The integral should be recovered if $p_i$ were uniformly distributed over a $d$-dimensional hypercube. This is analogous to the self-averaging phenomenon in condensed-matter physics.

In order to show how Eq. (14.9) works, let us demonstrate how the planar diagrams of perturbation theory for the matrix-valued scalar theory (14.1) are recovered in the quenched reduced model.

The quenched reduced model (14.12) is of the general type discussed in Chapter 11. The propagator is given by

$$\left\langle \tilde{\varphi}_{ij}\tilde{\varphi}_{kl} \right\rangle_{\mathrm{Gauss}} \quad = \quad \frac{1}{N} G(p_i - p_j)\, \delta_{il}\delta_{kj} \qquad (14.14)$$

with

$$G(p_i - p_j) \quad = \quad \frac{1}{M - 2\sum_\mu \cos\left[(p_i^\mu - p_j^\mu)a\right]}\,. \qquad (14.15)$$

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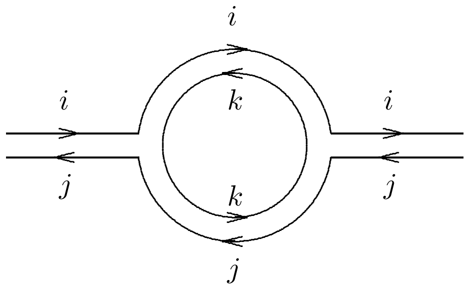


Fig. 14.1. The simplest planar diagram of second order in $\lambda_3$ for the propagator in the quenched reduced model (14.12). The momentum $p_i$ flows along the index line $i$. The momentum $p_i - p_j$ is associated with the double line $ij$.

It is convenient to associate the momenta $p_i$ and $p_j$ in Eq. (14.15) with each of the two index lines representing the propagator and carrying, respectively, indices $i$ and $j$. Remember, that these lines are oriented for a Hermitian matrix $\tilde{\varphi}$ and their orientation can be associated naturally with the direction of the flow of the momentum. The total momentum carried by the double line is $p_i - p_j$.

The simplest diagram which represents the second order in $\lambda_3$ correction to the propagator is depicted Fig. 14.1. The momenta $p_i$ and $p_j$ flow along the index lines $i$ and $j$, while the momentum $p_k$ circulates along the index line $k$. The contribution of the diagram in Fig. 14.1 is given by

$$\text{Fig. 14.1} \;=\; \frac{\lambda_3^2}{N^2}\, G^2(p_i - p_j) \sum_k G(p_i - p_k)\, G(p_k - p_j)\,, \tag{14.16}$$

where the summation over the index $k$ is just a standard one over indices forming a closed loop.

In order to show that the quenched-model result (14.16) reproduces the second order in $\lambda_3$ correction to the propagator in the $d$-dimensional theory on an infinite lattice, we pass to the variables of the total momenta flowing along the double lines:

$$\left.\begin{aligned} p_i - p_j &= p\,, \\ p_j - p_k &= q\,, \\ p_i - p_k &= p + q\,. \end{aligned}\right\} \tag{14.17}$$

This is obviously consistent with the momentum conservation at each of the two vertices of the diagram in Fig. 14.1.

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Since $p_k$ are uniformly distributed over the hypercube, the summation over $k$ can be substituted as $N \to \infty$ by the integral

$$\frac{1}{N}\sum_k f(p_k) \quad \Rightarrow \quad a^d \int\limits_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^d q}{(2\pi)^d} f(q)\,. \tag{14.18}$$

The prescription (14.9) then gives the correct expression

$$G^{(2)}(p) \quad = \quad a^d \frac{\lambda_3^2}{N}\, G^2(p) \int\limits_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^d q}{(2\pi)^d}\, G(q)\, G(p+q) \tag{14.19}$$

for the second-order contribution of the perturbation theory for the propagator on the lattice.

It is now clear how a generic planar diagram is recovered by the reduced model. We first represent the diagram by the double lines and associate the momentum $p_i$ to an index line carrying the index $i$. Then we write down an expression for the diagram in the reduced model with the propagator (14.15). Passing to momenta flowing along the double lines, similarly to Eq. (14.17), we obtain an expression which coincides with the *integrand* of the Feynman diagram for the theory on the $d$-dimensional lattice. It is crucial that such a change of variables can always be made for a planar diagram consistently with momentum conservation at each vertex. The last step is that a summation over indices of closed index lines reproduces an integration over momenta associated with each of the loops according to Eq. (14.18). It is assumed that the number of loops is much less than $N$ which is always true for a given diagram since $N$ is infinite.

Thus, we have shown how planar diagrams of the lattice theory defined by the partition function (14.1) are recovered in the reduced model (14.12). The lattice was needed only as a regularization to make all integrals well-defined and was not crucial in the consideration. In the next section we shall see how this construction can be formulated directly for the continuum theory.

### *Remark on large but finite $N$*

If $N$ is large but finite, the summation on the LHS of Eq. (14.18) runs over $N$ different momenta. Similarly, if a theory is defined on a periodic lattice of size $La$, the momentum takes on $L^d$ different values. One might think, therefore, that the quenched reduced model at very large but finite $N$ can be associated with a quantum field theory on a periodic lattice with $L = N^{1/d}$.

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## 14.2 Reduction of the scalar field (continuum)

The quenched reduced model can be formulated directly for the continuum theory. The proper formulas can be easily obtained from those of the previous section by setting $a \to 0$.

Equations (14.4)–(14.7) remain the same, while the derivative $\partial_\mu$ in the kinetic part of the continuum action

$$S[\varphi] \;=\; \int \mathrm{d}^d x \, N \, \mathrm{tr} \left\{ \frac{1}{2} (\partial_\mu \varphi)^2 + V(\varphi) \right\} \tag{14.20}$$

is substituted by $\mathrm{i}P_\mu$ acting in the adjoint representation.

The continuum reduced action

$$S_{\mathrm{R}} \;=\; v \, N \, \mathrm{tr} \left\{ -\frac{1}{2} [P_\mu, \tilde{\varphi}]^2 + \widetilde{V}(\tilde{\varphi}) \right\} \tag{14.21}$$

determines the propagator of the matrix $\tilde{\varphi}_{ij}$ in the continuum quenched reduced model, which is given by Eq. (14.14) with

$$G(p_i - p_j) \;=\; \frac{v^{-1}}{(p_i - p_j)^2 + m^2} \, . \tag{14.22}$$

The dimension of the reduced field $\tilde{\varphi}$ is $[\mathrm{mass}]^{(d-2)/2}$, i.e. the same as that of the field $\varphi(x)$ in the $d$-dimensional theory.

The normalizing factor of $v$ on the RHS of Eq. (14.21) (and therefore Eq. (14.22)) plays the role of the volume element for a given regularization and depends on the region for integration over the momenta $p_i$. If $p_i$ are restricted to a hypercube of size $2\Lambda$, the proper formulas look like their lattice counterparts (cf. Eq. (14.18)) with

$$v \;=\; a^d \qquad \boxed{\text{lattice regularization}} \tag{14.23}$$

and

$$a \;=\; \frac{\pi}{\Lambda} \, . \tag{14.24}$$

Lorentz invariance is restored as $\Lambda \to \infty$ at least for renormalizable theories.

A Lorentz-invariant regularization can be achieved by choosing $p_i$ inside a hypersphere. Alternatively, one can include the regularizing factor of $\exp\left(-p_i^2/\Lambda^2\right)$ in the integral over $p_i$ [GK82] by defining

$$\int^{\Lambda} \mathrm{d}^d p \cdots \;=\; \int \frac{\mathrm{d}^d p}{(\Lambda\sqrt{\pi})^d} \, \mathrm{e}^{-p^2/\Lambda^2} \cdots \, . \tag{14.25}$$

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Then,

$$v \;=\; \left(\frac{2\pi}{\Lambda^2}\right)^{d/2} \qquad \boxed{\text{regularization (14.25)}} \tag{14.26}$$

and

$$\Big\langle F[\varphi(x)]\Big\rangle \;\stackrel{\text{red.}}{=}\; \int^{\Lambda}\prod_{i=1}^{N}\mathrm{d}^d p_i\,\Big\langle F[D^\dagger(x)\tilde{\varphi}D(x)]\Big\rangle_{\text{RM}} \tag{14.27}$$

which is similar to Eq. (14.9) on the lattice.

