

# A NOTE RELATED TO THE CS DECOMPOSITION AND THE BK INEQUALITY FOR DISCRETE DETERMINANTAL PROCESSES

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## Abstract

We prove that for a discrete determinantal process the BK inequality occurs for increasing events generated by simple points. We also give some elementary but nonetheless appealing relationships between a discrete determinantal process and the well-known CS decomposition.

*Keywords:* Point processes; negative dependence; spanning trees; exterior algebra; principal angles

2020 Mathematics Subject Classification: Primary 60K99; 60G55; 15A23  
Secondary 68U99; 15A75

## 1. Introduction

There is an extensive mathematical literature, in several theoretical and applied areas, related to determinantal point processes; we cite, to mention a few recent applied works, [2, 7, 8, 16, 21, 22]. A good overview of the main conceptual basis and properties can be found in [18] and in the bibliography therein.

From the theoretical point of view, determinantal point processes could be defined (in a Bourbaki-like spirit) in the general locally compact Polish spaces setting, as point processes associated with some locally square integrable, Hermitian, positive semidefinite, locally trace-class operators, and thereafter specialized for particular cases, namely to discrete determinantal processes. Regarding the latter, the approach of [18], which consists of constructing such processes, first in the most elementary discrete context and then gradually extending them to the general situation, provides, in our opinion, many advantages. It also turns out that some results for the most general processes are proved only [9, 18], or more simply [19], indirectly from the corresponding results of the basic processes.

The basic elementary determinantal point process can be described via the exterior product concept, as follows. Fix  $1 < p < N$  and let  $\mathfrak{Z} = \{z^1, \dots, z^p\}$ ,  $1 < p < N$ , be a set of orthonormal vectors in  $\mathbb{C}^N$ . We write  $z^i = (z_1^i, \dots, z_N^i)^t$ ,  $i = 1, \dots, p$ , and  $z_i = (z_1^i, \dots, z_i^i)$ ,  $i = 1, \dots, N$ .

The associated determinantal process  $\phi(\mathfrak{Z})$  is a point process, view as a random subset of  $\mathcal{N} = \{1, \dots, N\}$  of cardinality  $|\phi(\mathfrak{Z})| = p$ , characterized [17, 18] by the formula  $\mathbb{P}\{\{i_1, \dots, i_p\} = \phi\} = |(\bigwedge_{i=1}^p z^i)_{\{i_1, \dots, i_p\}}|^2 = [\det((z_{ij}^k)_{k,j=1, \dots, p})]^2$  for all subsets  $\{i_1, \dots, i_p\} \subset \mathcal{N}$ . Note also that this formula implies  $\mathbb{P}\{\{i_1, \dots, i_k\} \subset \phi\} = \|\bigwedge_{j=1}^k z_{i_j}\|^2$  for all  $1 \leq k \leq p$ .

Received 1 June 2021; revision received 16 March 2022.

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Let  $E = E(\mathfrak{Z}) \subset \mathbb{C}^N$  be the vector space spanned by  $\mathfrak{Z}$ . For all sets of linearly independent vectors  $v^i \in E, i = 1, \dots, p$ , we have  $\bigwedge_{i=1}^p v^i = a \bigwedge_{i=1}^p z^i$  with  $a \neq 0$ ; thus, in particular, if  $\tilde{\mathfrak{Z}} = \{\tilde{z}^1, \dots, \tilde{z}^p\}$  is another orthonormal basis of  $E = E(\mathfrak{Z})$  then  $|\left(\bigwedge_{i=1}^p z^i\right)_{\{i_1, \dots, i_p\}}| = \left|\left(\bigwedge_{i=1}^p \tilde{z}^i\right)_{\{i_1, \dots, i_p\}}\right|$  for every  $\{i_1, \dots, i_p\} \subset \mathcal{N}$ , and consequently  $\phi(\mathfrak{Z}) = \phi(\tilde{\mathfrak{Z}})$ .

Note also that if  $\mathfrak{Z}^\perp = \{z^{p+1}, \dots, z^N\}$  is an orthonormal basis of the orthogonal complement  $E(\mathfrak{Z})^\perp$  of  $E(\mathfrak{Z})$  in  $\mathbb{C}^N$  then obviously  $\phi(\mathfrak{Z}^\perp) = \{1, \dots, N\} \setminus \phi(\mathfrak{Z})$ . A remarkable example of a non-trivial basic determinantal process is given by uniform spanning tree measure on a finite connected graph  $G$ . Roughly speaking, if  $G$  is fixed and arbitrarily edge-oriented, and  $M$  is the vertex–edge incidence matrix (the columns being indexed by vertices), then the determinantal process associated with the vector space spanned by all the column vectors but one provides a uniform probability on spanning trees. This result, due to [5], is called the Transfer Current Theorem. For more details, with clever short proofs, see [18, Section 2.6, p. 8]. Some extensions of this result are given in [4] with a series of open questions and conjectures, among them Conjecture 4.6, related to the van den Berg–Kesten (BK) inequality,

