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A NECESSARY AND SUFFICIENT CONDITION FOR CERTAIN MARTINGALE INEQUALITIES IN BANACH FUNCTION SPACES

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Abstract. Let X be a Banach function space over a nonatomic probability space. We investigate certain martingale inequalities in X that generalize those studied by A. M. Garsia. We give necessary and sufficient conditions on X for the inequalities to be valid.

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1. Introduction. It is well known that, for each $p \in [1, \infty)$, the Hardy space $\mathcal{H}_p$ of martingales consists of those $f = (f_n)_{n\in\mathbb{Z}_+}$ for which $Sf \in L_p$, where Sf denotes the square function of f. It is also known to many researchers of martingale theory that, for each $q \in [2, \infty]$, the space $\mathcal{K}_q$ consists of those $f = (f_n)_{n\in\mathbb{Z}_+}$ for which there exists a random variable $\gamma \in L_q$ satisfying

$$\mathbb{E}[\,|f_\infty - f_{n-1}|^2 \mid \mathcal{F}_n\,] \le \mathbb{E}[\,\gamma^2 \mid \mathcal{F}_n\,]$$

almost surely (a.s.) for all $n \in \mathbb{Z}_+$, where $f_{-1} \equiv 0$. The norm of $f \in \mathcal{K}_q$ is defined to be the infimum of $\|\gamma\|_q$ over all $\gamma \in L_q$ satisfying the inequality above.

The space $\mathcal{K}_q$ plays a crucial role in studying the dual space of $\mathcal{H}_p$. In fact, Garsia [**5**] proved that if $1 \le p \le 2$ and q is the conjugate exponent of p, then the dual space of $\mathcal{H}_p$ is isomorphic to $\mathcal{K}_q$. Since $\mathcal{K}_\infty$ coincides with BMO (the space of martingales of bounded mean oscillation), Garsia's result includes Fefferman's duality theorem which asserts that the dual space of $\mathcal{H}_1$ is isomorphic to BMO. On the other hand, Garsia also proved that if $2 \le q < \infty$, then $\mathcal{H}_q$ and $\mathcal{K}_q$ coincide, and for all $f \in \mathcal{K}_q$,

$$\sqrt{2/q}\,\|Sf\|_q \le \|f\|_{\mathcal{K}_q} \le \|Sf\|_q. \tag{1.1}$$

Moreover, combining (1.1) with the Burkholder square function inequality ([**2**, Theorem 9]), we see that if $2 \le q < \infty$, then there exists a constant $C_q > 0$ such that for any $f = (f_n) \in \mathcal{K}_q$,

$$C_q^{-1}\,\|f_\infty\|_q \le \|f\|_{\mathcal{K}_q} \le C_q\,\|f_\infty\|_q, \tag{1.2}$$

where $f_\infty := \lim_n f_n$ a.s.

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In this paper, we consider more general inequalities similar to those in (1.2). Given a Banach function space X (see Definition 1 below) and a filtration $\mathcal{F} = (\mathcal{F}_n)$, we introduce a Banach space of martingales, which we denote by $\mathcal{K}(X, \mathcal{F})$, and give necessary and sufficient conditions on X for the inequalities

$$C^{-1} \|f_\infty\|_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C \|f_\infty\|_X$$

and

$$C^{-1} \underline{\lim}_{n\to\infty} \|f_n\|_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C \overline{\lim}_{n\to\infty} \|f_n\|_X$$

to be valid. For a fixed filtration $\mathcal{F} = (\mathcal{F}_n)$, the definition of $\mathcal{K}(L_q, \mathcal{F})$ is slightly different from that of $\mathcal{K}_q$ (*cf*. Definition 3 in Section 3). However, $\mathcal{K}(L_q, \mathcal{F})$ and $\mathcal{K}_q$ in fact coincide for all $q \in [2, \infty]$.

2. Preliminaries. We deal with martingales on a *nonatomic* probability space $(\Omega, \Sigma, \mathbb{P})$. The assumption that Ω is nonatomic is essential. In addition, we have to deal with another probability space; let I be the interval $(0, 1]$ and let μ be Lebesgue measure on the σ-algebra $\mathfrak{M}$ consisting of all Lebesgue measurable subsets of I. The reader may assume that these two probability spaces are the same. However, our argument will not be very simple by doing so.

Let X and Y be normed linear spaces. We write $X \hookrightarrow Y$ if X is continuously embedded in Y, that is, if $X \subset Y$ and the inclusion map is continuous.

Definition 1. Let $(X, \|\cdot\|_X)$ be a Banach space of (equivalence classes of) random variables on Ω, or measurable functions on I. We call $(X, \|\cdot\|_X)$ a *Banach function space* if it satisfies the following conditions:

(B1) $L_\infty \hookrightarrow X \hookrightarrow L_1$;

(B2) if $|x| \leq |y|$ a.s. and $y \in X$, then $x \in X$ and $\|x\|_X \leq \|y\|_X$;

(B3) if $0 \leq x_n \uparrow x$ a.s., $x_n \in X$ for all n, and $\sup_n \|x_n\|_X < \infty$, then $x \in X$ and $\|x\|_X = \sup_n \|x_n\|_X$.

If $x \notin X$, we let $\|x\|_X := \infty$.

Note that, in Definition 1, we may replace (B3) by the condition that

(B3′) if $0 \leq x_n \in X$ for all n and $\underline{\lim}_n \|x_n\|_X < \infty$, then $\underline{\lim}_n x_n \in X$ and $\|\underline{\lim}_n x_n\|_X \leq \underline{\lim}_n \|x_n\|_X$.

Let x and y be random variables on Ω, or measurable functions on I. We write $x \simeq_d y$ to mean that x and y have the same distribution.

Definition 2. A Banach function space $(X, \|\cdot\|_X)$ is said to be *rearrangement-invariant* (r.i.) provided that

(RI) if $x \simeq_d y$ and $y \in X$, then $x \in X$ and $\|x\|_X = \|y\|_X$.

A rearrangement-invariant Banach function space will be simply called a *rearrangement-invariant space* or an *r.i. space*.

Typical examples of r.i. spaces are Lebesgue spaces L_p, Orlicz spaces L_Φ, Lorentz spaces $L_{p,q}$, and so on. An example of a Banach function space that is not r.i. is a weighted Lebesgue space. Let w be a strictly positive random variable such that $\mathbb{E}[w] = 1$, and let $1 < p < \infty$. If $w^{-1/(p-1)}$ is integrable, then the Lebesgue space L_p^w with respect to the measure $w\,d\mathbb{P}$ satisfies (B1)–(B3), and thus it is a Banach function space (with respect to $\mathbb{P}$). It is known that L_p^w can be renormed so as to be r.i. if and only if $0 < \operatorname{ess\,inf} w \le \operatorname{ess\,sup} w < \infty$ (*cf.* [**6**, Section 4]).

Let x be a random variable on Ω. The *nonincreasing rearrangement* of x, which we denoted by x^*, is the nonincreasing right-continuous function on $I = (0, 1]$ defined by

$$x^*(t) := \inf\{\lambda > 0 \mid \mathbb{P}(|x| > \lambda) \le t\} \quad \text{for all } t \in I,$$

with the convention that $\inf \emptyset = \infty$. Note that x^* is characterized as the nonincreasing right-continuous function that has the same distribution (with respect to μ) as $|x|$.

If ϕ is a measurable function on I, then the nonincreasing rearrangement ϕ^* is defined by regarding ϕ as a random variable on the probability space $(I, \mathfrak{M}, \mu)$.

Let x and y be integrable random variables on Ω, or measurable functions on I. We write $x \prec y$ if

$$\int_0^t x^*(s)\, ds \le \int_0^t y^*(s)\, ds \quad \text{for all } t \in I.$$

Then it is obvious that $x \simeq_d y$ if and only if $x \prec y \prec x$.

A Banach function space $(X, \|\cdot\|_X)$ is said to be *universally rearrangement-invariant* (u.r.i) provided that

(URI) if $x \prec y$ and $y \in X$, then $x \in X$ and $\|x\|_X \le \|y\|_X$.

Clearly condition (URI) implies condition (RI), while the converse is not true in general. However, if the underlying measure space is nonatomic, then condition (RI) implies condition (URI) (*cf.* [**1**, Theorem 4.6, p. 61]). Thus, in our argument, we need not distinguish u.r.i. spaces from r.i. spaces.