Analogously to the lattice case, we expect the self-averaging phenomenon as $N \to \infty$ so that, instead of integration over $p_i$ according to Eq. (14.25), we can simply choose them at $N = \infty$ to be distributed spherically symmetrically with Gaussian weight

$$\rho(p) \;=\; \left(\sqrt{\pi}\Lambda\right)^{-d}\,\mathrm{e}^{-p^2/\Lambda^2}\,. \tag{14.28}$$

A comment is needed concerning the normalization factors. A consideration similar to the topological analysis of Sect. 11.4 leads us to the conclusion that a planar diagram with $n_2$ loops possesses a factor of $v^{-n_2}$ in the reduced model. It will normalize correctly the integral over momenta circulating along the $n_2$ loops, which remains to be done after the $N - n_2$ momenta $p_i$ (which do not appear in the diagram) are integrated out. The analogous factor for the free energy is $v^{-n_2+1}$ owing to the extra $v$ in the definition (14.21).

**Problem 14.1** Substituting (14.22) into Eq. (14.27), obtain a regularized propagator in the $d$-dimensional theory.

**Solution** Inserting (14.22) into Eq. (14.27), we find explicitly

$$\begin{aligned} G(x) \;&\equiv\; \Big\langle\varphi_{ij}(x)\varphi_{ji}(0)\Big\rangle \;\stackrel{\text{red.}}{=}\; \int^{\Lambda}\prod_{k=1}^{N}\mathrm{d}^d p_k\,\mathrm{e}^{\mathrm{i}(p_i-p_j)x}\Big\langle\tilde{\varphi}_{ij}\tilde{\varphi}_{ji}\Big\rangle_{\text{RM}} \\ &=\; \left(\frac{\Lambda^2}{2\pi}\right)^{d/2}\int\frac{\mathrm{d}^d p_i}{\left(\Lambda\sqrt{\pi}\right)^d}\frac{\mathrm{d}^d p_j}{\left(\Lambda\sqrt{\pi}\right)^d}\,\mathrm{e}^{-(p_i^2+p_j^2)/\Lambda^2}\,\frac{\mathrm{e}^{\mathrm{i}(p_i-p_j)x}}{\left(p_i-p_j\right)^2+m^2}\,. \end{aligned} \tag{14.29}$$

Introducing $p_\pm = p_i \pm p_j$ and accounting for a Jacobian, we have

$$\begin{aligned} G(x) \;&=\; \int\frac{\mathrm{d}^d p_+}{\left(2\pi\Lambda^2\right)^{d/2}}\,\mathrm{e}^{-p_+^2/2\Lambda^2}\int\frac{\mathrm{d}^d p_-}{(2\pi)^d}\,\mathrm{e}^{-p_-^2/2\Lambda^2}\,\frac{\mathrm{e}^{\mathrm{i}p_-x}}{p_-^2+m^2} \\ &=\; \int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{-p^2/2\Lambda^2}\,\frac{\mathrm{e}^{\mathrm{i}px}}{p^2+m^2} \end{aligned} \tag{14.30}$$

which gives a regularized propagator with the correct normalization.

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*Remark on higher genera*

The quenched reduced model reproduces only planar graphs of the original theory and fails to reproduce nonplanar graphs. This can be easily seen for the simplest nonplanar graph depicted in Fig. 11.3 where the same momentum circulates along the closed index line, so that the total momentum flowing along each of the two crossing double lines is zero in the reduced model. Of course, this is not the case for the original $d$-dimensional theory.

Similarly, the quenched reduced model reproduces only the factorized part of the correlators of "colorless" operators, for example $\operatorname{tr}\varphi^2(x_i)/N$, and cannot reproduce the connected correlators.

## 14.3 Reduction of the Yang–Mills field

The large-$N$ reduction of the Yang–Mills fields has its specific features owing to gauge invariance. In order to make the results rigorous, we begin in this section with the lattice formulation of Yang–Mills theory introduced in Chapter 6 and then describe the continuum case in the next section.

The general prescription (14.4) and (14.5) of the large-$N$ reduction is applicable for gauge fields. For the lattice gauge field $U_\mu(x)$, it gives

$$U_\mu(x) \quad \stackrel{\text{red.}}{\to} \quad D^\dagger(x)\widetilde{U}_\mu D(x)\,, \tag{14.31}$$

where $d$ unitary $N \times N$ matrices $\widetilde{U}_\mu^{ij}$ $(\mu = 1, \ldots, d)$ are $x$-independent.

It is easy to deduce what transformation of the reduced gauge field $\widetilde{U}_\mu$ is compatible with the lattice gauge transformation (6.13), where $\Omega(x)$ is to be reduced by

$$\Omega(x) \quad \stackrel{\text{red.}}{\to} \quad D^\dagger(x)\widetilde{\Omega}D(x)\,. \tag{14.32}$$

Here $\widetilde{\Omega}$ is again an $x$-independent unitary matrix.

If we first perform the gauge transformation (6.13) and then the reduction of the gauge-transformed field $U_\mu(x)$, we obtain

$$\begin{aligned}
&\Omega(x+a\hat{\mu})\, U_\mu(x)\, \Omega^\dagger(x) \\
&\quad \stackrel{\text{red.}}{\to} \quad D^\dagger(x+a\hat{\mu})\,\widetilde{\Omega}\, D(x+a\hat{\mu})\, D^\dagger(x)\widetilde{U}_\mu D(x) D^\dagger(x)\,\widetilde{\Omega}^\dagger D(x) \\
&\quad = \quad D^\dagger(x) D_\mu^\dagger \widetilde{\Omega} D_\mu \widetilde{U}_\mu \widetilde{\Omega}^\dagger D(x)\,,
\end{aligned} \tag{14.33}$$

where

$$D_\mu \quad \stackrel{\text{def}}{=} \quad D(x+a\hat{\mu})\, D^\dagger(x) \quad = \quad \mathrm{e}^{-\mathrm{i}P_\mu a} \tag{14.34}$$

for $D(x)$ given by Eqs. (14.5) and (14.6).

This determines the proper transformation of the reduced field $\widetilde{U}_\mu$ to be

$$\widetilde{U}_\mu \quad \xrightarrow{\text{g.t.}} \quad D_\mu^\dagger \widetilde{\Omega} D_\mu \widetilde{U}_\mu \widetilde{\Omega}^\dagger \,. \tag{14.35}$$

This transformation is referred to as the gauge transformation of the reduced gauge field.

The substitution of (14.31) into the Wilson action (6.18) results in the reduced action

$$\begin{aligned} S_\mathrm{R} &= \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}\,\mathrm{tr}\left[\widetilde{U}_\nu^\dagger D_\nu^\dagger \widetilde{U}_\mu^\dagger D_\nu D_\mu^\dagger \widetilde{U}_\nu D_\mu \widetilde{U}_\mu\right]\right\} \\ &= \frac{1}{2}\sum_{\mu\neq\nu}\left\{1-\frac{1}{N}\,\mathrm{tr}\left[\left(\widetilde{U}_\nu^\dagger D_\nu^\dagger\right)\left(\widetilde{U}_\mu^\dagger D_\mu^\dagger\right)\left(D_\nu \widetilde{U}_\nu\right)\left(D_\mu \widetilde{U}_\mu\right)\right]\right\}, \end{aligned} \tag{14.36}$$

where the equality between the first and second lines is because $D_\mu$ and $D_\nu^\dagger$ commute.

The structure of the RHS of Eq. (14.36) prompts us to introduce a new variable

$$U_\mu \;=\; D_\mu \widetilde{U}_\mu \,. \tag{14.37}$$

Then we obtain for the reduced action

$$S_\mathrm{R}[U] \;=\; \frac{1}{2}\sum_{\mu\neq\nu}\left(1-\frac{1}{N}\,\mathrm{tr}\, U_\nu^\dagger U_\mu^\dagger U_\nu U_\mu\right) \tag{14.38}$$

and the gauge transformation (14.35) also simplifies to

$$U_\mu \quad \xrightarrow{\text{g.t.}} \quad \widetilde{\Omega}\, U_\mu\, \widetilde{\Omega}^\dagger \,. \tag{14.39}$$

If the measure $\mathrm{d}\widetilde{U}_\mu$ for averaging over $\widetilde{U}_\mu$ is the Haar measure, it is not changed under (left) multiplication by a unitary matrix $D_\mu$: $\mathrm{d}\widetilde{U}_\mu = \mathrm{d}U_\mu$. Finally, we arrive at the reduced model discovered originally by Eguchi and Kawai [EK82].