Recall that an event  $\mathfrak{A} \subset 2^{\mathcal{N}}, \mathcal{N} = \{1, \dots, N\}$ , is called increasing if, whenever  $A \in \mathfrak{A}$  and  $n \in \mathcal{N}$ , we also have  $A \cup \{n\} \in \mathfrak{A}$ . For a pair  $\mathfrak{A}, \mathfrak{B} \subset 2^{\mathcal{N}}$  of increasing events, the disjoint intersection  $\mathfrak{A} \circ \mathfrak{B}$  is then defined [26] by  $\mathfrak{A} \circ \mathfrak{B} = \{K \subset \mathcal{N} : \text{there exist } L \in \mathfrak{A}, M \in \mathfrak{B}, L, M \neq \emptyset \text{ such that } L \cap M = \emptyset, K \supset L \cup M\}$ . A point process  $\psi$  on  $\mathcal{N}$  is said to have the BK property if

$$\mathbb{P}\{\psi \in \mathfrak{A} \circ \mathfrak{B}\} \leq \mathbb{P}\{\psi \in \mathfrak{A}\} \times \mathbb{P}\{\psi \in \mathfrak{B}\} \tag{1}$$

for every pair of increasing events. In [26] it was proved that (1) is satisfied when  $\psi$  is related to a product probability on  $2^{\mathcal{N}}$ . In the basic determinantal process setting, Conjecture 4.6, which states that the same is true for the spanning trees determinantal point processes, is still unsolved. The question of whether general determinantal processes have the BK property was raised in [17].

The purpose of this note is twofold. First, we introduce a new method to investigate discrete determinantal processes using the CS decomposition (CSD) of a partitioned unitary matrix, which is a useful non-trivial tool in numerical linear algebra; a precise statement of CSD is given in Section 2. We show that the CSD gives a pertinent description of conditioning and provides (at least in our opinion) a suggestive perspective for future investigations; see, for example, the result given by Proposition 2, which seems to us to be new, and the results of [10]. Furthermore, this should be an appropriate framework for computational needs.

Second, we study the BK inequality. We prove that the BK inequality (1) is satisfied for all discrete determinantal processes when the increasing events  $\mathfrak{A}$  and  $\mathfrak{B}$  are generated by simple points: Theorem 3 in Sections 4 and 5. We also conjecture the following.

**Conjecture 1.** For all  $n \geq 2$ ,

$$\mathbb{P}\{A \not\subset \phi \mid A_i \not\subset \phi \text{ for all } i = 1, \dots, n\} \leq \mathbb{P}\{A \not\subset \phi \mid A_i \not\subset \phi \text{ for all } i = 1, \dots, n - 1\} \tag{2}$$

for every choice of  $A, A_i, i = 1, \dots, n$ , of disjoint subsets of  $\{1, \dots, N\}$  such that  $\mathbb{P}\{A_i \not\subset \phi \text{ for all } i = 1, \dots, n\} > 0$ .

If Conjecture 1 holds then it can be shown that the BK inequality (1) is satisfied for increasing events  $\mathfrak{A}$  and  $\mathfrak{B}$  generated by disjoint sets: Theorem 2 in Section 3. When the sets above are reduced to being simple points then the inequality (2) is a well-known result. For general sets, note that  $\mathbb{P}\{A \not\subset \phi \mid A_1 \not\subset \phi\} \leq \mathbb{P}\{A \not\subset \phi\}, A \cap A_1 = \emptyset$ , is the classical correlation inequality

[17], and that (2) was obtained in [10] for  $n = 2, 3$  with precise values of the conditional probabilities.

**Remark 1.** Note that for the process  $\psi$  related to a product probability on  $2^{\mathcal{N}}$ , the counting random variables  $|\psi \cap A_i|$  (the sets  $A_i, i = 1, \dots, n$ , being disjoint) are independent and thus the inequality (2) becomes trivial. However, the situation is less obvious if the process  $\psi$  is conditioned to have exactly  $k$  points,  $1 < k < N$ . In the particular case when the conditioned process  $\psi_k$  assigns equal probability to all subsets  $\{i_1, \dots, i_k\} \subset \mathcal{N}$ , i.e. if  $\mathbb{P}\{\psi_k = \{i_1, \dots, i_k\}\} = 1/\binom{N}{k}$ , it was proved in [25] that  $\psi_k$  has the BK property. As regards the inequality (2), we have, with the choice  $\mathbb{P}\{i \notin \psi_k, i_j \notin \psi_k \text{ for all } j = 1, \dots, n\} > 0, \mathbb{P}\{i \notin \psi_k \mid i_j \notin \psi_k \text{ for all } j = 1, \dots, n\} = \binom{N-n-1}{k} / \binom{N-n}{k}$  and, consequently, the inequality (2) is equivalent, for simple points, to the well-known log-concave inequality  $\binom{N-n-1}{N-k} \times \binom{N-n+1}{N-k} \leq \binom{N-n}{N-k}^2$ , and thus is fulfilled. Likewise, for general sets, the correlation inequality  $\mathbb{P}\{A \not\subset \psi_k \mid A_1 \not\subset \psi_k\} \leq \mathbb{P}\{A \not\subset \psi_k\}, A \cap A_1 = \emptyset, |A| = n, |A_1| = m$ , with (the non-trivial case)  $n + m \leq k$ , follows from the BK property and is equivalent to the log-concave inequality  $\binom{N}{N-k} \times \binom{N-n-m}{N-k} \leq \binom{N-n}{N-k} \times \binom{N-m}{N-k}$ . For  $n \geq 2$  it is easy to see that the validity of the inequality (2) depends on whether or not functions of the form