Now let us recall Luxemburg's representation theorem. If X is an r.i. space over Ω, then there exists a unique Banach function space $\widehat{X}$ over I such that:

- $x \in X$ if and only if $x^* \in \widehat{X}$;
- $\|x\|_X = \|x^*\|_{\widehat{X}}$ for all $x \in X$.

In fact $\widehat{X}$ consists of those functions ϕ for which

$$\|\phi\|_{\widehat{X}} := \sup\left\{\int_0^1 \phi^*(s)\, y^*(s)\, ds \,\middle|\, \|y\|_{X'} \le 1\right\} < \infty,$$

where

$$\|y\|_{X'} := \sup\{\mathbb{E}[|xy|] \mid x \in X,\ \|x\|_X \le 1\}. \tag{2.1}$$

We call $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ the *Luxemburg representation* of $(X, \|\cdot\|_X)$. For example, the Luxemburg representation of $L_p(\Omega)$ is $L_p(I)$. For more details, see [**1**, pp. 62–64].

Now let Z_1 and Z_2 be r.i. spaces over I, and let T be a linear operator whose domain contains Z_1. We write $T \in B(Z_1, Z_2)$ to mean that the restriction of T to Z_1 is a bounded operator on Z_1 into Z_2. If $Z_1 = Z_2 = Z$, we also write $T \in B(Z)$ for $T \in B(Z, Z)$.

In order to state our results, we need the notion of Boyd indices, which are defined as follows. Given a measurable function ϕ on I, we define a function $D_s\phi$ on I by setting

$$(D_s\phi)(t) := \begin{cases} \phi(st) & \text{if } st \in I, \\ 0 & \text{otherwise.} \end{cases}$$

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If Z is an r.i. space over I, then $D_s \in B(Z)$ and $\| D_s \|_{B(Z)} \le (1/s) \vee 1$ for all $s > 0$, where $\| D_s \|_{B(Z)}$ denotes the operator norm of D_s (restricted to Z). The *lower* and *upper Boyd indices* of an r.i. space Z are defined by

$$\alpha_Z := \sup_{0<s<1} \frac{\log \| D_{s^{-1}} \|_{B(Z)}}{\log s} \quad \text{and} \quad \beta_Z := \inf_{1<s<\infty} \frac{\log \| D_{s^{-1}} \|_{B(Z)}}{\log s},$$

respectively. Then we have

$$\alpha_Z = \lim_{s\downarrow 0} \frac{\log \| D_{s^{-1}} \|_{B(Z)}}{\log s}, \quad \beta_Z = \lim_{s\uparrow\infty} \frac{\log \| D_{s^{-1}} \|_{B(Z)}}{\log s}$$

and

$$0 \le \alpha_Z \le \beta_Z \le 1.$$

If X is an r.i. space over Ω, we define the Boyd indices of X by $\alpha_X := \alpha_{\widehat{X}}$ and $\beta_X := \beta_{\widehat{X}}$, where $\widehat{X}$ is the Luxemburg representation of X. For instance, $\alpha_{L_p} = \beta_{L_p} = 1/p$ for all $p \in [1, \infty]$. See [**1**, pp. 148–149] for details.

We conclude this section by introducing operators $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. For a measurable function ϕ on I, we define

$$(\mathcal{P}\phi)(t) := \frac{1}{t}\int_0^t \phi(s)\,ds, \quad t \in I,$$

$$(\mathcal{Q}\phi)(t) := \int_t^1 \frac{\phi(s)}{s}\,ds, \quad t \in I,$$

and

$$(\mathcal{R}\phi)(t) := \int_0^1 \frac{\phi(s)}{s+t}\,ds, \quad t \in I,$$

provided that these integrals exist for all $t \in I$. It is easy to verify that if ϕ is nonnegative and integrable, then

$$\frac{1}{2}(\mathcal{P}\phi + \mathcal{Q}\phi) \le \mathcal{R}\phi \le \mathcal{P}\phi + \mathcal{Q}\phi \quad \text{on } I, \tag{2.2}$$

$$\mathcal{P}(\mathcal{Q}\phi) = \mathcal{P}\phi + \mathcal{Q}\phi \quad \text{on } I, \tag{2.3}$$

and

$$\mathcal{Q}(\mathcal{P}\phi) = \mathcal{P}\phi + \mathcal{Q}\phi - \int_0^1 \phi(s)\,ds \quad \text{on } I. \tag{2.4}$$

Note that each of the operators $\mathcal{P}$ and $\mathcal{Q}$ is the (formal) adjoint of the other. It is known that $\mathcal{P} \in B(Z)$ (resp. $\mathcal{Q} \in B(Z)$) if and only if $\beta_Z < 1$ (resp. $\alpha_Z > 0$). Furthermore, by (2.2) we have that $\mathcal{R} \in B(Z)$ if and only if $\alpha_Z > 0$ and $\beta_Z < 1$. See [**1**, p. 150] for details (*cf.* [**10**]).

3. Results. Let $\mathbb{F}$ denote the collection of all filtrations of $(\Omega, \Sigma, \mathbb{P})$, where by *filtration* of $(\Omega, \Sigma, \mathbb{P})$ we mean a nondecreasing sequence of sub-σ-algebras of Σ. Given $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$, we denote by $\mathcal{M}(\mathcal{F})$ the space of all martingales with respect

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to $\mathcal{F}$ and $\mathbb{P}$, and we denote by $\mathcal{M}_u(\mathcal{F})$ the linear subspace of $\mathcal{M}(\mathcal{F})$ consisting of all uniformly integrable martingales. Recall that every $f=(f_n) \in \mathcal{M}_u(\mathcal{F})$ converges a.s.; we let $f_\infty := \lim_n f_n$ a.s. for each $f=(f_n) \in \mathcal{M}_u(\mathcal{F})$.

Henceforth we adopt the convention that $f_{-1} \equiv 0$ for any $f=(f_n) \in \mathcal{M}(\mathcal{F})$.

DEFINITION 3. Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. We denote by $\Gamma_f(X, \mathcal{F})$ the set of all nonnegative, $\sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$-measurable random variables $\gamma \in X$ satisfying

$$\sup_{m\geq n} \mathbb{E}[\,|f_m - f_{n-1}|\,|\,\mathcal{F}_n\,] \leq \mathbb{E}[\,\gamma\,|\,\mathcal{F}_n\,] \quad \text{a.s.,} \quad n \in \mathbb{Z}_+. \tag{3.1}$$

The space $\mathcal{K}(X, \mathcal{F})$ is defined to be the set of $f=(f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$ for which $\Gamma_f(X, \mathcal{F}) \neq \emptyset$. The norm of $f \in \mathcal{K}(X, \mathcal{F})$ is given by

$$\|f\|_{\mathcal{K}(X,\mathcal{F})} := \inf\{\|\gamma\|_X \mid \gamma \in \Gamma_f(X, \mathcal{F})\}.$$

For martingales $f \in \mathcal{M}(\mathcal{F})$ that are not in $\mathcal{K}(X, \mathcal{F})$, we let $\|f\|_{\mathcal{K}(X,\mathcal{F})} := \infty$.

Note that if $f=(f_n) \in \mathcal{M}_u(\mathcal{F})$, then (3.1) can be rewritten as

$$\mathbb{E}[\,|f_\infty - f_{n-1}|\,|\,\mathcal{F}_n\,] \leq \mathbb{E}[\,\gamma\,|\,\mathcal{F}_n\,] \quad \text{a.s.,} \quad n \in \mathbb{Z}_+.$$

Note also that $\mathcal{K}(X, \mathcal{F})$ is a Banach space. Indeed, it is not hard to show that $\mathcal{K}(X, \mathcal{F})$ has the Riesz-Fischer property, that is, that if $\{f^{(k)}\}$ is a sequence in $\mathcal{K}(X, \mathcal{F})$ such that $\sum_{k=1}^{\infty} \|f^{(k)}\|_{\mathcal{K}(X,\mathcal{F})} < \infty$, then the series $\sum_{k=1}^{\infty} f^{(k)}$ converges in $\mathcal{K}(X, \mathcal{F})$. As is well known, a normed linear space that has the Riesz-Fischer property is complete. Thus $\mathcal{K}(X, \mathcal{F})$ is a Banach space.