Its partition function

$$Z_\mathrm{EK} \;=\; \int \prod_\mu \mathrm{d}U_\mu \,\mathrm{e}^{-NS_\mathrm{R}[U]/g^2} \tag{14.40}$$

is of the same type as Wilson's lattice gauge theory on a unit hypercube with periodic boundary conditions. There is no dependence on the

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quenched momenta $p_i$ in the action of the Eguchi–Kawai model since the $D$s have mutually canceled owing to the local gauge invariance of the lattice action (6.16).

In addition to the gauge symmetry (14.39), the Eguchi–Kawai model possesses symmetry under multiplication of $U_\mu$ by an element of $U(1)$, the center of $U(N)$, which depends on the direction $\mu$:*

$$U_\mu \quad \to \quad Z_\mu U_\mu \qquad (Z_\mu \in U(1))\,. \tag{14.41}$$

Such a global symmetry is also present, of course, for the Wilson action (6.16) but plays no special role there because of local gauge invariance. It will be crucial in providing the equivalence of the $d$-dimensional theory and the Eguchi–Kawai model at large $N$.

The equivalence of the $d$-dimensional lattice gauge theory and the Eguchi–Kawai model at $N = \infty$ states

$$\Big\langle F[U_\mu(x)] \Big\rangle \stackrel{\text{red.}}{=} a^{Nd} \int\limits_{-\pi/a}^{\pi/a} \prod_{\mu=1}^{d} \prod_{i=1}^{N} \frac{\mathrm{d}p_i^\mu}{2\pi} \Big\langle F\Big[D^\dagger(x+a\hat\mu)\, U_\mu D(x)\Big] \Big\rangle_{\mathrm{EK}}\,, \tag{14.42}$$

which is similar to Eq. (14.9) for scalars. Here the LHS is given by Eq. (6.39) and the RHS is calculated in the Eguchi–Kawai model:

$$\Big\langle F[U_\mu] \Big\rangle_{\mathrm{EK}} \;=\; Z_{\mathrm{EK}}^{-1} \int \prod_\mu \mathrm{d}U_\mu \, \mathrm{e}^{-NS_{\mathrm{R}}[U]/g^2}\, F[U_\mu]\,. \tag{14.43}$$

The commutativity of $D_\mu$ was used in representing the argument of $F$ on the RHS of Eq. (14.42) as $D^\dagger(x+a\hat\mu)\, U_\mu D(x)$. Note that it looks like a gauge transformation of a constant field in the $d$-dimensional theory.

For the latter reason Eq. (14.42) simplifies for the averages of gauge-invariant quantities when it takes the form

$$\Big\langle F[U_\mu(x)] \Big\rangle \stackrel{\text{red.}}{=} \Big\langle F[U_\mu] \Big\rangle_{\mathrm{EK}} \qquad \boxed{\text{gauge invariant } F} \tag{14.44}$$

as $N \to \infty$. In this formula there is no dependence on $D(x)$ and correspondingly the quenched momenta $p_i$ because $F[U_\mu(x)]$ is gauge invariant.

As has already been explained in Sect. 12.1, gauge-invariant observables in Yang–Mills theory can be expressed via the Wilson loops. Applying Eq. (14.44) for the Wilson loop averages (6.42), we obtain

$$\left\langle \frac{1}{N} \operatorname{tr} U(C) \right\rangle \stackrel{\text{red.}}{=} \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \prod_i U_{\mu_i} \right\rangle_{\mathrm{EK}} \tag{14.45}$$

* For the gauge group $SU(N)$, it is an element of $Z(N)$ rather than $U(1)$ for the Haar measure $\mathrm{d}U_\mu$ to be invariant.

as $N \to \infty$. In other words, the Wilson loop in the Eguchi–Kawai model is constructed as an (ordered) product of the constant matrices $U_{\mu_i}$ along the links forming the contour $C$. It is nontrivial since $U_\mu$ do not commute.

The equality (14.45) is possible since the Wilson loop averages in the original theory do not depend on the position of the beginning of the contour $C$ owing to translational invariance.

The simplest example of the Wilson loop average is that for a rectangular contour depicted in Fig. 6.6 on p. 111. It is represented in the Eguchi–Kawai model by

$$W(R \times \mathcal{T}) \;=\; \left\langle \frac{1}{N} \mathrm{tr}\, U_d^{\dagger\,\mathcal{T}} U_1^{\dagger\,R} U_d^{\mathcal{T}} U_1^{R} \right\rangle_{\mathrm{EK}} . \tag{14.46}$$

There is an important difference between the averages of open Wilson loops in the $d$-dimensional theory and the Eguchi–Kawai model. In the former case, the averages of open Wilson loops always vanish because of the local gauge invariance which cannot be broken spontaneously owing to Elitzur's theorem, which was already mentioned in Sect. 7.3. In the latter case, open Wilson loops are invariant under the transformation (14.39) since $\widetilde{\Omega}$ is the same at the beginning and the end of the contour:

$$\frac{1}{N} \mathrm{tr}\, \boldsymbol{P} \prod_i U_{\mu_i} \;\stackrel{(14.39)}{\longrightarrow}\; \frac{1}{N} \mathrm{tr} \left( \widetilde{\Omega}\, \boldsymbol{P} \prod_i U_{\mu_i} \widetilde{\Omega}^{\dagger} \right) \;=\; \frac{1}{N} \mathrm{tr}\, \boldsymbol{P} \prod_i U_{\mu_i} . \tag{14.47}$$

They are *not* invariant however under the $U(1)^d$ transformation (14.41):

$$\frac{1}{N} \mathrm{tr}\, \boldsymbol{P} \prod_i U_{\mu_i} \;\stackrel{(14.41)}{\longrightarrow}\; \prod_i Z_{\mu_i} \frac{1}{N} \mathrm{tr}\, \boldsymbol{P} \prod_i U_{\mu_i} . \tag{14.48}$$

Only closed Wilson loops, where each link occurs with an equal number of positive and negative orientations, are invariant. This symmetry is global and can be broken spontaneously as $N \to \infty$.

It is easy to see that no such breaking occurs within the strong-coupling expansion of the Eguchi–Kawai model, which is pretty much similar to that described in Sect. 6.5. For this reason the equivalence (14.44) holds at least for large enough values of $g^2 N$. It was shown [BHN82] that the $U(1)^d$ symmetry *is* spontaneously broken for small values of $g^2 N$ and therefore in the continuum. We shall return to this point in Sect. 14.5.

Two modifications of the Eguchi–Kawai model were proposed: the quenched Eguchi–Kawai model (described later in Sect. 14.6) and the twisted Eguchi–Kawai model (described later in Sect. 15.3). These two models are equivalent, in the large-$N$ limit, to the $d$-dimensional theory both on the lattice and in the continuum.

**Problem 14.2** Derive the loop equation for the Eguchi–Kawai model.

**Solution** The derivation is similar to Problem 12.6 on p. 265. We perform the shift of $U_\mu$:

$$U_\mu \;\to U_\mu\,(1-\mathrm{i}\epsilon_\mu)\,, \qquad U_\mu^\dagger \;\to (1+\mathrm{i}\epsilon_\mu)\,U_\mu^\dagger \tag{14.49}$$

which is same as in Eq. (6.22) for $x$-independent $\epsilon$. The resulting loop equation is given as [EK82]

$$\sum_p \left[W_{\mathrm{EK}}(C\,\partial p) - W_{\mathrm{EK}}\big(C\,\partial p^{-1}\big)\right] \;=\; g^2 N \sum_{l\in C} \tau_\nu(l)\, W_{\mathrm{EK}}(C_{yx})\, W_{\mathrm{EK}}(C_{xy})\,. \tag{14.50}$$

It is similar to Eq. (12.65) except the Kronecker symbol $\delta_{xy}$ is missing on the RHS of Eq. (14.50). It is restored if the averages of the open Wilson loops vanish, as prescribed by the unbroken $U(1)^d$ symmetry, since then we can substitute

$$W_{\mathrm{EK}}(C_{xy}) \;=\; \delta_{xy} W_{\mathrm{EK}}(C_{xx})\,. \tag{14.51}$$

The coincidence of the loop equations proves the equivalence of the two theories at $N=\infty$.