$$u \rightarrow \sum_{i_1=0}^{n_1} \dots \sum_{i_M=0}^{n_M} \binom{N-u-(i_1+\dots+i_M)}{N-k-M}$$

are log-concave, a question which does not seem to me to have been really investigated. Finally, the occurrence of log-concave criteria for negative dependence properties is not quite a surprise; see, for example, [24].

## 2. The CS decomposition and the basic determinantal point process

Following [23], the general CSD for a matrix  $Q$  from the unitary group  $U(N)$  specifies that, for any  $2 \times 2$  partitioning

$$Q = \begin{bmatrix} c_1 & c_2 \\ Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix}$$

with  $N = r_1 + r_2 = c_1 + c_2$ , there exist unitary matrices  $U_1, U_2, V_1, V_2$  such that (here, all unnamed blocks of the matrices are always zero, and the superscript  $H$  represents the conjugate transpose)

$$\begin{bmatrix} U_1^H & \\ & V_1^H \end{bmatrix} Q \begin{bmatrix} U_2 & \\ & V_2 \end{bmatrix} = \begin{bmatrix} U_1^H Q_{11} U_2 & U_1^H Q_{12} V_2 \\ V_1^H Q_{21} U_2 & V_1^H Q_{22} V_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{matrix} r_1 \\ r_2 \end{matrix}, \tag{3}$$

where the matrices

$$D_{11} = \begin{bmatrix} I & & \\ & C & \\ & & \mathbf{0}_c \end{bmatrix}, D_{12} = \begin{bmatrix} \mathbf{0}_s^H & & \\ & S & \\ & & I \end{bmatrix}, D_{21} = \begin{bmatrix} \mathbf{0}_s & & \\ & S & \\ & & I \end{bmatrix}, D_{22} = \begin{bmatrix} I & & \\ & -C & \\ & & \mathbf{0}_c^H \end{bmatrix}$$

are diagonal with  $C \equiv \text{diag}(\cos \theta_1, \dots, \cos \theta_s), S \equiv \text{diag}(\sin \theta_1, \dots, \sin \theta_s), 1 > \cos \theta_1 > \dots > \cos \theta_s > 0$ . In some cases the matrices of zeros  $\mathbf{0}_s$  and  $\mathbf{0}_c$ , as well as the unit matrices  $I$ ,

could be nonexistent. See [23, Theorem 1] and the discussion that follows it for the full statement, and below for a detailed description given from Jordan’s geometrical point of view.

The CS decomposition is a deep result which has a long history going back to the work of Camille Jordan in 1875 on angles between subspaces in  $\mathbb{R}^n$  [15]. Nowadays it is a popular tool in numerical linear algebra, useful for solving various questions such as, for example, constrained least squares problems, computing principal angles between subspaces, the generalized singular value decomposition, quantum computing, and more [3, 6, 11, 12, 23].

Now, let  $E \subset \mathbb{C}^N$  be a vector space of dimension  $1 < p < N$ ,  $\mathfrak{Z} = \{z^1, \dots, z^p\}$  an orthonormal basis of  $E$ , and  $\mathfrak{Z}^\perp = \{z^{p+1}, \dots, z^N\}$  an orthonormal basis of the orthogonal complement  $E(\mathfrak{Z})^\perp$ . Fix  $1 \leq n \leq p$  and consider the CSD of the partitioned unitary matrix  $Q = (z^1, \dots, z^p, z^{p+1}, \dots, z^N)$ :

$$Q = \begin{matrix} & \begin{matrix} p & N-p \end{matrix} \\ \begin{matrix} n \\ N-n \end{matrix} & \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \end{matrix}$$

It follows from (3) that the column vectors of these two matrices,

$$\begin{bmatrix} U_1 D_{11} \\ V_1 D_{21} \end{bmatrix}, \quad \begin{bmatrix} U_1 D_{12} \\ V_1 D_{22} \end{bmatrix}$$

are respectively orthonormal bases of  $E$  and  $E(\mathfrak{Z})^\perp$ .

Now we will detail the different cases given by these column vectors, which need to be distinguished. The description given here is somewhat lengthy but, in our opinion, useful for both theoretical and computational purposes. We denote by  $e(k)$ ,  $k = 1, \dots, N$ , the null vector of the space  $\mathbb{C}^k$ . Note also the slight change with regard to angles appearing in CSD (3) which allows values 0 and  $\pi/2$  in order to recover all Jordan’s principal angles.