We can now state the main result of this paper.

THEOREM 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. Then the following are equivalent*:

(i) *there exists a positive constant C such that for any $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f=(f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$C^{-1}\underline{\lim}_{n\to\infty} \|f_n\|_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C\,\overline{\lim}_{n\to\infty} \|f_n\|_X\,; \tag{3.2}$$

(ii) *there exists a positive constant C such that for any $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f=(f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$,*

$$C^{-1}\,\|f_\infty\|_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C\,\|f_\infty\|_X\,; \tag{3.3}$$

(iii) *there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space such that $\alpha_X > 0$ and $\beta_X < 1$.*

REMARK 1. Suppose that (iii) of Theorem 1 holds. Then (3.2) can be rewritten as

$$K^{-1} \sup_{n\in\mathbb{Z}_+} |||f_n|||_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} \leq K \sup_{n\in\mathbb{Z}_+} |||f_n|||_X, \tag{3.4}$$

where K is a positive constant, independent of f. To see this, let $\mathcal{F}=(\mathcal{F}_n) \in \mathbb{F}$ and $f=(f_n) \in \mathcal{M}(\mathcal{F})$. Then $f_n \prec f_{n+1}$ for all n (see [**7**, Remark 4.3]), and hence (URI) with $\|\cdot\|_X$ replaced by $|||\cdot|||_X$ implies that $|||f_n|||_X \leq |||f_{n+1}|||_X$ for all n. Thus we may replace

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both $\overline{\lim}\,\|f_n\|_X$ and $\underline{\lim}\,\|f_n\|_X$ in (3.2) with a constant multiple of $\sup |||f_n|||_X$ to obtain (3.4).

As we shall see in the last section, Theorem 1 is a consequence of Propositions 1, 2, and 3 below.

PROPOSITION 1. *Let $(X, \|\cdot\|_X)$ be a Banach function space over Ω. Suppose that one of the following four conditions holds:*

(i) *the first inequality of* (3.2) *holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;*

(ii) *the second inequality of* (3.2) *holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;*

(iii) *the first inequality of* (3.3) *holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$;*

(iv) *the second inequality of* (3.3) *holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$.*

Then there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is a rearrangement-invariant space.

PROPOSITION 2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be rearrangement-invariant spaces over Ω, and let $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ and $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ be their Luxemburg representations. Then the following are equivalent*:

(i) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$\|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y\,; \tag{3.5}$$

(ii) *there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$,*

$$\|f\|_{\mathcal{K}(X,\mathcal{F})} \leq C \|f_\infty\|_Y\,; \tag{3.6}$$

(iii) *$\mathcal{P} \in B(\widehat{Y}, \widehat{X})$.*

REMARK 2. As mentioned before, $\mathcal{P} \in B(\widehat{X})$ if and only if $\beta_{\widehat{X}} < 1$. Hence by Propositions 1 and 2, the following are equivalent:

- the second inequality of (3.2) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$;
- the second inequality of (3.3) holds for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$;
- X can be renormed so that it is an r.i. space with $\beta_X < 1$.

PROPOSITION 3. *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(\widehat{X}, \|\cdot\|_{\widehat{X}})$, and $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ be as in Proposition 2.*

(i) *Suppose that $\mathcal{R} \in B(\widehat{Y}, \widehat{X})$. Then there exists a positive constant C such that for any $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$,*

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \leq \|Mf\|_X \leq C\|f\|_{\mathcal{K}(Y,\mathcal{F})}\,. \tag{3.7}$$

Here Mf denotes the maximal function of f, that is, $Mf := \sup_{n\in\mathbb{Z}_+} |f_n|$.

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(ii) *Suppose that there exists a positive constant C such that the inequality*

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \leq C\,\|f\|_{\mathcal{K}(Y,\mathcal{F})} \tag{3.8}$$

holds for any $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{Z}_+}\in\mathbb{F}$ and any $f=(f_n)_{n\in\mathbb{Z}_+}\in\mathcal{M}(\mathcal{F})$. Then $\mathcal{Q}\in B(\widehat{Y},\widehat{X})$.

REMARK 3. From (2.2), (2.3), and (2.4), we see that the hypothesis $\mathcal{R}\in B(\widehat{Y},\widehat{X})$ in (i) of Proposition 3 is equivalent to each of the following:
(a) $\mathcal{P}\in B(\widehat{Y},\widehat{X})$ and $\mathcal{Q}\in B(\widehat{Y},\widehat{X})$;
(b) $\mathcal{P}\mathcal{Q}\in B(\widehat{Y},\widehat{X})$;
(c) $\mathcal{Q}\mathcal{P}\in B(\widehat{Y},\widehat{X})$.
Incidentally, in order to prove that (c) implies (a), we have to use the hypotheses $L_\infty(I)\hookrightarrow\widehat{X}$ and $\widehat{Y}\hookrightarrow L_1(I)$ (*cf.* (B1)).

REMARK 4. Let $(X,\|\cdot\|_X)$ be an r.i. space over Ω. There is a characterization of those r.i. spaces Y for which $\mathcal{P}\in B(\widehat{Y},\widehat{X})$. Define $H(X)$ to be the set of all $x\in L_1(\Omega)$ such that $\|x\|_{H(X)}:=\|\mathcal{P}x^*\|_{\widehat{X}}<\infty$. Then $\big(H(X),\|\cdot\|_{H(X)}\big)$ is an r.i. space and $\mathcal{P}\in B(\widehat{H(X)},\widehat{X})$. Moreover $\mathcal{P}\in B(\widehat{Y},\widehat{X})$ if and only if $Y\hookrightarrow H(X)$.

There is a similar result concerning the boundedness of $\mathcal{Q}$. Define $K(X)$ to be the set of all $x\in L_1(\Omega)$ such that $\|x\|_{K(X)}:=\|\mathcal{Q}x^*\|_{\widehat{X}}<\infty$. If the function $t\mapsto-\log t$ belongs to $\widehat{X}$, then $K(X)$ is an r.i. space and $\mathcal{Q}\in B\big(\widehat{K(X)},\widehat{X}\big)$. Moreover $\mathcal{Q}\in B(\widehat{Y},\widehat{X})$ if and only if $Y\hookrightarrow K(X)$, provided that $-\log t\in\widehat{X}$. See **[7]** for details.

4. Proof of Proposition 1. We begin with a lemma.

LEMMA 1. *Let $(X,\|\cdot\|_X)$ be a Banach function space over Ω, and let S_+ be the set of all nonnegative simple random variables on Ω. Then the following are equivalent:*
(i) *there is a constant $c>0$ such that if $x,y\in S_+$, $x\simeq_d y$, and $x\wedge y\equiv0$, then $\|y\|_X\leq c\|x\|_X$;*
(ii) *there is a constant $c>0$ such that if $x,y\in X$ and $x\simeq_d y$, then $\|y\|_X\leq c\|x\|_X$;*
(iii) *there is a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space.*

A complete proof of this lemma can be found in **[8]**. For convenience, we sketch the proof here (cf. **[9]**).

Proof. (iii) $\Rightarrow$ (i). Obvious.

(i) $\Rightarrow$ (ii). Assume that (i) holds. We first show that if $x,y\in X$, $x\simeq_d y$, and $|x|\wedge|y|\equiv0$, then $\|y\|_X\leq c\|x\|_X$. For such x and y, there are sequences $\{x_n\}$ and $\{y_n\}$ in S_+ such that $x_n\simeq_d y_n$ for all $n\in\mathbb{Z}_+$, and such that $0\leq x_n\uparrow|x|$ and $0\leq y_n\uparrow|y|$. Since by assumption $\|y_n\|_X\leq c\|x_n\|_X$ for all n, we can apply (B3) to obtain $\|y\|_X\leq c\|x\|_X$. Next we show that (ii) holds, or equivalently, that

$$\sup\{\|y\|_X \mid x,y\in X,\ x\simeq_d y,\ \|x\|_X\leq1\}<\infty. \tag{4.1}$$

Suppose $x,y\in X$, $x\simeq_d y$, and $\|x\|_X\leq1$. We choose a positive number λ so that $\mathbb{P}(|x|>\lambda)=\mathbb{P}(|y|>\lambda)\leq1/3$, and let $x':=|x|1_{\{|x|>\lambda\}}$ and $y':=|y|1_{\{|y|>\lambda\}}$. Here, and in what follows, 1_A denotes the indicator function of A. Then, since $\mathbb{P}(x'=0,\ y'=0)\geq1/3$, there exists a random variable z such that $\{z\neq0\}\subset\{x'=0,\ y'=0\}$ and $z\simeq_d x'$ (*cf.* **[4**, (5.6), p. 44**]**). Since $x'\wedge z\equiv0$ and $y'\wedge z\equiv0$, we have that

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$\|y'\|_X \leq c\|z\|_X \leq c^2\|x'\|_X \leq c^2$. Hence, letting **1** denote the constant function with value one, we obtain

$$\|y\|_X \leq \|y'\|_X + \lambda\,\|\mathbf{1}\|_X \leq c^2 + \lambda\,\|\mathbf{1}\|_X,$$

which proves (4.1).

