ON EXCEPTIONAL VALUES OF A MEROMORPHIC FUNCTION

MAKOTO OHTSUKA

1. M. Brelot [1] has shown that if $u(z)$ is subharmonic in an open set D in the z-plane with boundary C and is bounded from above in a neighborhood of a boundary point z_0, which is contained in a set $E \subset C$ of inner harmonic measure zero with respect to D, and such that z_0 is a regular point for Dirichlet problem in D, then

$$(1) \qquad \varlimsup_{\substack{z \to z_0 \\ z \in D}} u(z) = \varlimsup_{\substack{z' \to z_0 \\ z' \in C-E}} \left(\varlimsup_{\substack{z \to z' \\ z \in D}} u(z) \right).$$

Furthermore, it was shown that if $f(z)$ is meromorphic in D, then, for any z_0 of E, which is in the closure of $C-E$, whether a regular point or not, the same relation holds when $u(z)$ is replaced by $|f(z)|$ whenever the left side of (1) is finite. It is easy to see that this last relation is equivalent to the relation:[1]

$$(2) \qquad \textit{boundary of } S_{z_0}^{(D)} \subset S_{z_0}^{(C-E)},$$

where the cluster set $S_{z_0}^{(D)}$ is the set of values approached sequencewise by $f(z)$ in any neighborhood of z_0 and the boundary cluster set $S_{z_0}^{(C-E)}$ from $C-E$ is the limit of the closure of $\bigcup_{z' \in (C-E)_r} S_{z'}^{(D)}$ as $r \to 0$, $(C-E)_r$ being that part of $C-E$ in $|z - z_0| < r$.

Later M. Tsuji [5] showed that in the special case that D is a domain and E is a closed set of logarithmic capacity zero, the exceptional values in $\Omega = S_{z_0}^{(D)} - S_{z_0}^{(C-E)}$, that is, the set of values in Ω which $f(z)$ does not assume in some neighborhood of z_0 form a set of inner logarithmic capacity zero.

2. In this note we shall prove that this is true in the general case.

THEOREM. *Let D be an open set in the z-plane, C its boundary, $E \subset C$ a set of inner harmonic measure zero with respect to D, z_0 a point of E in the closure of $C-E$, and $f(z)$ a meromorphic function in D. Then every value of*

Received May 30, 1955.

[1] See [4], for instance.

$S_{z_0}^{(D)} - S_{z_0}^{(C-E)}$ is assumed by $f(z)$ in any neighborhood of z_0 except for a set of (outer) logarithmic capacity zero.

Proof. If there is a disc: $|z - z_0| < r$ such that the part C_r of C in this disc is contained in E, this part is of harmonic measure zero with respect to this disc minus C_r and hence of logarithmic capacity zero. Hence in any neighborhood of z_0 in D $f(z)$ takes on every complex value except for a closed set of logarithmic capacity zero, or else $f(z)$ is continuous at z_0, by a theorem of Kametani [3]. Thus our theorem is true in this case.

Next we consider the case when z_0 is in the closure of $C - E$ and suppose that the exceptional values in $S_{z_0}^{(D)} - S_{z_0}^{(C-E)}$ form a set of positive inner logarithmic capacity. Then there exists a closed bounded set F of positive logarithmic capacity lying in a component Ω_1 of $S_{z_0}^{(D)} - S_{z_0}^{(C-E)}$ such that the values of F are not assumed by $f(z)$ in D_{r_0}: $D \cap \{|z - z_0| < r_0\}$. Let K be a compact set in Ω_1 containing a closed subset $F_1 \subset F$ of positive logarithmic capacity in its interior and bounded by a smooth curve γ. If we take $r_1 < r_0$ sufficiently small, K is disjoint from the closure of $\bigcup_{z' \in (C-E)_{r_1}} S_{z'}^{(D)}$. Brelot's result (2) shows that $S_{z'}^{(D)} \cap K = \phi$ or $S_{z'}^{(D)} \supset K$ at any regular point $z' \in E_{r_1}$: $E \cap \{|z - z_0| < r_1\}$. However, the latter case cannot occur. For, if we apply Brelot's result (1) to the composed function in D_{r_1} of $f(z)$ with the equilibrium potential of F_1, we get a contradiction. Therefore if we exclude all regular points from E and denote the remaining set by E_1, that component of $S_{z_0}^{(D)} - S_{z_0}^{(C-E_1)}$ containing K is equal to Ω_1; that is, this set remains unchanged. Let us consider the inverse image in D_{r_1} of the interior of K and denote it by D_0. The boundary of D_0 consists of (i) part of $|z - z_0| = r_1$, (ii) certain arcs in D_{r_1} on which $f(z) \in \gamma$ and (iii) a closed subset E_2 of E_1 of logarithmic capacity zero. If there is no connected component of D_0 containing z_0 on its boundary, then z_0 is a regular point with respect to D_0 and the reasoning used above is applied again. If there is a domain containing z_0 on its boundary, we can apply the result of Tsuji, stated at the beginning, to obtain a contradiction to the fact that K contains a closed set of exceptional values of positive logarithmic capacity. Since the set B_n of values in $S_{z_0}^{(D)} - S_{z_0}^{(C-E)}$ not taken by $f(z)$ in $D_{1/n}$ is a countable union of closed sets of logarithmic capacity zero, it is of (outer) logarithmic capacity zero. Hence the set of exceptional values which is equal to the union $\bigcup_n B_n$ is of (outer) logarithmic capacity zero. Thus our proof is completed.

Finally we remark that if we use the ramified topology in D (see [2] for this) and define the vanishing of harmonic measure and the cluster sets with respect to this topology, then we can extend our result to this case.

Bibliography

[1] M. Brelot: Sur l'allure à la frontière des fonctions harmoniques, sousharmoniques ou holomorphes, Bull. Soc. Roy. Liège (1939), pp. 468–477.

[2] M. Brelot: Le problème de Dirichlet ramifié, Ann. Univ. Grenoble, **22** (1946), pp. 167–200.

[3] S. Kametani: The exceptional values of functions with the set of capacity zero of essential singularities, Proc. Imp. Acad. Tokyo, **17** (1941), pp. 429–433.

[4] K. Noshiro: On the singularities of analytic functions, Jap. Journ. Math., **17** (1940), 37–96.

[5] M. Tsuji: On the cluster set of a meromorphic function, Proc. Imp. Acad. Tokyo, **19** (1943), pp. 60–65.

Mathematical Institute,
Nagoya University