Case I:  $n < p$  and  $p + n < N$ . There exist

- a sequence  $u^1, \dots, u^n$  of orthonormal vectors in  $\mathbb{C}^n$ ;
- three sequences of mutually orthonormal vectors in  $\mathbb{R}^{N-n}$ ,  $\mathfrak{V} = \{V^1, \dots, V^n\}$ ,  $\mathfrak{W} = \{W^1, \dots, W^{p-n}\}$ , and  $\tilde{\mathfrak{W}} = \{\tilde{W}^1, \dots, \tilde{W}^{N-p-n}\}$ ;
- Jordan angles  $0 \leq \theta_1 \leq \dots \leq \theta_n \leq \pi/2$

such that, noting that

$$\begin{aligned} z^i &= \begin{bmatrix} u^i \cos \theta_i \\ V^i \sin \theta_i \end{bmatrix}, & i = 1, \dots, n; \\ z^i &= \begin{bmatrix} e(n) \\ W^i \end{bmatrix}, & i = n + 1, \dots, p; \\ z^{p+i} &= \begin{bmatrix} u^i \sin \theta_i \\ -V^i \cos \theta_i \end{bmatrix}, & i = 1, \dots, n; \\ z^{p+n+i} &= \begin{bmatrix} e(n) \\ \tilde{W}^i \end{bmatrix}, & i = 1, \dots, N - p - n, \end{aligned}$$

the sequence  $\mathfrak{Z} = \{z^1, \dots, z^p\}$  is an orthonormal basis of  $E$  and the sequence  $\mathfrak{Z} = \{z^{p+1}, \dots, z^N\}$  is an orthonormal basis of the orthogonal complement  $E^\perp$ .

Case II:  $n < p$  and  $p + n > N$ . There exist

- a sequence  $u^1, \dots, u^n$  of orthogonal vectors in  $\mathbb{C}^n$ ;
- two sequences of mutually orthogonal vectors in  $\mathbb{C}^{N-n}$ ,  $\mathfrak{V} = \{V^1, \dots, V^{N-p}\}$  and  $\mathfrak{W} = \{W^1, \dots, W^{p-n}\}$ ;
- Jordan angles  $0 = \theta_1 = \dots = \theta_{n+p-N} \leq \dots \leq \theta_n \leq \pi/2$

such that, noting that

$$z^i = \begin{bmatrix} u^i \\ e(N-n) \end{bmatrix}, \quad i = 1, \dots, n+p-N,$$

$$z^i = \begin{bmatrix} u^i \cos \theta_i \\ V^{i-n-p+N} \sin \theta_i \end{bmatrix}, \quad i = n+p-N+1, \dots, n;$$

$$z^{n+i} = \begin{bmatrix} e(n) \\ W^i \end{bmatrix}, \quad i = 1, \dots, p-n;$$

$$z^{p+i} = \begin{bmatrix} u^{n+p-N+i} \sin \theta_{n+p-N+i} \\ -V^i \cos \theta_{n+p-N+i} \end{bmatrix}, \quad i = 1, \dots, N-p,$$

the set  $\mathfrak{Z} = \{z^1, \dots, z^p\}$  is an orthonormal basis of  $E$  and the set  $\mathfrak{Z} = \{z^{p+1}, \dots, z^N\}$  is an orthonormal basis of  $E^\perp$ .

Case III:  $n < p$  and  $p + n = N$ . There exist

- a sequence  $u^1, \dots, u^n$  of orthogonal vectors in  $\mathbb{C}^n$ ;
- two sequences of mutually orthogonal vectors in  $\mathbb{C}^{N-n}$ ,  $\mathfrak{V} = \{V^1, \dots, V^n\}$  and  $\mathfrak{W} = \{W^1, \dots, W^{p-n}\}$ ;
- Jordan angles  $0 \leq \theta_1 \leq \dots \leq \theta_n \leq \pi/2$

such that, noting that

$$z^i = \begin{bmatrix} u_i \cos \theta_i \\ V^i \sin \theta_i \end{bmatrix}, \quad i = 1, \dots, n;$$

$$z^{n+i} = \begin{bmatrix} e(n) \\ W^i \end{bmatrix}, \quad i = 1, \dots, p-n;$$

$$z^{p+i} = \begin{bmatrix} u^i \sin \theta_i \\ -V^i \cos \theta_i \end{bmatrix}, \quad i = 1, \dots, n,$$

the set  $\mathfrak{Z} = \{z^1, \dots, z^p\}$  is an orthonormal basis of  $E$  and the set  $\mathfrak{Z} = \{z^{p+1}, \dots, z^N\}$  is an orthonormal basis of  $E^\perp$ .