(ii) ⇒ (iii). For each $x \in L_1(\Omega)$, we define

$$|||x|||_X := \sup\left\{\int_0^1 x^*(s)\,y^*(s)\,ds \mid \|y\|_{X'} \leq 1\right\},$$

where $\|y\|_{X'}$ is defined as in (2.1). Then the set of all $x \in L_1(\Omega)$ such that $|||x|||_X < \infty$ forms an r.i. space. Moreover, under the assumption that (ii) holds, one can show that $|||x|||_X < \infty$ if and only if $x \in X$, and in this case $\|x\|_X \leq |||x|||_X \leq c\,\|x\|_X$ for all $x \in X$. □

Proof of Proposition 1. Suppose first that (ii) of Proposition 1 holds. We show that (i) of Lemma 1 holds. Let x and y be nonnegative simple random variables such that $x \simeq_d y$ and $x \wedge y \equiv 0$. Then we can write

$$x = \sum_{j=1}^{\ell} \alpha_j 1_{A_j} \quad \text{and} \quad y = \sum_{j=1}^{\ell} \alpha_j 1_{B_j},$$

where $\alpha_j > 0$ for each $j \in \{1, 2, \ldots, \ell\}$, and where $\{A_j\}_{j=1}^{\ell}$ and $\{B_j\}_{j=1}^{\ell}$ are sequences of sets in Σ such that:

- $\mathbb{P}(A_j) = \mathbb{P}(B_j)$ for each $j \in \{1, 2, \ldots, \ell\}$;
- $A_j \cap A_k = B_j \cap B_k = \emptyset$ whenever $j \neq k$;
- $\left(\bigcup_{j=1}^{\ell} A_j\right) \cap \left(\bigcup_{j=1}^{\ell} B_j\right) = \emptyset$.

Let $\Lambda_j := A_j \cup B_j$ for each $j \in \{1, 2, \ldots, \ell\}$. We define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \begin{cases} \sigma\{\Lambda_j \mid j = 1, 2, \ldots, \ell\} & \text{if } n = 0, \\ \Sigma & \text{if } n \geq 1. \end{cases} \quad \text{and} \quad f := \mathbb{E}[\,x \mid \mathcal{F}_n\,],\ n \in \mathbb{Z}_+.$$

Suppose that $\gamma \in \Gamma_f(X, \mathcal{F})$. Then since $f_0 = 2^{-1}(x + y)$ and $f_n = x$ for $n \geq 1$,

$$\frac{y}{2} \leq f_0 = \mathbb{E}[\,|f_1 - f_0| \mid \mathcal{F}_1\,] \leq \mathbb{E}[\,\gamma \mid \mathcal{F}_1\,] = \gamma \quad \text{a.s.}$$

Hence $\|y\|_X \leq 2\,\|\gamma\|_X$, which implies $\|y\|_X \leq 2\,\|f\|_{\mathcal{K}(X,\mathcal{F})}$. Combining this with the second inequality of (3.2), we obtain $\|y\|_X \leq 2C\,\|x\|_X$. Thus (i) of Lemma 1 holds.

If (iv) of Proposition 1 holds, we can use exactly the same argument as above to show that (i) of Lemma 1 holds.

Suppose next that (i) of Proposition 1 holds. Let x and y be as above, and let C be the constant appearing in (3.2). Of course, we may assume that $C \geq 1$. (In fact, one can deduce that $C \geq 1$.) For each $j \in \{1, 2, \ldots, \ell\}$, we choose $B_j' \in \Sigma$ so that $B_j' \subset B_j$ and $\mathbb{P}(B_j') = C^{-1}\mathbb{P}(B_j)$. This is possible, since $(\Omega, \Sigma, \mathbb{P})$ is nonatomic. Now let $\Lambda_j := A_j \cup B_j'$ for each $j \in \{1, 2, \ldots, \ell\}$, and define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$

as above. Then, letting $y' := \sum_{j=1}^{\ell} \alpha_j 1_{B_j'}$, we have

$$f_n = \begin{cases} C(C+1)^{-1}(x+y') & \text{if } n = 0, \\ x & \text{if } n \geq 1. \end{cases}$$

It is easy to see that $(C+1)^{-1}(x + C^2 y') \in \Gamma_f(X, \mathcal{F})$. Since $y' \leq y$, we have

$$\|f\|_{\mathcal{K}(X,\mathcal{F})} \leq \frac{1}{C+1}\,\|x\|_X + \frac{C^2}{C+1}\,\|y\|_X.$$

On the other hand, the first inequality of (3.2) implies $\|x\|_X \leq C\,\|f\|_{\mathcal{K}(X,\mathcal{F})}$. Therefore

$$\|x\|_X \leq \frac{C}{C+1}\,\|x\|_X + \frac{C^3}{C+1}\,\|y\|_X,$$

which implies $\|x\|_X \leq C^3\,\|y\|_X$. Thus (i) of Lemma 1 holds. Exactly the same argument applies if (iii) holds, and Proposition 1 is proved. □

5. Proof of Proposition 2. In order to prove Proposition 2, we need four lemmas, which will also be used in the proof of Proposition 3.

LEMMA 2. *Let* $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(\widehat{X}, \|\cdot\|_{\widehat{X}})$, *and* $(\widehat{Y}, \|\cdot\|_{\widehat{Y}})$ *be as in Proposition* 2, *and let* $f = (f_n)_{n\in\mathbb{Z}_+}$ *be a martingale.*

(i) *If* $f = (f_n)_{n\in\mathbb{Z}_+}$ *is uniformly integrable, then*

$$\|f_\infty\|_X = \lim_{n\to\infty} \|f_n\|_X = \sup_{n\in Z} \|f_n\|_X.$$

(ii) *If* $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$ *and* $\sup_{n\in\mathbb{Z}_+} \|f_n\|_Y < \infty$, *then* $Mf = \sup_{n\in\mathbb{Z}_+} |f_n| \in X$ *and*

$$\|Mf\|_X \leq \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})} \cdot \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y,$$

where $\|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})}$ *stands for the operator norm of* $\mathcal{P}\colon \widehat{Y} \to \widehat{X}$.

Proof. (i) Assume that $f = (f_n)$ is uniformly integrable. Then $f_n \prec f_{n+1} \prec f_\infty$ for all $n \in \mathbb{Z}_+$ (see [**7**, Remark 4.3]). Hence, by (URI) and (B3$'$),

$$\sup_{n\in\mathbb{Z}_+} \|f_n\|_X \leq \|f_\infty\|_X \leq \lim_{n\to\infty} \|f_n\|_X = \sup_{n\in\mathbb{Z}_+} \|f_n\|_X,$$

as desired. Of course, if $f_\infty \notin X$, then $\|f_\infty\|_X = \sup_n \|f_n\|_X = \infty$.