**Problem 14.3** Verify Eq. (14.51) by an explicit calculation to zeroth order in $g^2$.

**Solution** Extrema of the Eguchi–Kawai action (14.38) are given modulo a gauge transformation by diagonal matrices

$$U_\mu^{\mathrm{cl}} \;=\; \mathrm{e}^{-\mathrm{i}P_\mu a}\,. \tag{14.52}$$

This determines the Wilson loop average to zeroth order in $g^2$ to be

$$\begin{aligned} W_{\mathrm{EK}}(C_{yx}) \;&=\; a^{Nd} \int\limits_{-\pi/a}^{\pi/a} \prod_{i=1}^{N} \frac{\mathrm{d}^d p_i}{(2\pi)^d} \frac{1}{N} \sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}p_k(x-y)} \\ &=\; a^d \int\limits_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^d p}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}p(x-y)} \;=\; \delta_{xy}\,, \end{aligned} \tag{14.53}$$

where the integration over $P_\mu$ accounts for equivalent classical extrema. The Kronecker symbol in Eq. (14.53) appears because of the translational symmetry in momentum space.

## 14.4 The continuum Eguchi–Kawai model

The Eguchi–Kawai reduced model can be formulated directly for the continuum theory. The proper formulas can be derived from their lattice counterparts of the previous section by substituting

$$U_\mu \;=\; \mathrm{e}^{\mathrm{i}aA_\mu} \tag{14.54}$$

with $a \to 0$.

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The continuum Eguchi–Kawai model describes a reduction of the $d$-dimensional Yang–Mills theory at $N=\infty$ to a point. The action of the continuum Eguchi–Kawai model is given by

$$S_{\rm EK}[A] \;=\; -\left(\frac{2\pi}{\Lambda^2}\right)^{d/2}\frac{1}{4g^2}\,{\rm tr}\,[A_\mu,A_\nu]^2\,, \tag{14.55}$$

where $A_\mu$ are $d$ space-independent matrices.

The parameter $\Lambda$ has the dimension of mass, same as has $A_\mu$ in $d=4$. As we shall see in a moment, $\Lambda$ is to be associated with a momentum-space ultraviolet cutoff in the spirit of Sect. 14.2. In this chapter we assume the Lorentz-invariant regularization (14.25) when the normalization factor in Eq. (14.55) is given by Eq. (14.26). For the lattice regularization, $\Lambda$ is related to the lattice spacing $a$ by Eq. (14.24) and the normalization factor in Eq. (14.55) is to be changed according to Eq. (14.23).

Therefore, the very formulation of the continuum Eguchi–Kawai model implies a regularization.

The action (14.55) is obviously invariant under the gauge transformation

$$A_\mu \;\stackrel{\rm g.t.}{\longrightarrow}\; \widetilde{\Omega}\,A_\mu\,\widetilde{\Omega}^\dagger\,. \tag{14.56}$$

It is worth noting that, owing to Eqs. (14.37), (6.10), and (14.34), $A_\mu$ is associated with the reduction of the covariant derivative ${\rm i}\partial_\mu+\mathcal{A}_\mu(x)$ rather than the field $\mathcal{A}_\mu(x)$ itself:

$${\rm i}\partial_\mu+\mathcal{A}_\mu(x) \;\stackrel{\rm red.}{\rightarrow}\; D^\dagger(x)\,A_\mu\,D(x)\,. \tag{14.57}$$

This explains why Eq. (14.56) is consistent with the gauge transformation of the covariant derivative

$${\rm i}\partial_\mu+\mathcal{A}_\mu(x) \;\stackrel{\rm g.t.}{\longrightarrow}\; \Omega(x)\,[{\rm i}\partial_\mu+\mathcal{A}_\mu(x)]\,\Omega^\dagger(x) \tag{14.58}$$

rather than $\mathcal{A}_\mu(x)$ itself.

Similarly to Eq. (14.44),

$$\Big\langle F[{\rm i}\partial_\mu+\mathcal{A}_\mu(x)]\Big\rangle \;\stackrel{\rm red.}{=}\; \Big\langle F[A_\mu]\Big\rangle_{\rm EK} \qquad \boxed{\text{gauge invariant } F} \tag{14.59}$$

as $N\to\infty$ for gauge-invariant functionals $F$, where the LHS is calculated using the action (11.72) and the RHS is calculated using the Eguchi–Kawai action (14.55). For instance, the averages of closed Wilson loops coincide in both cases

$$\left\langle \frac{1}{N}\,{\rm tr}\,\boldsymbol{P}\,{\rm e}^{{\rm i}\oint {\rm d}\xi^\mu \mathcal{A}_\mu(\xi)}\right\rangle \;\stackrel{\rm red.}{=}\; \left\langle \frac{1}{N}\,{\rm tr}\,\boldsymbol{P}\,{\rm e}^{{\rm i}\oint {\rm d}\xi^\mu A_\mu}\right\rangle_{\rm EK}. \tag{14.60}$$

This is a continuum version of Eq. (14.45).

The continuum analog of the $U(1)^d$ symmetry (14.41) is the invariance of the Eguchi–Kawai action (14.55) under the shift of $A_\mu$ by a unit matrix:

$$A_\mu^{ij} \quad \to \quad A_\mu^{ij} + r_\mu \delta^{ij}, \tag{14.61}$$

where $r_\mu$ is a parameter of the transformation. It is often called the $R^d$ symmetry.

Under the transformation (14.61), an open Wilson loop is transformed as

$$\frac{1}{N}\mathrm{tr}\,\left(\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}}\mathrm{d}\xi^\mu A_\mu}\right) \quad \to \quad \mathrm{e}^{\mathrm{i}(y^\mu - x^\mu)r_\mu}\frac{1}{N}\mathrm{tr}\,\left(\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}}\mathrm{d}\xi^\mu A_\mu}\right). \tag{14.62}$$

This guarantees, if the symmetry is not broken, the vanishing of the averages of open Wilson loops

$$W_{\mathrm{EK}}(C_{yx}) \equiv \left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}}\mathrm{d}\xi^\mu A_\mu}\right\rangle_{\mathrm{EK}} \;=\; 0 \qquad \boxed{\text{for } y \neq x} \tag{14.63}$$

in the Eguchi–Kawai model.

Such vanishing in the $d$-dimensional theory is provided by the local gauge invariance under which

$$\left(\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}}\mathrm{d}\xi^\mu \mathcal{A}_\mu(\xi)}\right)_{ij} \quad \to \quad \left(\Omega(y)\,\boldsymbol{P}\,\mathrm{e}^{\mathrm{i}\int_{C_{yx}}\mathrm{d}\xi^\mu \mathcal{A}_\mu(\xi)}\Omega^\dagger(x)\right)_{ij}. \tag{14.64}$$

In contrast, the global symmetry (14.56) does *not* guarantee the vanishing of the averages of open Wilson loops in the Eguchi–Kawai model.

When and only when the $R^d$ symmetry (14.61) is not broken spontaneously, is the Eguchi–Kawai model equivalent to the $d$-dimensional Yang–Mills theory at large $N$.

The equivalence of the two theories can then be shown using the loop equation which is given for the Eguchi–Kawai model by

$$\begin{aligned}
\partial_\mu^x \frac{\delta}{\delta\sigma_{\mu\nu}(x)} W_{\mathrm{EK}}(C) \;&=\; \mathrm{i}\left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\,[A_\mu,[A_\mu,A_\nu]]\;\mathrm{e}^{\mathrm{i}\oint_{C_{xx}}\mathrm{d}\xi^\mu A_\mu}\right\rangle_{\mathrm{EK}} \\
&=\; -\mathrm{i}\lambda\left(\frac{\Lambda^2}{2\pi}\right)^{d/2}\left\langle \frac{1}{N}\,\mathrm{tr}\,\boldsymbol{P}\frac{\partial}{\partial A_\nu}\,\mathrm{e}^{\mathrm{i}\oint_{C_{xx}}\mathrm{d}\xi^\mu A_\mu}\right\rangle_{\mathrm{EK}} \\
&=\; \lambda\left(\frac{\Lambda^2}{2\pi}\right)^{d/2}\oint_C \mathrm{d}y_\nu\, W_{\mathrm{EK}}(C_{yx})\, W_{\mathrm{EK}}(C_{xy})\,,
\end{aligned} \tag{14.65}$$

where $\lambda = g^2 N$. The RHS is pretty much similar to that in Eq. (12.59), while $(\Lambda^2/2\pi)^{d/2}$ is present instead of $\delta^{(d)}(x-y)$.