Case IV:  $n = p$ . With the notations of cases I–III:

- For  $2p < N$ ,

$$z^i = \begin{bmatrix} u^i \cos \theta_i \\ V^i \sin \theta_i \end{bmatrix}, \quad i = 1, \dots, p;$$

$$z^{p+i} = \begin{bmatrix} u^i \sin \theta_i \\ -V^i \cos \theta_i \end{bmatrix}, \quad i = 1, \dots, p;$$

$$z^{2p+i} = \begin{bmatrix} e(n) \\ W^i \end{bmatrix}, \quad i = 1, \dots, N - 2p.$$

- For  $2p > N$ ,

$$z^i = \begin{bmatrix} u^i \\ e(N - n) \end{bmatrix}, \quad i = 1, \dots, 2p - N;$$

$$z^i = \begin{bmatrix} u^i \cos \theta_i \\ V^{i-2p+N} \sin \theta_i \end{bmatrix}, \quad i = 2p - N + 1, \dots, p;$$

$$z^i = \begin{bmatrix} u^{i+p-N} \sin \theta_{i+p-N} \\ -V^{i-p} \cos \theta_{i+p-N} \end{bmatrix}, \quad i = p + 1, \dots, N.$$

- For  $2p = N$ ,

$$z^i = \begin{bmatrix} u^i \cos \theta_i \\ V^i \sin \theta_i \end{bmatrix}, \quad z^{p+i} = \begin{bmatrix} u^i \sin \theta_i \\ -V^i \cos \theta_i \end{bmatrix}, \quad i = 1, \dots, p.$$

By reordering the rows of  $Q$ , the procedure described above works for every subset  $J = \{x_1, \dots, x_n\} \subset \{1, \dots, N\}$ ,  $1 \leq n \leq p$ , and gives related bases of the spaces  $E$  and  $E^\perp$ . Note that in the Euclidean context, i.e. for  $E \subset \mathbb{R}^N$  and the CSD applied to orthogonal matrices, the angles appearing in the CS decomposition (related to  $J$ ) are the principal Jordan angles between the space  $E$  and the basic subspace  $\mathbb{R}_J^N = \{x = (x_k) \in \mathbb{R}^N : x_k = 0 \text{ if } k \notin J\}$ .

An important statistical application of principal angles is the canonical correlation analysis (CCA) of [13]. In order to develop a unified algebraic formulation of concepts in multivariate analysis (like, e.g., CCA), [1] thoroughly studied (see also [20]) the geometry of subspaces in  $\mathbb{R}^N$  in terms of orthogonal and oblique projectors, and introduced, among others, the notation of so-called *multiplicative cosine and sine*:  $\cos\{E, \mathbb{R}_J^N\} = \prod_{i=1}^n \cos \theta_i$ ,  $\sin\{E, \mathbb{R}_J^N\} = \prod_{i=1}^n \sin \theta_i$ .

The basis of  $E$  given by the CSD is a pertinent tool for the study of the associated determinantal process. For example, it immediately gives the following proposition.

**Proposition 1.** For a set  $J = \{x_1, \dots, x_n\}$ ,  $n \leq p$ , we have:

(a)  $\mathbb{P}\{|J \cap \phi| = n\} = \prod_{i=1}^n \cos^2 \theta_i$ , and, for  $k = 1, \dots, n - 1$ ,

$$\mathbb{P}\{|J \cap \phi| = k\} = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \cos^2 \theta_{i_j} \prod_{j \notin \{i_1, \dots, i_k\}} \sin^2 \theta_j. \tag{4}$$

(b)

$$\mathbb{P}\{|J \cap \phi^c| = n\} = \prod_{i=1}^n \sin^2 \theta_i \tag{5}$$

and, for  $k = 1, \dots, n - 1$ ,

$$\mathbb{P}\{|J \cap \phi^c| = k\} = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k \sin^2 \theta_{i_j} \prod_{j \notin \{i_1, \dots, i_k\}} \cos^2 \theta_j.$$

(c) If  $n < p$  and  $\mathbb{P}\{J \subset \phi\} > 0$ , then the conditioned process  $\{\phi \mid J \subset \phi\} \setminus J$  is determinantal such that  $\{\phi \mid J \subset \phi\} \setminus J = \phi(\mathfrak{W})$ .

(d) If  $N - p > n$  and  $\mathbb{P}\{J \subset \phi^c\} > 0$ , then the conditioned process  $\{\phi \mid J \subset \phi^c\}$  is determinantal such that  $\{\phi \mid J \subset \phi^c\} = \phi(\mathfrak{Y} \cup \mathfrak{W})$ .

(e) If  $\mathbb{P}\{J \subset \phi\} > 0$  then, for all  $K \subset \{1, \dots, N\} \setminus J$ ,  $\mathbb{P}\{K \subset \phi(\mathfrak{W})\} \leq \mathbb{P}\{K \subset \phi\}$ , and if  $\mathbb{P}\{J \subset \phi^c\} > 0$  then  $\mathbb{P}\{K \subset \phi\} \leq \mathbb{P}\{K \subset \phi(\mathfrak{Y} \cup \mathfrak{W})\}$ .

**Remark 2.** The fact that the conditioned processes  $\{\phi \mid J \subset \phi\} \setminus J$  and  $\{\phi \mid J \subset \phi^c\}$  are determinantal, as well as the inequalities in Proposition 1(e), are well-known results proved in [17].