(ii) As shown in the proof of [**6**, Proposition 3], for each $n \in \mathbb{Z}_+$,

$$(M_n f)^*(t) \leq (\mathcal{P} f_n^*)(t), \qquad t \in I,$$

where $M_n f := \sup_{0\leq m\leq n} |f_m|$. Therefore

$$\|M_n f\|_X = \|(M_n f)^*\|_{\widehat{X}} \leq \|\mathcal{P} f_n^*\|_{\widehat{X}} \leq \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})} \cdot \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y < \infty.$$

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Since $M_nf \uparrow Mf$, it follows from (B3) that $Mf \in X$ and

$$\|Mf\|_X \leq \|\mathcal{P}\|_{B(\widehat{Y},\widehat{X})} \cdot \sup_{n\in\mathbb{Z}_+} \|f_n\|_Y,$$

as desired. □

LEMMA 3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be r.i. spaces over Ω.*

(i) *If (3.6) holds for any $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f=(f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}_u(\mathcal{F})$, then $\widehat{Y} \hookrightarrow \widehat{X}$, or equivalently $Y \hookrightarrow X$.*

(ii) *If (3.8) holds for any $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and any $f=(f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$, then $\widehat{Y} \hookrightarrow \widehat{X}$, or equivalently $Y \hookrightarrow X$.*

Proof. Let $x \in Y$. We define $\mathcal{F}=(\mathcal{F}_n) \in \mathbb{F}$ and $f=(f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \begin{cases} \{\emptyset, \Omega\} & \text{if } n=0, \\ \Sigma & \text{if } n \geq 1, \end{cases} \qquad \text{and} \qquad f_n := \mathbb{E}[x \mid \mathcal{F}_n], \quad n \in \mathbb{Z}_+.$$

Then, for any $\gamma \in \Gamma_f(X, \mathcal{F})$,

$$|x - \mathbb{E}[x]| = \mathbb{E}[|f_1 - f_0| \mid \mathcal{F}_1] \leq \mathbb{E}[\gamma \mid \mathcal{F}_1] = \gamma \quad \text{a.s.},$$

and hence $|x| \leq \gamma + \|x\|_1 \leq \gamma + d\|x\|_Y$ a.s., where d is a positive constant such that $\|\cdot\|_1 \leq d\|\cdot\|_Y$ on Y. Therefore $\|x\|_X \leq \|\gamma\|_X + d\|\mathbf{1}\|_X\|x\|_Y$, which implies

$$\|x\|_X \leq \|f\|_{\mathcal{K}(X,\mathcal{F})} + d\|\mathbf{1}\|_X\|x\|_Y.$$

Suppose that (3.6) holds for this $f=(f_n)$. Then

$$\|x\|_X \leq C\|f_\infty\|_Y + d\|\mathbf{1}\|_X\|x\|_Y = (C + d\|\mathbf{1}\|_X)\|x\|_Y,$$

which shows that $Y \hookrightarrow X$. Moreover, if $\phi \in \widehat{Y}$, then there exists $y \in Y$ such that $y^* = \phi^*$ on I (see [**4**, (5.6), p. 44]). Hence $\|\phi\|_{\widehat{X}} = \|y\|_X \leq C'\|y\|_Y = C'\|\phi\|_{\widehat{Y}}$, where $C' := C + d\|\mathbf{1}\|_X$. This shows that $\widehat{Y} \hookrightarrow \widehat{X}$ and (i) is proved.

To prove (ii), suppose that (3.8) holds for $f=(f_n)$ defined above. If we let $\eta := |x| + \|x\|_1$, then $\eta \in \Gamma_f(Y, \mathcal{F})$ and hence

$$\|f\|_{\mathcal{K}(Y,\mathcal{F})} \leq \|\eta\|_Y \leq \|x\|_Y + \|\mathbf{1}\|_Y\|x\|_1 \leq (1 + d\|\mathbf{1}\|_Y)\|x\|_Y.$$

Then by (i) of Lemma 2 and (3.8),

$$\|x\|_X = \sup_{n\in\mathbb{Z}_+} \|f_n\|_X \leq C(1 + d\|\mathbf{1}\|_Y)\|x\|_Y.$$

Thus $Y \hookrightarrow X$ and $\widehat{Y} \hookrightarrow \widehat{X}$. This completes the proof. □

Before stating the next lemma, we introduce the following notation: if Z is a Banach function space over I, then $\mathcal{D}(Z)$ denotes the set of all functions in Z that are *nonnegative*, *nonincreasing*, and *right-continuous*.

LEMMA 4. *Let $(Z_1, \|\cdot\|_{Z_1})$ and $(Z_2, \|\cdot\|_{Z_2})$ be r.i. spaces over I.*

(i) *If there is a constant $c > 0$ such that $\|\mathcal{P}\phi\|_{Z_2} \leq c\|\phi\|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, then $\mathcal{P} \in B(Z_1, Z_2)$ and $\|\mathcal{P}\|_{B(Z_1,Z_2)} \leq c$.*

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(ii) *If there is a constant* $c > 0$ *such that* $\| \mathcal{Q}\phi \|_{Z_2} \le c \| \phi \|_{Z_1}$ *for all* $\phi \in \mathcal{D}(Z_1)$, *then* $\mathcal{Q} \in B(Z_1, Z_2)$ *and* $\| \mathcal{Q} \|_{B(Z_1, Z_2)} \le c$.

Proof. (i) Suppose $\| \mathcal{P}\phi \|_{Z_2} \le c \| \phi \|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, and let $\psi \in Z_1$ be arbitrary. According to [**1**, Lemma 2.1, p. 44], we have $|\mathcal{P}\psi| \le \mathcal{P}\psi^*$. Since $\psi^* \in \mathcal{D}(Z_1)$, it follows that

$$\| \mathcal{P}\psi \|_{Z_2} \le \| \mathcal{P}\psi^* \|_{Z_2} \le c \| \psi^* \|_{Z_1} = c \| \psi \|_{Z_1},$$

as desired.

(ii) Suppose $\| \mathcal{Q}\phi \|_{Z_2} \le c \| \phi \|_{Z_1}$ for all $\phi \in \mathcal{D}(Z_1)$, and let $\psi \in Z_1$ be arbitrary. As shown in the proof of [**6**, Lemma 3], $|\mathcal{Q}\psi| \le \mathcal{Q}|\psi| \prec \mathcal{Q}\psi^*$. Hence

$$\| \mathcal{Q}\psi \|_{Z_2} \le \| \mathcal{Q}\psi^* \|_{Z_2} \le c \left\| \psi^* \right\|_{Z_1} = c \| \psi \|_{Z_1},$$

as desired. □

Note that since $(\Omega, \Sigma, \mathbb{P})$ is nonatomic, there exists a random variable ξ such that

$$\xi^*(t) = 1 - t \quad \text{for all } t \in I. \tag{5.1}$$

It is easy to prove the following:

LEMMA 5. *Let* ξ *be a random variable satisfying* (5.1), *and define a family of sets* $\{A(t) \in \Sigma \mid t \in [0, 1]\}$ *by setting*

$$A(t) := \{\omega \in \Omega \mid \xi(\omega) > 1 - t\} \quad \textit{for each } t \in [0, 1].$$

Let $\phi \in L_1(I)$ *and let* $x := \phi(1 - \xi)$. *Then*:

(i) $x^*(t) = \phi^*(t)$ *for all* $t \in I$;
(ii) $A(s) \subset A(t)$ *whenever* $0 \le s \le t \le 1$;
(iii) $\mathbb{P}\big(A(t)\big) = t$ *for all* $t \in [0, 1]$;
(iv) $\int_{A(t)} x \, d\mathbb{P} = \int_0^t \phi(s) \, ds$ *for all* $t \in [0, 1]$.

We are now ready to prove Proposition 2.

Proof of Proposition 2. (iii) ⇒ (i). Assume that $\mathcal{P} \in B(\widehat{Y}, \widehat{X})$. Let $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and let $f = (f_n) \in \mathcal{M}(\mathcal{F})$. To prove (3.5), we may assume $\sup_n \| f_n \|_Y < \infty$. Then by (ii) of Lemma 2, $Mf \in X$ and

$$\| Mf \|_X \le \| \mathcal{P} \|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \| f_n \|_Y.$$

On the other hand, since $2Mf \in \Gamma_f(X, \mathcal{F})$, we have that $\| f \|_{\mathcal{K}(X, \mathcal{F})} \le 2 \| Mf \|_X$. Therefore

$$\| f \|_{\mathcal{K}(X, \mathcal{F})} \le 2 \| \mathcal{P} \|_{B(\widehat{Y}, \widehat{X})} \cdot \sup_{n \in \mathbb{Z}_+} \| f_n \|_Y,$$

as desired.