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In order to show how the two RHSs are essentially equal to each other providing the $R^d$ symmetry is not broken, we need to remember that the continuum Eguchi–Kawai model is, in fact, somehow regularized.

While the action (14.55) is formally invariant under the transformation (14.61) for arbitrary $r_\mu$, admitable values of $r_\mu$ should be much smaller than the cutoff $\Lambda$. This is clear, in particular, from the lattice formula (14.41) where

$$Z_\mu \;=\; \mathrm{e}^{\mathrm{i}ar_\mu} \tag{14.66}$$

and to obtain Eq. (14.61) we expand in $a$ which destroys the compactness.

For this reason we expect the average of an open Wilson loop to vanish in the continuum Eguchi–Kawai model only when the distance $|y-x|$ between the end points $x$ and $y$ is much larger than the ultraviolet cutoff $1/\Lambda$. Otherwise, we may regard the loop to be essentially closed since distances smaller than the cutoff make no sense in the theory.

Introducing a smeared delta-function $\delta_\Lambda^{(d)}(x-y)$, for example, by

$$\delta_\Lambda^{(d)}(x) \;=\; \left(\frac{\Lambda^2}{2\pi}\right)^{d/2} \mathrm{e}^{-x^2\Lambda^2/2}\,, \tag{14.67}$$

we therefore expect something like*

$$W_{\mathrm{EK}}(C_{yx}) \;\approx\; \frac{\delta^{(d)}_{\Lambda/\sqrt{2}}(x-y)}{\delta^{(d)}_{\Lambda/\sqrt{2}}(0)} W_{\mathrm{EK}}(C_{xx}) \tag{14.68}$$

for the averages of open Wilson loops in the continuum Eguchi–Kawai model.

Finally, the delta function is recovered on the RHS of Eq. (14.65) as

$$\left(\frac{\Lambda^2}{2\pi}\right)^{d/2} \left[\frac{\delta^{(d)}_{\Lambda/\sqrt{2}}(x)}{\delta^{(d)}_{\Lambda/\sqrt{2}}(0)}\right]^2 \;=\; \left(\frac{\Lambda^2}{2\pi}\right)^{d/2} \mathrm{e}^{-x^2\Lambda^2/2} \;=\; \delta_\Lambda^{(d)}(x) \;\to\; \delta^{(d)}(x)\,, \tag{14.69}$$

reproducing the delta function on the RHS of Eq. (12.59).

This demonstrates the equivalence of the continuum Eguchi–Kawai model and the $d$-dimensional Yang–Mills theory at large $N$ under the assumption that the $R^d$ symmetry is not broken. The consideration simply repeats the proof of the equivalence given in Problem 14.2 on p. 336 by using the lattice regularization.

* Why it should be $\delta^{(d)}_{\Lambda/\sqrt{2}}$ rather than $\delta^{(d)}_\Lambda$ is clear from Eq. (14.69) and Problem 14.4.

**Problem 14.4** Verify Eq. (14.68) by explicit calculation to zeroth order in $g^2$, regularizing the integral over the zero modes of $A_\mu$ by Eq. (14.25).

**Solution** The calculation is similar to that in Problem 14.3 for the lattice case. Extrema of the continuum Eguchi–Kawai action (14.55) are given modulo a gauge transformation by diagonal matrices $A_\mu^{\rm cl} = -P_\mu$. This determines the Wilson loop average to zeroth order in $g^2$ to be

$$\begin{aligned} W_{\rm EK}(C_{yx}) &= \int^\Lambda \prod_{i=1}^N {\rm d}^d p_i \frac{1}{N} \sum_{k=1}^N {\rm e}^{{\rm i}p_k(x-y)} \\ &= \int \frac{{\rm d}^d p}{(\Lambda\sqrt{\pi})^d}\, {\rm e}^{-p^2/\Lambda^2}\, {\rm e}^{{\rm i}p(x-y)} = \frac{\delta^{(d)}_{\Lambda/\sqrt{2}}(x-y)}{\delta^{(d)}_{\Lambda/\sqrt{2}}(0)}, \end{aligned} \tag{14.70}$$

where the integration over $P_\mu$ accounts for the zero modes of $A_\mu$.

The $R^d$ symmetry *is*, in fact, broken spontaneously in the continuum Eguchi–Kawai model for $d > 2$ as is discussed in the next section. For this reason the equivalence between the $d$-dimensional theory and the naive continuum Eguchi–Kawai model described in this section is valid, strictly speaking, only in $d = 2$. The reduced model should be slightly modified to be equivalent to the $d$-dimensional theory for $d > 2$. Such a modification, which is based on the quenched momentum prescription, is described later in Sect. 14.6.

## 14.5 $R^d$ symmetry in perturbation theory

Since $N$ is infinite, the $R^d$ symmetry can be broken spontaneously. The point is that the large-$N$ limit plays the role of a statistical average, as has already been mentioned in Sect. 11.8, and phase transitions are possible for an infinite number of degrees of freedom. This phenomenon occurs [BHN82] in perturbation theory for the naive Eguchi–Kawai model with $d > 2$.

A perturbation theory can be constructed by expanding the fields around solutions of the classical equation

$$[A_\mu,[A_\mu,A_\nu]] = 0\,. \tag{14.71}$$

An arbitrary diagonal matrix

$$A_\mu^{\rm cl} = -P_\mu \tag{14.72}$$

is a solution to Eq. (14.71) associated with the minimal value $S_{\rm EK} = 0$ of the action (14.55).

The perturbation theory of the reduced model can be constructed by expanding around the classical solution (14.72):

$$A_\mu = A_\mu^{\rm cl} + g b_\mu\,, \tag{14.73}$$

where $b_\mu$ is off-diagonal.

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Substituting (14.73) into the action (14.55), we obtain

$$S_{\mathrm{EK}} = -\left(\frac{2\pi}{\Lambda^2}\right)^{d/2} \operatorname{tr}\left\{\frac{1}{2}[P_\mu, b_\nu]^2 - \frac{1}{2}[P_\mu, b_\mu]^2\right\} + \text{higher orders}\,. \tag{14.74}$$

To fix the gauge symmetry (14.56), it is convenient to add

$$S_{\mathrm{gf}} = -\left(\frac{2\pi}{\Lambda^2}\right)^{d/2} \operatorname{tr}\left\{\frac{1}{2}[P_\mu, b_\mu]^2 + [P_\mu, \bar{c}][P_\mu, c]\right\}, \tag{14.75}$$

where $c$ and $\bar{c}$ are ghosts.

The sum of (14.74) and (14.75) gives

$$S_2 = -\left(\frac{2\pi}{\Lambda^2}\right)^{d/2} \operatorname{tr}\left\{\frac{1}{2}[P_\mu, b_\nu]^2 + [P_\mu, \bar{c}][P_\mu, c]\right\} \tag{14.76}$$

to quadratic order in $b_\mu$.

Performing the Gaussian integral over $b_\nu$, we find at the one-loop level:

$$\int \mathrm{d}P_\mu\, \mathrm{d}b_\mu\, \mathrm{e}^{-S_2} \cdots = \int \prod_{k=1}^{N} \mathrm{d}^d p_k \prod_{i<j} \left[(p_i - p_j)^2\right]^{2-d} \cdots, \tag{14.77}$$

where the integration over $P_\mu$ accounts for the moduli space of classical solutions.

For $d = 1$ the product on the RHS of Eq. (14.77) reproduces the square of the Vandermonde determinant (13.14). For $d = 2$ the exponent $2 - d$ vanishes so that the product equals unity and does not affect the dynamics. For $d \geq 3$ the measure is singular and the eigenvalues collapse. This leads us to a spontaneous breakdown of the $R^d$ symmetry in perturbation theory.