**Remark 3.** Regarding Proposition 1(a) and (b), it was proved more generally in [14, Theorem 5] that, for general determinantal processes with trace-class (both discrete and continuous case) kernels, the number of points in the process has the distribution of a sum of independent Bernoulli random variables.

More elaborate information can be obtained from this point of view.

**Proposition 2.** Consider the discrete determinantal process  $\phi = \phi(\mathfrak{Z})$  associated with a set  $\mathfrak{Z} = \{z^1, \dots, z^p\}$ ,  $1 < p < N$ , of orthonormal vectors in  $\mathbb{C}^N$ . Fix points  $J = \{x_1, \dots, x_n\} \subset \{1, \dots, N\}$ ,  $1 \leq n \leq p$ , such that  $\mathbb{P}\{x_2, \dots, x_n \subset \phi^c\} > 0$ . With the choice (to simplify the notation)  $x_i = i$ ,  $i = 1, \dots, n$ , we have

$$\begin{aligned} & \left| \langle z_1, z_n \rangle + \sum_{k=1}^{n-2} (-1)^k \sum_{2 \leq i_1 < \dots < i_k \leq n-1} \left\langle z_1 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right), z_n \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right) \right\rangle \right|^2 \\ &= \mathbb{P}\{x_2, \dots, x_n \subset \phi^c\} \times \mathbb{P}\{x_2, \dots, x_{n-1} \subset \phi^c\} \\ & \quad \times \left[ \mathbb{P}\{x_1 \in \phi \mid \{x_2, \dots, x_n\} \subset \phi^c\} - \mathbb{P}\{x_1 \in \phi \mid \{x_2, \dots, x_{n-1}\} \subset \phi^c\} \right]. \tag{6} \end{aligned}$$

*Proof.* The left- and right-hand sides of (6) do not depend of the choice of the basis of  $E$ . Choose the basis given by the CS decomposition related to the set  $J = \{2, \dots, n - 1\}$ , with the

reordering  $(2, \dots, n - 1, 1, n)^t$  and  $N - p - n + 2 > 0$  (the general situation, case I). The first  $n$  coordinates of these bases have the following form:

$$\begin{matrix} 2 \\ \vdots \\ n-1 \\ 1 \\ n \end{matrix} \begin{bmatrix} \cos \theta_1 u_1^1 \dots \cos \theta_{n-2} u_1^{n-2} & 0 & \dots & 0 \\ \vdots & \vdots & & \\ \cos \theta_1 u_{n-2}^1 \dots \cos \theta_{n-2} u_{n-2}^{n-2} & 0 & \dots & 0 \\ \sin \theta_1 V_1^1 \dots \sin \theta_{n-2} V_1^{n-2} & W_1^1 \dots W_1^{p-n+2} & & \\ \sin \theta_1 V_2^1 \dots \sin \theta_{n-2} V_2^{n-2} & W_2^1 \dots W_2^{p-n+2} & & \end{bmatrix} \cdot \begin{bmatrix} \sin \theta_1 u_1^1 \dots \sin \theta_{n-2} u_1^{n-2} & 0 & \dots & 0 \\ \vdots & \vdots & & \\ \sin \theta_1 u_{n-2}^1 \dots \sin \theta_{n-2} u_{n-2}^{n-2} & 0 & \dots & 0 \\ -\cos \theta_1 V_1^1 \dots -\cos \theta_{n-2} V_1^{n-2} & \tilde{W}_1^1 \dots \tilde{W}_1^{N+2-n-p} & & \\ -\cos \theta_1 V_2^1 \dots -\cos \theta_{n-2} V_2^{n-2} & \tilde{W}_2^1 \dots \tilde{W}_2^{N+2-n-p} & & \end{bmatrix}$$

It follows from Proposition 1 that

- (a)  $\mathbb{P}\{\{x_2, \dots, x_{n-1}\} \subset \phi^c\} = \prod_{i=1}^{n-2} \sin^2 \theta_i.$
- (b)  $\mathbb{P}\{\{x_2, \dots, x_n\} \subset \phi^c\} = \prod_{i=1}^{n-2} \sin^2 \theta_i \|\tilde{W}_2\|^2.$
- (c)  $\mathbb{P}\{x_1 \in \phi | \{x_2, \dots, x_{n-1}\} \subset \phi^c\} = \|V_1\|^2 + \|W_1\|^2.$
- (d)  $\mathbb{P}\{x_1 \in \phi | \{x_2, \dots, x_n\} \subset \phi^c\} =$

$$\begin{aligned}
 & \mathbb{P}\{x_1 \in \phi, x_n \in \phi^c | \{x_2, \dots, x_{n-1}\} \subset \phi^c\} \times \frac{\mathbb{P}\{\{x_2, \dots, x_{n-1}\} \subset \phi^c\}}{\mathbb{P}\{\{x_2, \dots, x_n\} \subset \phi^c\}} \\
 &= [\mathbb{P}\{x_1 \in \phi | \{x_2, \dots, x_{n-1}\} \subset \phi^c\} - \mathbb{P}\{x_1 \in \phi, x_n \in \phi | \{x_2, \dots, x_{n-1}\} \subset \phi^c\}] \\
 & \quad \times \frac{\mathbb{P}\{\{x_2, \dots, x_{n-1}\} \subset \phi^c\}}{\mathbb{P}\{\{x_2, \dots, x_n\} \subset \phi^c\}} \\
 &= [\|V_1\|^2 + \|W_1\|^2 - \|(V_1, W_1) \wedge (V_2, W_2)\|^2] \times \frac{1}{\|\tilde{W}_2\|^2}.
 \end{aligned}$$