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(i) $\Rightarrow$ (ii). This is an immediate consequence of (i) of Lemma 2.

(ii) $\Rightarrow$ (iii). Assume that (3.6) holds for any $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and any $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$. In view of Lemma 4, it suffices to show that for all $\phi \in \mathcal{D}(\widehat{Y})$,

$$\|\mathcal{P}\phi\|_{\widehat{X}} \leq k\,\|\phi\|_{\widehat{Y}} \tag{5.2}$$

with some constant $k > 0$, independent of ϕ. Let $\phi \in \mathcal{D}(\widehat{Y})$ and define $(\mathcal{P}\phi)(0) := \lim_{t\downarrow 0}(\mathcal{P}\phi)(t)$. Then $(\mathcal{P}\phi)(0)$ is finite if and only if $\phi \in L_\infty(I)$, and in this case $(\mathcal{P}\phi)(0) = \|\phi\|_\infty$. Bearing this in mind, we define a nonincreasing sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ in $[0,\ 1]$ by setting

$$t_0 := 1 \quad \text{and} \quad t_n := \inf\{s \in [0,\ 1] \mid (\mathcal{P}\phi)(s) \leq 2(\mathcal{P}\phi)(t_{n-1})\}, \quad n \geq 1.$$

Then $t_n \to 0$ and

$$(\mathcal{P}\phi)(t_n) \leq 2(\mathcal{P}\phi)(t_{n-1}) \quad \text{for all } n \geq 1. \tag{5.3}$$

(In fact, equality holds if and only if $2(\mathcal{P}\phi)(t_{n-1}) \leq (\mathcal{P}\phi)(0)$.)

Let x and $\{A(t) \mid t \in [0,\ 1]\}$ be as in Lemma 5. Define $\mathcal{F} = (\mathcal{F}_n) \in \mathbb{F}$ and $f = (f_n) \in \mathcal{M}_u(\mathcal{F})$ by

$$\mathcal{F}_n := \sigma\{\Lambda \setminus A(t_n) \mid \Lambda \in \Sigma\},\ \ n \in \mathbb{Z}_+, \quad \text{and} \quad f_n := \mathbb{E}[x \mid \mathcal{F}_n],\ \ n \in \mathbb{Z}_+. \tag{5.4}$$

Then by Lemma 5

$$f_n = \frac{1_{A(t_n)}}{\mathbb{P}(A(t_n))}\int_{A(t_n)} x\,d\mathbb{P} + x\,1_{\Omega\setminus A(t_n)} = (\mathcal{P}\phi)(t_n)\,1_{A(t_n)} + x\,1_{\Omega\setminus A(t_n)} \ \text{(a.s.)},$$

for each $n \in \mathbb{Z}_+$. Since $A(t_n) \downarrow \emptyset$ a.s., we have $f_\infty = x$ a.s. and hence for each $n \geq 1$,

$$\begin{aligned}\{(\mathcal{P}\phi)(t_{n-1}) - |x|\}1_{A(t_{n-1})\setminus A(t_n)} &\leq |f_\infty - f_{n-1}|\,1_{A(t_{n-1})\setminus A(t_n)} \\ &= \mathbb{E}[\,|f_\infty - f_{n-1}|\mid \mathcal{F}_n]\,1_{A(t_{n-1})\setminus A(t_n)} \ \text{(a.s.)}.\end{aligned} \tag{5.5}$$

Now let $\gamma \in \Gamma_f(X,\ \mathcal{F})$. Then for each $n \geq 1$,

$$\begin{aligned}\mathbb{E}[\,|f_\infty - f_{n-1}|\mid \mathcal{F}_n]\,1_{A(t_{n-1})\setminus A(t_n)} &\leq \mathbb{E}[\gamma \mid \mathcal{F}_n]\,1_{A(t_{n-1})\setminus A(t_n)} \\ &= \gamma\,1_{A(t_{n-1})\setminus A(t_n)} \quad \text{(a.s.)}.\end{aligned} \tag{5.6}$$

From (5.5) and (5.6), it follows that

$$\sum_{n=1}^{\infty}(\mathcal{P}\phi)(t_{n-1})\,1_{A(t_{n-1})\setminus A(t_n)} \leq \gamma + |x| \quad \text{(a.s.)}.$$

We write η for the sum on the left-hand side (which is a finite sum if $\phi \in L_\infty(I)$). Then by (5.3), we have for each $t \in I$,

$$\begin{aligned}(\mathcal{P}\phi)(t) &\leq \sum_{n=1}^{\infty}(\mathcal{P}\phi)(t_n)\,1_{[t_n,\ t_{n-1})}(t) \\ &\leq 2\sum_{n=1}^{\infty}(\mathcal{P}\phi)(t_{n-1})\,1_{[t_n,\ t_{n-1})}(t) = 2\eta^*(t) \leq 2(\gamma + |x|)^*(t).\end{aligned}$$

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Therefore

$$\|\mathcal{P}\phi\|_{\widehat{X}} \le 2\|(\gamma + |x|)^*\|_{\widehat{X}} = 2\|\gamma + |x|\|_X \le 2(\|\gamma\|_X + \|x\|_X),$$

which implies

$$\|\mathcal{P}\phi\|_{\widehat{X}} \le 2\,\|f\|_{\mathcal{K}(X,\mathcal{F})} + 2\,\|x\|_X.$$

By Lemma 3, we may replace $\|x\|_X$ with $d\,\|x\|_Y = d\,\|\phi\|_{\widehat{Y}}$, where d is a positive constant that is independent of x, and by (3.6) we may replace $\|f\|_{\mathcal{K}(X,\mathcal{F})}$ with $C\,\|f_\infty\|_Y = C\,\|\phi\|_{\widehat{Y}}$. Thus (5.2) holds with $k = 2(C+d)$. This completes the proof. □

6. Proof of Proposition 3. In addition to lemmas in the previous section, we need one more lemma.

LEMMA 6. *Let $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{Z}_+} \in \mathbb{F}$ and $f = (f_n)_{n\in\mathbb{Z}_+} \in \mathcal{M}(\mathcal{F})$. If $\gamma \in \Gamma_f(L_1, \mathcal{F})$ and if $g = (g_n)_{n\in\mathbb{Z}_+}$ is the martingale defined by $g_n = \mathbb{E}[\gamma \mid \mathcal{F}_n]$, $n \in \mathbb{Z}_+$, then*

$$\mathbb{E}[Mf] \le 16\,\mathbb{E}[Mg].$$

Proof. Let $0 < \delta < 1 < b < \infty$ and let $0 < \lambda < \infty$. We define stopping times ρ, σ, and τ by

$$\rho := \min\{n \in \mathbb{Z}_+ \mid g_n > \delta\lambda\}, \quad \sigma := \min\{n \in \mathbb{Z}_+ \mid |f_n| > \lambda\},$$

and

$$\tau := \min\{n \in \mathbb{Z}_+ \mid |f_n| > b\lambda\}.$$

Here we follow the usual convention that $\min\emptyset = \infty$. Then, on the one hand,

$$\begin{aligned}\{Mf > b\lambda,\ Mg \le \delta\lambda\} &= \{\tau < \infty,\ \rho = \infty\} \\ &\subset \{|f_\tau - f_{\sigma-1}| \ge (b-1)\lambda,\ \sigma < \rho\}.\end{aligned} \tag{6.1}$$

On the other hand, by assumption,

$$\mathbb{E}[\,|f_\tau - f_{\sigma-1}|\,1_{\{\sigma<\rho\}} \mid \mathcal{F}_\sigma] \le g_\sigma 1_{\{\sigma<\rho\}} \le \delta\lambda 1_{\{\sigma<\infty\}} = \delta\lambda 1_{\{Mf>\lambda\}} \quad \text{(a.s.)}. \tag{6.2}$$

Using (6.1) and (6.2), we have that

$$\begin{aligned}\mathbb{P}(Mf > b\lambda,\ Mg \le \delta\lambda) &\le \mathbb{P}(|f_\tau - f_{\sigma-1}| \ge (b-1)\lambda,\ \sigma < \rho) \\ &\le \frac{1}{(b-1)\lambda}\,\mathbb{E}\big[\,|f_\tau - f_{\sigma-1}|\,1_{\{\sigma<\rho\}}\big] \\ &\le \frac{\delta}{b-1}\,\mathbb{P}(Mf > \lambda).\end{aligned}$$