*Remark on supersymmetric case*

In a supersymmetric gauge theory, there is an extra contribution from fermions to the exponent on the RHS of Eq. (14.77). Since the integration over fermions results in the extra factor of $[(p_i - p_j)^2]^{\operatorname{tr}\mathbb{I}}$, this finally yields the exponent $2 - d + \operatorname{tr}\mathbb{I}$. It vanishes in $d = 4$ for either Majorana or Weyl fermions and in $d = 10$ for the Majorana–Weyl fermions. This explicit calculation [IKK97] confirms, at first sight, the claim [MK83] that $R^d$ symmetry is not broken perturbatively in supersymmetric Yang–Mills theory and no quenching is needed in the supersymmetric case. This statement seems, in fact, to be not quite correct because of fermionic zero modes [AIK00].

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### 14.6 Quenched Eguchi–Kawai model

Soon after the breakdown of the $R^d$ symmetry in perturbation theory was discovered for the Eguchi–Kawai model, a cure for the problem was proposed [BHN82]. The idea was to treat the eigenvalues of the Hermitian matrix $A_\mu$ as being quenched rather than dynamical variables.

In order to separate the degrees of freedom associated with the eigenvalues, we represent $A_\mu$ in a canonical form

$$A_\mu \;=\; -V_\mu P_\mu V_\mu^\dagger\,, \tag{14.78}$$

where $P_\mu$ is diagonal and $V_\mu$ is unitary. The measure for integration over $A_\mu$ is then represented in a standard Weyl form

$$\mathrm{d}A_\mu \;=\; \mathrm{d}P_\mu\,\mathrm{d}V_\mu\,\Delta^2(P_\mu)\,, \tag{14.79}$$

where $\mathrm{d}V_\mu$ denotes the Haar measure* on $U(N)$ and $\Delta(P_\mu)$ is the Vandermonde determinant defined by Eq. (13.14). Equation (14.79) is the same as Eq. (13.13) for the one-matrix case.

Note that the substitution (14.78) is consistent with the gauge symmetry (14.56), which is equivalent to the left multiplication

$$V_\mu \;\to\; \widetilde{\Omega}\,V_\mu\,. \tag{14.80}$$

The Haar measure $\mathrm{d}V_\mu$ is invariant under such a multiplication.

In the quenched Eguchi–Kawai model, $A_\mu$ is substituted by Eq. (14.78) both in the reduced action (14.55) and in the averaging functionals. But the averaging is taken only with respect to the $V_\mu$ variables considering $P_\mu$ as quenched variables. The averages are then integrated over $P_\mu$ which is quite analogous to Eq. (14.27):

$$\Big\langle F[\mathrm{i}\partial_\mu + \mathcal{A}_\mu(x)]\Big\rangle \;\stackrel{\text{red.}}{=}\; \int^\Lambda \prod_{i=1}^N \mathrm{d}^d p_i \left\langle F\Big[-D^\dagger(x)\,V_\mu P_\mu V_\mu^\dagger D(x)\Big]\right\rangle_{\mathrm{QEK}}. \tag{14.81}$$

The average on the RHS of Eq. (14.81) is defined for the quenched Eguchi–Kawai model by

$$\begin{aligned}&\left\langle F\Big[V_\mu P_\mu V_\mu^\dagger\Big]\right\rangle_{\mathrm{QEK}}\\ &\quad=\; Z_{\mathrm{QEK}}^{-1}\int\prod_\nu \mathrm{d}V_\nu\,\Delta^2(P_\nu)\,\mathrm{e}^{-S_{\mathrm{EK}}\left[V_\mu P_\mu V_\mu^\dagger\right]}\,F\Big[V_\mu P_\mu V_\mu^\dagger\Big]\end{aligned} \tag{14.82}$$

* Strictly speaking, $V_\mu$ in Eq. (14.78) should be off-diagonal to match the number of degrees of freedom, so the measure $\mathrm{d}V_\mu$ should be the Haar measure on the coset $U(N)/U(1)^N$. But nothing depends on these diagonal degrees of freedom of $V_\mu$ since $P_\mu$ is diagonal. We simply normalize the proper (compact) integrals over these diagonal degrees of freedom of a unitary matrix to unity.

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and

$$Z_{\mathrm{QEK}} = \int \prod_{\nu} \mathrm{d}V_\nu \, \Delta^2(P_\nu) \, \mathrm{e}^{-S_{\mathrm{EK}}\left[V_\mu P_\mu V_\mu^\dagger\right]} \tag{14.83}$$

is the partition function of the quenched Eguchi–Kawai model.

Similarly to Eq. (14.13), the free energy per unit volume is given as $N \to \infty$ by

$$\frac{1}{N^2}\frac{\ln Z}{V} = \left(\frac{\Lambda^2}{2\pi}\right)^{d/2} \int^{\Lambda} \prod_{i=1}^{N} \mathrm{d}^d p_i \, \frac{1}{N^2} \ln Z_{\mathrm{QEK}} \, . \tag{14.84}$$

This prescription for constructing the quenched Eguchi–Kawai model is very similar to what is described in Sect. 14.2 for scalars. The measure $\mathrm{d}A_\mu$ is split according to Eq. (14.79) but the integration in Eq. (14.82) or Eq. (14.83) is solely over $V_\mu$, keeping $P_\mu$ quenched. Only these averages of the quenched Eguchi–Kawai model (or the logarithm of the partition function in Eq. (14.84)) are integrated over the quenched momenta $p_i$ according to Eq. (14.81).

This is crucial to cure the breakdown of the $R^d$ symmetry in perturbation theory. The perturbative calculation in the quenched Eguchi–Kawai model looks like that of the previous section since now the classical vacuum is associated with $V_\mu = 1$ (modulo a gauge transformation). Instead of integrating over the distinct classical vacua as in the naive Eguchi–Kawai model, we have in the quenched Eguchi–Kawai model integration over the quenched variables $p_i$ which enters differently. The factor of $\prod_{i<j}[(p_i - p_j)^2]^{2-d}$, which resulted in the breaking of the $R^d$ symmetry in Eq. (14.77), appears now both in the numerator and denominator of the averages and thus cancels. Similarly, its logarithm is integrated over $p_i$ in Eq. (14.84) which does not result in a collapse of eigenvalues in the quenched Eguchi–Kawai model. The $R^d$ symmetry is *not* broken perturbatively in the quenched Eguchi–Kawai model and it is equivalent to the $d$-dimensional Yang–Mills theory in the $N = \infty$ limit.

Just as in the scalar case, we can substitute the integration over $p_i^\mu$ in Eq. (14.81) at $N = \infty$ by distributing them with a proper weight. It is again convenient to choose the weight (14.28) as is prescribed by Eq. (14.25). In contrast to the momentum regularization in the $d$-dimensional gauge theory, this results in a gauge-invariant regularization of perturbation theory since the eigenvalues of $A_\mu$ are gauge invariant (cf. Eq. (14.57)).

In fact, the precise form of the measure for integrating over $p_i$ on the RHS of Eq. (14.81) is not essential as $N \to \infty$. All that is needed from the measure is for the integral over $p_i$ to converge, which would protect the eigenvalues from collapsing in perturbation theory. Any other measure performing the same job is as good as this one.

For the same reason, the precise form of the distribution of the quenched momenta, substituting the integration at $N = \infty$, is not essential if it is smooth. The distribution (14.28) simply provides a nice gauge-invariant regularization of perturbation theory which is of the same type as the proper-time regularization.

Given a distribution of the eigenvalues $p_i^\mu$, Eq. (14.81) simplifies to

$$\left\langle F[\mathrm{i}\partial_\mu + \mathcal{A}_\mu(x)] \right\rangle \stackrel{\text{red.}}{=} \left\langle F\left[-D^\dagger(x) V_\mu P_\mu V_\mu^\dagger D(x)\right] \right\rangle_{\text{QEK}}. \tag{14.85}$$

In particular, the averages of closed Wilson loops are given by

$$\left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, \mathrm{e}^{\mathrm{i} \oint \mathrm{d}\xi^\mu \mathcal{A}_\mu(\xi)} \right\rangle \stackrel{\text{red.}}{=} \left\langle \frac{1}{N} \operatorname{tr} \boldsymbol{P} \, \mathrm{e}^{-\mathrm{i} \oint \mathrm{d}\xi^\mu V_\mu P_\mu V_\mu^\dagger} \right\rangle_{\text{QEK}}. \tag{14.86}$$

The averages of open Wilson loops in the quenched Eguchi–Kawai model obey Eq. (14.68).