From (a)–(d), an elementary computation gives the right-hand side of (6) (note that  $\|V_2\|^2 + \|W_2\|^2 + \|\tilde{W}_2\|^2 = 1$ ). Indeed, we get

$$\begin{aligned}
 & \mathbb{P}\{\{x_2, \dots, x_n\} \subset \phi^c\} \times \mathbb{P}\{\{x_2, \dots, x_{n-1}\} \subset \phi^c\} \\
 & \quad \times [\mathbb{P}\{x_1 \in \phi | \{x_2, \dots, x_n\} \subset \phi^c\} - \mathbb{P}\{x_1 \in \phi | \{x_2, \dots, x_{n-1}\} \subset \phi^c\}] \\
 &= \prod_{i=2}^{n-2} \sin^4 \theta_i [(\|V_1\|^2 + \|W_1\|^2)(1 - \|\tilde{W}_2\|^2) - \|(V_1, W_1) \wedge (V_2, W_2)\|^2] \\
 &= \prod_{i=1}^{n-2} \sin^4 \theta_i | \langle (V_1, W_1), (V_2, W_2) \rangle |^2. \tag{7}
 \end{aligned}$$



To compute the left-hand side of (6), we write  $z_1^0 = (\sin \theta_1 V_1^1, \dots, \sin \theta_{n-2} V_1^{n-2})$ ,  $z_n^0 = (\sin \theta_1 V_2^1, \dots, \sin \theta_{n-2} V_2^{n-2})$ , and  $\tilde{z}^i = (\cos \theta_i u_1^i, \dots, \cos \theta_i u_{n-2}^i, 0)^t$ . Observe that

$$\langle z_1, z_n \rangle + \sum_{k=1}^{n-2} (-1)^k \sum_{2 \leq i_1 < \dots < i_k \leq n-1} \left\langle z_1 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right), z_n \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right) \right\rangle = A + B, \tag{8}$$

with

$$A = \langle z_1^0, z_n^0 \rangle + \sum_{k=1}^{n-2} (-1)^k \sum_{2 \leq i_1 < \dots < i_k \leq n-1} \left\langle z_1^0 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right), z_n^0 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right) \right\rangle, \tag{9}$$

$$B = \langle W_1, W_2 \rangle \left( 1 + \sum_{k=1}^{n-2} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n-2} \left\| \bigwedge_{j=1}^k \tilde{z}^{i_j} \right\|^2 \right).$$

Obviously,  $\| \bigwedge_{j=1}^k \tilde{z}^{i_j} \|^2 = \prod_{j=1}^k \cos^2 \theta_{i_j}$ , and thus

$$1 + \sum_{k=1}^{n-2} (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq n-2} \left\| \bigwedge_{j=1}^k \tilde{z}^{i_j} \right\|^2 = \prod_{i=1}^{n-2} (1 - \cos^2 \theta_i) = \prod_{i=1}^{n-2} \sin^2 \theta_i,$$

and consequently

$$B = \langle W_1, W_2 \rangle \prod_{i=1}^{n-2} \sin^2 \theta_i. \tag{10}$$

In order to compute  $A$  we introduce  $\tilde{z}^{i,l} = (\cos \theta_i u_1^i, \dots, \cos \theta_i u_{n-2}^i, \sin \theta_i V_l^i)^t$ ,  $l = 1, 2$  and  $i = 1, \dots, n - 2$ . A little thought shows that, for  $k \geq 1$ ,

$$\begin{aligned} \sum_{2 \leq i_1 < \dots < i_k \leq n-1} \left\langle z_1^0 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right), z_n^0 \wedge \left( \bigwedge_{j=1}^k z_{i_j} \right) \right\rangle \\ = \sum_{1 \leq i_1 < \dots < i_{k+1} \leq n-2} \left[ \left\langle \bigwedge_{j=1}^{k+1} \tilde{z}^{i_j,1}, \bigwedge_{j=1}^{k+1} \tilde{z}^{i_j,2} \right\rangle - \left\| \bigwedge_{j=1}^{k+1} \tilde{z}^{i_j} \right\|^2 \right]. \end{aligned} \tag{11}$$