Hence, by [**3**, Lemma 7.1],

$$\mathbb{E}[Mf] \le \frac{b(b-1)}{\delta(b - b\delta - 1)}\,\mathbb{E}[Mg],$$

provided $b - b\delta - 1 > 0$. Setting $b = 2$ and $\delta = 1/4$ gives the desired result. □

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Let $\mathcal{F}=(\mathcal{F}_n)$, $f=(f_n)$, and $g=(g_n)$ be as in Lemma 6. Given $n\in\mathbb{Z}_+$ and $A\in\mathcal{F}_n$, we define $\mathcal{F}'_k:=\mathcal{F}_{k+n}$, $f'_k:=(f_{k+n}-f_{n-1})\,1_A$, and $g'_k:=g_{k+n}\,1_A=\mathbb{E}[\,\gamma\,1_A\,|\,\mathcal{F}'_k\,]$ for each $k\in\mathbb{Z}_+$. Then $\mathcal{F}'=(\mathcal{F}'_k)_{k\in\mathbb{Z}_+}\in\mathbb{F}$, $f'=(f'_k)_{k\in\mathbb{Z}_+}\in\mathcal{M}(\mathcal{F}')$, $g'=(g'_k)_{k\in\mathbb{Z}_+}\in\mathcal{M}(\mathcal{F}')$, and $\gamma\,1_A\in\Gamma_{f'}(L_1,\,\mathcal{F}')$. Hence by Lemma 6,

$$\mathbb{E}[(Mf-M_{n-1}f)\,1_A\,]\le\mathbb{E}[\,Mf'\,]\le 16\,\mathbb{E}[\,Mg'\,]\le 16\,\mathbb{E}[\,(Mg)\,1_A\,].$$

Thus, under the same assumption as Lemma 6,

$$\mathbb{E}[\,Mf-M_{n-1}f\,|\,\mathcal{F}_n\,]\le 16\,\mathbb{E}[\,Mg\,|\,\mathcal{F}_n\,]\quad\text{a.s. for all } n\in\mathbb{Z}_+. \tag{6.3}$$

Proof of Proposition 3. (i) The first inequality of (3.7) is obvious. To prove the second inequality, suppose $\mathcal{R}\in B(\widehat{Y},\,\widehat{X})$. Then $\mathcal{QP}\in B(\widehat{Y},\,\widehat{X})$. Let $f=(f_n)\in\mathcal{K}(Y,\,\mathcal{F})$, let $\gamma\in\Gamma_f(Y,\,\mathcal{F})$, and let $g=(g_n)$ be the martingale defined as in Lemma 6. Then (6.3) holds. According to [**7**, Theorem 3.3] (or [**6**, Lemma 4]), we have that $(Mf)^*\prec 16\,\mathcal{Q}(Mg)^*$. Furthermore we know that $(Mg)^*\le\mathcal{P}g^*_\infty=\mathcal{P}\gamma^*$ on I (see the proof of [**6**, Proposition 3]). Therefore $(Mf)^*\prec 16\,\mathcal{Q}(\mathcal{P}\gamma^*)$, which implies that

$$\begin{aligned}\|\,Mf\,\|_X=\|\,(Mf)^*\,\|_{\widehat{X}}&\le 16\|\,\mathcal{Q}(\mathcal{P}\gamma^*)\,\|_{\widehat{X}}\\ &\le 16\,\|\,\mathcal{QP}\,\|_{B(\widehat{Y},\,\widehat{X})}\,\|\,\gamma^*\,\|_{\widehat{Y}}=16\,\|\,\mathcal{QP}\,\|_{B(\widehat{Y},\,\widehat{X})}\,\|\,\gamma\,\|_Y\,.\end{aligned}$$

Thus the second inequality of (3.7) holds with $C=16\,\|\,\mathcal{QP}\,\|_{B(\widehat{Y},\,\widehat{X})}$.

(ii) Suppose that (3.8) holds for any $\mathcal{F}=(\mathcal{F}_n)\in\mathbb{F}$ and any $f=(f_n)\in\mathcal{M}(\mathcal{F})$. In view of Lemma 4, it suffices to show that for all $\psi\in\mathcal{D}(\widehat{Y})$,

$$\|\,\mathcal{Q}\psi\,\|_{\widehat{X}}\le k\,\|\,\psi\,\|_{\widehat{Y}} \tag{6.4}$$

with some constant $k>0$, independent of ψ. To this end, we may assume that $\psi\not\equiv 0$; then $\psi>0$ on some interval $(0,\,\delta]$, and hence $(\mathcal{Q}\psi)(t)\uparrow\infty$ as $t\downarrow 0$. Let $\varepsilon>0$ be given. Since $(\mathcal{Q}\psi)(t)$ is continuous, we can find a sequence $\{t_n\}_{n\in\mathbb{Z}_+}$ in I such that

$$t_0=1\quad\text{and}\quad(\mathcal{Q}\psi)(t_n)=(\mathcal{Q}\psi)(t_{n-1})+\varepsilon\quad n\ge 1. \tag{6.5}$$

It is obvious that $\{t_n\}$ is strictly decreasing and $t_n\to 0$. Let $\phi:=(\mathcal{Q}\psi)-\psi$, and let x and $\{A(t)\,|\,t\in[0,\,1]\}$ be as in Lemma 5. We again consider the martingale $f=(f_n)$ defined as in (5.4). Since $\mathcal{P}\phi=\mathcal{P}(\mathcal{Q}\psi)-\mathcal{P}\psi=\mathcal{Q}\psi$, we now have

$$f_n=(\mathcal{Q}\psi)(t_n)\,1_{A(t_n)}+x\,1_{\Omega\setminus A(t_n)}\quad\text{(a.s.) for all } n\in\mathbb{Z}_+.$$

It then follows that

$$\begin{aligned}\mathbb{E}[\,|f_\infty-f_{n-1}|\,|\,\mathcal{F}_n]&=\frac{1_{A(t_n)}}{t_n}\int_{A(t_n)}|x-(\mathcal{Q}\psi)(t_{n-1})|\,d\mathbb{P}\\ &\quad+|x-(\mathcal{Q}\psi)(t_{n-1})|\,1_{A(t_{n-1})\setminus A(t_n)}\quad\text{(a.s.) for all } n\in\mathbb{Z}_+.\end{aligned} \tag{6.6}$$

Since $(\mathcal{Q}\psi)(1-\xi)\,1_{A(t_n)}\ge(\mathcal{Q}\psi)(t_n)\,1_{A(t_n)}>(\mathcal{Q}\psi)(t_{n-1})1_{A(t_n)}$, we have that

$$|x-(\mathcal{Q}\psi)(t_{n-1})|\,1_{A(t_n)}\le\{(\mathcal{Q}\psi)(1-\xi)-(\mathcal{Q}\psi)(t_{n-1})+\psi(1-\xi)\}\,1_{A(t_n)},$$

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and hence

$$\begin{aligned}
&\frac{1}{t_n}\int_{A(t_n)} |x-(\mathcal{Q}\psi)(t_{n-1})|\,d\mathbb{P} \\
&\quad\le \frac{1}{t_n}\int_{\{1-\xi<t_n\}} (\mathcal{Q}\psi)(1-\xi)\,d\mathbb{P} + \frac{1}{t_n}\int_{\{1-\xi<t_n\}} \psi(1-\xi)\,d\mathbb{P} - (\mathcal{Q}\psi)(t_{n-1}) \\
&\quad= \frac{1}{t_n}\int_0^{t_n} (\mathcal{Q}\psi)(s)\,ds + \frac{1}{t_n}\int_0^{t_n} \psi(s)\,ds - (\mathcal{Q}\psi)(t_{n-1}).
\end{aligned}$$

By (6.5) the right-hand side is equal to

$$\begin{aligned}
&(\mathcal{P}(\mathcal{Q}\psi))(t_n) + (\mathcal{P}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) \\
&\qquad= 2(\mathcal{P}\psi)(t_n) + (\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) = 2(\mathcal{P}\psi)(t_n) + \varepsilon.
\end{aligned}$$