A formal proof of the equivalence of the $d$-dimensional Yang–Mills theory in the large-$N$ limit and the quenched Eguchi–Kawai model can be given [GK82, Mig82] using the loop equation. To derive the equation for the Wilson loops in the quenched Eguchi–Kawai model, which are defined by the RHS of Eq. (14.86), we perform the right shift of the unitary matrix $V_\mu$:

$$\delta V_\mu \;=\; \mathrm{i} V_\mu \epsilon_\mu \,, \tag{14.87}$$

where $\epsilon_\mu$ is Hermitian. Substituting into Eq. (14.78), we obtain

$$\delta A_\mu \;=\; \mathrm{i} V_\mu \left[P_\mu, \epsilon_\mu\right] V_\mu^\dagger \tag{14.88}$$

under the shift (14.87).

Using the gauge symmetry (14.80), we can always choose the gauge where $V_\mu = 1$ for the given $\mu$. Then

$$\delta A_\mu \;=\; \mathrm{i} \left[P_\mu, \epsilon_\mu\right] \tag{14.89}$$

does not depend on $V_\mu$.

The variation (14.89) is almost the same as that which resulted in the loop equation (14.65) of the Eguchi–Kawai model. The only difference resides in the fact that the variation (14.87) does not change the eigenvalues of $A_\mu$. When we expand the induced variation of $A_\mu$, given by Eq. (14.89), in the Lie algebra basis, no diagonal generators appear. But their number is $\sim N$ and hence $\mathcal{O}\big(N^{-1}\big)$ of the total number of generators. For this reason, the Wilson loop averages in the quenched Eguchi–Kawai model

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obey at $N = \infty$ the same loop equation as in the naive Eguchi–Kawai model. Additional terms of order $1/N$ appear in the loop equation of the quenched Eguchi–Kawai model since diagonal generators, which are needed for the completeness condition (11.6), are missing. Hence, corrections to the $N = \infty$ loop equation of the quenched Eguchi–Kawai model are $\sim 1/N$ rather than $\sim 1/N^2$ as in the $d$-dimensional Yang–Mills theory.

This demonstrates once again that quenched reduced models can reproduce only planar diagrams of the $d$-dimensional theories but cannot reproduce diagrams of higher genera.

The representation (14.82) of the averages in the quenched Eguchi–Kawai model does not look like that in gauge theories where the averaging is over quantum fluctuations of $A_\mu$. The quenched Eguchi–Kawai model can, however, be represented in such a form as is shown by Gross and Kitazawa [GK82].

Let us introduce

$$1 = \int \mathrm{d}A_\mu \, \delta\Big(A_\mu + V_\mu P_\mu V_\mu^\dagger\Big) \tag{14.90}$$

into the numerator and denominator on the RHS of Eq. (14.82).

Changing the order of integration over $\mathrm{d}A_\mu$ and $\mathrm{d}V_\mu$, we obtain

$$\Big\langle F[A] \Big\rangle_{\mathrm{QEK}} = \frac{\int \prod_\mu \mathrm{d}A_\mu \, C(A,P) \, \mathrm{e}^{-S_{\mathrm{EK}}[A]} \, F[A]}{\int \prod_\mu \mathrm{d}A_\mu \, C(A,P) \, \mathrm{e}^{-S_{\mathrm{EK}}[A]}}, \tag{14.91}$$

where

$$C(A;P) = \int \prod_\mu \mathrm{d}V_\mu \, \delta\Big(A_\mu + V_\mu P_\mu V_\mu^\dagger\Big) \, \Delta^2(P_\mu) \,. \tag{14.92}$$

And analogously,

$$Z_{\mathrm{QEK}} = \int \prod_\mu \mathrm{d}A_\mu \, C(A,P) \, \mathrm{e}^{-S_{\mathrm{EK}}[A]} \tag{14.93}$$

for the partition function of the quenched Eguchi–Kawai model from Eq. (14.83).

Substituting

$$A_\mu = -P_\mu + g b_\mu \,, \tag{14.94}$$

we can calculate $C(A,P)$ at least perturbatively in $g$. Evaluating the integral on the RHS of Eq. (14.92), we find

$$C(A,P) = \prod_\mu \prod_{i=1}^{N} \delta\left( b_{ii}^\mu + g \sum_{j \neq i} \frac{|b_{ij}^\mu|^2}{p_i^\mu - p_j^\mu} + \mathcal{O}\big(b_\mu^3\big) \right) \tag{14.95}$$

to quadratic order in $b_\mu$.

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The meaning of this constraint is obvious: diagonal elements of $b_\mu$ are expressed via off-diagonal elements for the eigenvalues of $A_\mu$ to coincide with $-p_i^\mu$. In particular, the diagonal elements of $b_\mu$ vanish to the leading order. This vanishing of $b_{ii}^\mu$ is however not a gauge-invariant condition to higher orders in $g$. The role of the higher terms in the argument of the delta-function in Eq. (14.95) is to ensure gauge invariance to all orders in $g$ as $C(A,P)$ is gauge invariant according to the definition (14.92).

The constraint (14.95) restricts only $N$ out of $N^2$ degrees of freedom, which explains why it is inessential, say, in the large-$N$ limit of the loop equations in the quenched Eguchi–Kawai model.

The presence of the delta-function affects, however, the dynamics of the degrees of freedom associated with the diagonal elements $A_{ii}$. In particular, the analog of the continuum propagator (14.22) is given by

$$\left\langle b_{ij}^\mu b_{ji}^\nu \right\rangle_{\mathrm{QEK}} = \begin{cases} \left(\dfrac{\Lambda^2}{2\pi}\right)^{d/2} \dfrac{\delta_{\mu\nu}}{(p_i - p_j)^2} & i \neq j \\ 0 & i = j \end{cases} \tag{14.96}$$

which cures the divergence of a massless propagator at $i = j$.

If the constraint (14.95) is solved for $b_{ii}^\mu$ versus off-diagonal components and the result is substituted into the action, this will generate new interactions. The diagrams of perturbation theory in the quenched Eguchi–Kawai model coincide with the integrands of the planar Feynman graphs in the $d$-dimensional Yang–Mills theory except for diagrams with the new vertices which are needed for gauge invariance of the quenched Eguchi–Kawai model. The sum of these additional diagrams vanishes [GK82] after averaging over the quenched momenta.

**Problem 14.5** Derive Eq. (14.95) to quadratic order in $b_\mu$.

**Solution** We need to solve the equation

$$P_\mu - g b_\mu = V_\mu P_\mu V_\mu^\dagger \tag{14.97}$$

for $V_\mu$ iteratively in $g$. Substituting

$$V_\mu = \mathrm{e}^{\mathrm{i} g h_\mu}\,, \tag{14.98}$$

we find that Eq. (14.97) is reduced to the linear order in $g$ to

$$b_{ij}^\mu = \mathrm{i}\left(p_i^\mu - p_j^\mu\right) h_{ij}^\mu\,. \tag{14.99}$$

This equation requires $b_{ii}^\mu = 0$ and fixes off-diagonal components of $h_{ij}^\mu$ to be

$$h_{ij}^\mu = -\mathrm{i}\frac{b_{ij}^\mu}{p_i^\mu - p_j^\mu}\,. \tag{14.100}$$

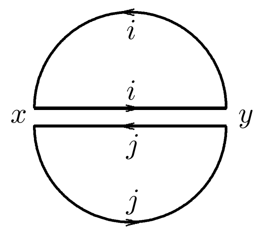


Fig. 14.2. Index-space diagram for the average of closed Wilson loop to order $g^2$. The momentum $p_i$ or $p_j$ flows along the index line $i$ or $j$. The momentum $p_i - p_j$ is associated with the double line $ij$. The diagram is associated with an analytic formula given in Eqs. (14.102) and (14.103).

To the quadratic order in $g$, only $h^\mu_{ij}$ to the linear order contributes to the diagonal components of Eq. (14.97) since a commutator with the diagonal matrix $P_\mu$ has no diagonal components. Then Eq. (14.97) yields

$$b^\mu_{ii} \;=\; g \sum_{j \neq i} \left(p^\mu_i - p^\mu_j\right) h^\mu_{ij} h^\mu_{ji} \;=\; -\, g \sum_{j \neq i} \frac{\left|b^\mu_{ij}\right|^2}{p^\mu_i - p^\mu_j} \tag{14.101}$$

which reproduces the argument of the delta-function in Eq. (14.95).