Thus

$$\frac{1_{A(t_n)}}{t_n}\int_{A(t_n)} |x-(\mathcal{Q}\psi)(t_{n-1})|\,d\mathbb{P} \le \{2(\mathcal{P}\psi)(t_n)+\varepsilon\}\,1_{A(t_n)}. \tag{6.7}$$

As for the second term on the right-hand side of (6.6), we have

$$\begin{aligned}
&|x-(\mathcal{Q}\psi)\,(t_{n-1})|\,1_{A(t_{n-1})\setminus A(t_n)} \\
&\qquad\le \{(\mathcal{Q}\psi)(t_n) - (\mathcal{Q}\psi)(t_{n-1}) + \psi(1-\xi)\}\,1_{A(t_{n-1})\setminus A(t_n)} \\
&\qquad= \{\psi(1-\xi)+\varepsilon\}\,1_{A(t_{n-1})\setminus A(t_n)}.
\end{aligned} \tag{6.8}$$

From (6.6), (6.7), and (6.8), we see that for each $n \ge 1$,

$$\mathbb{E}[\,|f_\infty - f_{n-1}|\,|\,\mathcal{F}_n\,] \le 2(\mathcal{P}\psi)(t_n)\,1_{A(t_n)} + \psi(1-\xi)\,1_{A(t_{n-1})\setminus A(t_n)} + \varepsilon \quad \text{(a.s.)}. \tag{6.9}$$

If we set $t_{-1} = 1$, then (6.9) remains valid for $n=0$. Indeed,

$$\mathbb{E}[\,|f_\infty|\,|\,\mathcal{F}_0\,] = \|x\|_1 \le \|\mathcal{Q}\psi\|_1 + \|\psi\|_1 = 2\,\|\psi\|_1 = 2(\mathcal{P}\psi)(t_0) \quad \text{(a.s.)}.$$

Let $\gamma_\varepsilon := 2\psi(1-\xi)+\varepsilon$. Then $\gamma_\varepsilon^* = 2\psi + \varepsilon \in \widehat{Y}$ and

$$\mathbb{E}[\,\gamma_\varepsilon\,|\,\mathcal{F}_n\,] = 2(\mathcal{P}\psi)(t_n)\,1_{A(t_n)} + 2\psi(1-\xi)\,1_{\Omega\setminus A(t_n)} + \varepsilon \quad \text{(a.s.)}.$$

Comparing this with (6.9), we see that $\gamma_\varepsilon \in \Gamma_f(Y, \mathcal{F})$. Thus

$$\|f\|_{\mathcal{K}(Y,\mathcal{F})} \le \|\gamma_\varepsilon\|_Y \le 2\,\|\psi\|_{\widehat{Y}} + \varepsilon\,\|\mathbf{1}\|_Y\,. \tag{6.10}$$

On the other hand, by (B3′) we have that

$$\|\mathcal{Q}\psi\|_{\widehat{X}} - \|\psi\|_{\widehat{X}} \le \|\phi\|_{\widehat{X}} = \|x\|_X \le \underline{\lim}_{n\to\infty}\|f_n\|_X \le \sup_{n\in\mathbb{Z}_+}\|f_n\|_X\,. \tag{6.11}$$

Using (3.8), (6.10) and (6.11), we obtain

$$\|\mathcal{Q}\psi\|_{\widehat{X}} \le C(2\,\|\psi\|_{\widehat{Y}} + \varepsilon\,\|\mathbf{1}\|_Y) + \|\psi\|_{\widehat{X}}\,.$$

According to Lemma 3, there is a positive constant d such that $\|\cdot\|_{\widehat{X}} \le d\,\|\cdot\|_{\widehat{Y}}$ on $\widehat{Y}$. Replacing $\|\psi\|_{\widehat{X}}$ by $d\,\|\psi\|_{\widehat{Y}}$ and letting $\varepsilon \downarrow 0$, we see that (6.4) holds with $k = 2C+d$. This completes the proof. □

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7. Proof of Theorem 1. We conclude the paper with the proof of our main theorem.

Proof of Theorem 1. (i) $\Rightarrow$ (iii). Suppose that (i) of Theorem 1 holds. Then by Proposition 1, there exists a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space. If $\mathcal{F}=(\mathcal{F}_n)\in\mathbb{F}$ and $f=(f_n)\in\mathcal{M}(\mathcal{F})$, then $|||f_n|||_X\le|||f_{n+1}|||_X$ for all $n\in\mathbb{Z}_+$ (*cf.* Remark 1), and hence by (3.2),

$$K^{-1}\sup_{n\in\mathbb{Z}_+}|||f_n|||_X\le|||f|||_{\mathcal{K}(X,\mathcal{F})}\le K\sup_{n\in\mathbb{Z}_+}|||f_n|||_X. \tag{7.1}$$

Here $|||f|||_{\mathcal{K}(X,\mathcal{F})}=\inf\{|||\gamma|||_X\mid\gamma\in\Gamma_f(X,\mathcal{F})\}$ and K is a constant that is independent of f. It then follows from Propositions 2 and 3 that $\mathcal{P}\in B(\widehat{X})$ and $\mathcal{Q}\in B(\widehat{X})$, where $\widehat{X}$ stands for the Luxemburg representation of $(X, |||\cdot|||_X)$. As mentioned at the end of Section 2, this means that $\alpha_X>0$ and $\beta_X<1$.

(iii) $\Rightarrow$ (ii). Suppose that (iii) holds. This implies that $\mathcal{P}\in B(\widehat{X})$ and $\mathcal{R}\in B(\widehat{X})$. It then follows from Propositions 2 and 3 that (7.1) holds for any $\mathcal{F}=(\mathcal{F}_n)\in\mathbb{F}$ and any $f=(f_n)\in\mathcal{M}(\mathcal{F})$. If $f=(f_n)$ is uniformly integrable, then $|||f_\infty|||_X=\sup_n|||f_n|||_X$ by (i) of Lemma 2. Since the norms $\|\cdot\|_X$ and $|||\cdot|||_X$ are equivalent, we obtain (3.3).

(ii) $\Rightarrow$ (i). Given a martingale $f=(f_n)$, we let $f^{\langle n\rangle}$ denote the stopped martingale $(f_{n\wedge k})_{k\in\mathbb{Z}_+}$. Let $\mathcal{F}=(\mathcal{F}_n)\in\mathbb{F}$ and $f=(f_n)\in\mathcal{M}(\mathcal{F})$, and suppose that (ii) holds. Then, by the first inequality of (3.3),

$$\|f_n\|_X\le C\|f^{\langle n\rangle}\|_{\mathcal{K}(X,\mathcal{F})}\le C\|f\|_{\mathcal{K}(X,\mathcal{F})}\quad\text{for all } n\in\mathbb{Z}_+,$$

where the second inequality follows from the fact that $\Gamma_f(X,\mathcal{F})\subset\Gamma_{f^{\langle n\rangle}}(X,\mathcal{F})$. From the inequality above, we easily obtain the first inequality of (3.2). Hence, by Proposition 1, there is a norm $|||\cdot|||_X$ on X which is equivalent to $\|\cdot\|_X$ and with respect to which X is an r.i. space.

We now turn our attention to the second inequality of (3.2). Note that if $f=(f_n)\in\mathcal{M}(\mathcal{F})$ is uniformly integrable, then by the second inequality of (3.3) and (B3′),

$$\|f\|_{\mathcal{K}(X,\mathcal{F})}\le C\|f_\infty\|_X\le C\underline{\lim}_{n\to\infty}\|f_n\|_X\le C\overline{\lim}_{n\to\infty}\|f_n\|_X. \tag{7.2}$$

Thus the required inequality holds for uniformly integrable martingales. Moreover, since the first inequality of (7.2) can be rewritten as $|||f|||_{\mathcal{K}(X,\mathcal{F})}\le K|||f_\infty|||_X$, Proposition 2 implies that $\mathcal{P}\in B(\widehat{X})$.

Finally, let $f=(f_n)\in\mathcal{M}(\mathcal{F})$ be such that $\overline{\lim}_n\|f_n\|_X<\infty$. Then, since $\sup_n|||f_n|||_X<\infty$, Lemma 2 shows that $Mf\in X$. Therefore f is uniformly integrable, and satisfies (7.2). This completes the proof. □

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