**Problem 14.6** Calculate the average of a closed Wilson loop in the quenched Eguchi–Kawai model to order $g^2$.

**Solution** The calculation is similar to that in Problem 14.1 on p. 331. Substituting Eq. (14.94) into the RHS of Eq. (14.86) and expanding to order $g^2$, we obtain

$$W^{(2)}_{\text{QEK}}(C) \;=\; -\frac{\lambda}{2} \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\nu \frac{1}{N^2} \sum_{i,j=1}^{N} \mathrm{e}^{\mathrm{i}(p_i - p_j)(y-x)} \left\langle b^\mu_{ij} b^\nu_{ji} \right\rangle_{\text{QEK}} \tag{14.102}$$

since $P_\mu$ and $b_\nu$ do not commute. The associated index-space diagram is depicted in Fig. 14.2. For the distribution of eigenvalues given as $N \to \infty$ by the Gaussian weight (14.28), we have using Eq. (14.96)

$$\begin{aligned} W^{(2)}_{\text{QEK}}(C) \;=\; & -\frac{\lambda}{2} \oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \\ & \times \left(\frac{\Lambda^2}{2\pi}\right)^{d/2} \int \frac{\mathrm{d}^d p_i}{(\Lambda\sqrt{\pi})^d} \frac{\mathrm{d}^d p_j}{(\Lambda\sqrt{\pi})^d} \, \mathrm{e}^{-p_i^2/\Lambda^2 - p_j^2/\Lambda^2} \, \frac{\mathrm{e}^{\mathrm{i}(p_i - p_j)(y-x)}}{(p_i - p_j)^2} \, . \end{aligned} \tag{14.103}$$

Introducing the variables $p_\pm = p_i \pm p_j$ and accounting for a Jacobian, we have

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quite similarly to Problem 14.1:

$$W^{(2)}_{\mathrm{QEK}}(C) \;=\; -\frac{\lambda}{2}\oint_C \mathrm{d}x_\mu \oint_C \mathrm{d}y_\mu \int \frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{-p^2/2\Lambda^2}\,\frac{\mathrm{e}^{\mathrm{i}p(y-x)}}{p^2} \tag{14.104}$$

which reproduces (a regularized version of) Eq. (12.17) with the correct normalization.

*Remark on the quenched Eguchi–Kawai model on the lattice*

The quenched Eguchi–Kawai model was originally formulated on a lattice [BHN82]. All the formulas are analogous to those given above in this section, while taking into account the fact that $U_\mu$ is compact on the lattice.

The analogs of Eqs. (14.78) and (14.79) are given by

$$U_\mu \;=\; V_\mu\,\mathrm{e}^{-\mathrm{i}P_\mu a}\,V_\mu^\dagger \tag{14.105}$$

and

$$\mathrm{d}U_\mu \;=\; \mathrm{d}P_\mu\,\mathrm{d}V_\mu\,\Delta^2\big(\,\mathrm{e}^{-\mathrm{i}P_\mu a}\big)\,, \tag{14.106}$$

where explicitly

$$\Delta\big(\,\mathrm{e}^{-\mathrm{i}P_\mu a}\big) \;=\; \prod_{i<j} 2\sin\Big(\frac{p_i^\mu - p_j^\mu}{2}a\Big)\,. \tag{14.107}$$

The quenched variables $p_i^\mu \in (-\pi/a, +\pi/a]$ play the role of the lattice momenta restricted to the Brillouin zone.

The measure $\mathrm{d}U_\mu$ of the naive Eguchi–Kawai model (14.43) is multiplied by

$$C(U,P) \;=\; \int \prod_\mu \mathrm{d}V_\mu\,\delta\Big(U_\mu - V_\mu\,\mathrm{e}^{-\mathrm{i}P_\mu a}V_\mu^\dagger\Big)\,\Delta^2\big(\,\mathrm{e}^{-\mathrm{i}P_\mu a}\big)\,. \tag{14.108}$$

Correspondingly, the partition function of the lattice quenched Eguchi–Kawai model is given by

$$Z_{\mathrm{QEK}} \;=\; \int \prod_\mu \mathrm{d}U_\mu\,C(U,P)\,\mathrm{e}^{-NS_{\mathrm{R}}[U]/g^2}, \tag{14.109}$$

where the eigenvalues $p_i^\mu$ are distributed uniformly over the hypercube.

The $U(1)^d$ symmetry is not broken in the lattice version of the quenched Eguchi–Kawai model for all values of the coupling $g^2N$. This is illustrated by the one-loop calculation in Problem 14.7. The lattice quenched Eguchi–Kawai model is equivalent to an $N=\infty$ Wilson lattice gauge theory on a $d$-dimensional lattice for all values of $g^2N$. This is verified, in particular, by numerical simulations.

**Problem 14.7** Calculate the partition function (14.109) to the leading order in $g^2$, fixing the gauge by $V_d = 1$.

**Solution** Since the gauge is fixed by $V_d = 1$, the vacuum state is simply

$$U_\mu^{\mathrm{cl}} = \mathrm{e}^{-\mathrm{i}P_\mu a} \tag{14.110}$$

or $V_\mu = 1$. This can be seen representing the action (14.38) in an equivalent form by rewriting

$$S_{\mathrm{R}} = \frac{1}{4N} \sum_{\mu\neq\nu} \mathrm{tr} \left| [U_\mu, U_\nu] \right|^2 . \tag{14.111}$$

We expand

$$V_{ij}^\mu = \delta_{ij} - \mathrm{i}ga \frac{b_{ij}^\mu}{S_{ij}^\mu} \qquad \mu = 1, \ldots, d-1\,, \tag{14.112}$$

where

$$S_{ij}^\mu = 2\sin\left(\frac{p_i^\mu - p_j^\mu}{2} a\right). \tag{14.113}$$

Here $b^\mu$ is the off-diagonal Hermitian matrix as has already been explained. Equation (14.112) reproduces the continuum equations (14.98) and (14.100) as $a \to 0$.

Keeping the terms which are quadratic in $b_\mu$ in the action, we have

$$S_2 = \frac{1}{2} \sum_{\mu,\nu=1}^{d-1} \sum_{i,j} \left| S_{ij}^\mu b_{ij}^\nu - S_{ij}^\nu b_{ij}^\mu \right|^2 , \tag{14.114}$$

while the measure is

$$\prod_{\mu=1}^{d-1} \mathrm{d}V_\mu = \prod_{\mu=1}^{d-1} \Delta^{-2}\big(\mathrm{e}^{-\mathrm{i}P_\mu a}\big)\, db_\mu \tag{14.115}$$

to this level of accuracy.

The calculation of the Gaussian integral over $b_\mu$ reduces for the given indices $i$ and $j$ to a calculation of the determinant of the $(d-1)\times(d-1)$ matrix

$$R_{\mu\nu} = \sum_{\rho=1}^{d} S_\rho^2 \delta_{\mu\nu} - S_\mu S_\nu\,, \tag{14.116}$$

which has one eigenvalue $S_d^2$ and $d-2$ eigenvalues $\sum_{\rho=1}^d S_\rho^2$. This can be easily seen using the rotational symmetry, which allows us to choose $S_\mu = 0$ for $\mu = 2, \ldots, d-1$. Therefore, we have

$$\det_{\mu\nu} R_{\mu\nu} = S_d^2 \left(\sum_{\rho=1}^{d} S_\rho^2\right)^{d-2} . \tag{14.117}$$

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Finally, we obtain

$$\int \prod_{\mu=1}^{d} \mathrm{d}V_\mu \Delta^2\left(\mathrm{e}^{-\mathrm{i}P_\mu a}\right) \mathrm{e}^{-S_2} = \prod_{i<j} \left( \sum_\mu 4 \sin^2 \frac{p_i^\mu - p_j^\mu}{2} a \right)^{2-d}, \qquad (14.118)$$

which reproduces the integrand in Eq. (14.77) as $a \to 0$. There is no collapse of eigenvalues of $U_\mu$ thanks to the quenching procedure.

In this Problem we have followed the calculation of [KM82].