Moreover, we have

$$\begin{aligned} \bigwedge_{j=1}^k \tilde{z}^{i_j,l} &= \bigwedge_{j=1}^k (\tilde{z}^{i_j} + (e(n-2), \sin \theta_{i_j} V_l^{i_j})^t) \\ &= \bigwedge_{j=1}^k \tilde{z}^{i_j} + \sum_{j=1}^k (-1)^{j+1} (e(n-2), \sin \theta_{i_j} V_l^{i_j})^t \wedge \left( \bigwedge_{s=1, s \neq j}^k \tilde{z}^{i_s} \right). \end{aligned}$$

From the orthogonality properties of the relevant multivectors we obtain, from the last equation,

$$\begin{aligned} & \left\langle \bigwedge_{j=1}^k \tilde{z}^{j,1}, \bigwedge_{j=1}^k \tilde{z}^{j,2} \right\rangle - \left\| \bigwedge_{j=1}^k \tilde{z}^{j,1} \right\|^2 \\ &= \sum_{j=1}^k \left\langle (e(n-2), \sin \theta_{ij} V_1^{ij})^t \wedge \left( \bigwedge_{s=1, s \neq j}^k \tilde{z}^{i_s} \right), (e(n-2), \sin \theta_{ij} V_2^{ij})^t \wedge \left( \bigwedge_{s=1, s \neq j}^k \tilde{z}^{i_s} \right) \right\rangle \\ &= \sum_{j=1}^k V_1^{ij} V_2^{ij} \sin^2 \theta_{ij} \left\| \bigwedge_{s=1, s \neq j}^k \tilde{z}^{i_s} \right\|^2 = \sum_{j=1}^k V_1^{ij} V_2^{ij} \sin^2 \theta_{ij} \prod_{s=1, s \neq j}^k \cos^2 \theta_{i_s}. \end{aligned} \tag{12}$$

From (9), (11), and (12), an elementary computation gives

$$A = \sum_{i=1}^{n-2} V_1^i V_2^i \sin^2 \theta_i \prod_{j=1, j \neq i}^{n-2} (1 - \cos^2 \theta_j) = \langle V_1, V_2 \rangle \prod_{i=1}^{n-2} \sin^2 \theta_i,$$

and, with (10),  $A + B = \langle (V_1, W_1), (V_2, W_2) \rangle \prod_{i=1}^{n-2} \sin^2 \theta_i$ . This and (7) prove Proposition 2. Note that from the last equation we also get that (8) is identified as a scalar product.  $\square$

For further results by using the CSD, and for some extensions of Proposition 2, see [10].

### 3. The BK inequality for increasing events generated by disjoint sets

Let  $\mathfrak{A}, \mathfrak{B} \subset 2^{\mathcal{N}}, \mathcal{N} = \{1, \dots, N\}$ , be a pair of increasing events, and suppose (obviously) that  $\emptyset \notin \mathfrak{A} \cup \mathfrak{B}$ . The events being increasing, there exist two minimal sets  $S_1 = S(\mathfrak{A}) = \{A_i, i = 1, \dots, n_1\} \subset \mathfrak{A}$  and  $S_2 = S(\mathfrak{B}) = \{B_i, i = 1, \dots, n_2\} \subset \mathfrak{B}$  such that  $A \in \mathfrak{A}$  if and only if there exists  $A_i$  such that  $A \supset A_i$ , and  $B \in \mathfrak{B}$  if and only if there exists  $B_i$  such that  $B \supset B_i$ . The sets  $A_i$  and  $B_i$  are minimal in the sense that none of  $A \in \mathfrak{A}$  (resp.  $B \in \mathfrak{B}$ ) is strictly included in  $A_i$  (resp. in  $B_i$ ).

Consider now a basic determinantal process  $\phi$  on  $\mathcal{N}$ . In the particular case when  $A \cap B = \emptyset$  for all  $A \in S_1$  and  $B \in S_2$ , we at once have  $\mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\} = \mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{B}\}$ , and thus the BK inequality (1) becomes  $\mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\} \leq \mathbb{P}\{\phi \in \mathfrak{A}\} \times \mathbb{P}\{\phi \in \mathfrak{B}\}$ , which is a negative association inequality. It was proved in [17, 18] that determinantal processes have negative association, meaning that this inequality is fulfilled.

In the general situation it is helpful to reformulate the BK inequality (1) as follows.

**Proposition 3.** *The inequality (1) is satisfied if and only if*

$$\mathbb{P}\{\phi \notin \mathfrak{A} \cup \mathfrak{B}\} \leq \mathbb{P}\{\phi \notin \mathfrak{A}\} \times \mathbb{P}\{\phi \notin \mathfrak{B}\} + \mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\} - \mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{B}\}. \tag{13}$$

*Proof.* Observe that

$$\begin{aligned} \mathbb{P}\{\phi \notin \mathfrak{A} \cup \mathfrak{B}\} &= 1 - \mathbb{P}\{\phi \in \mathfrak{A} \cup \mathfrak{B}\} = 1 - \mathbb{P}\{\phi \in \mathfrak{A}\} - \mathbb{P}\{\phi \in \mathfrak{B}\} + \mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\} \\ &= \mathbb{P}\{\phi \notin \mathfrak{A}\} \times \mathbb{P}\{\phi \notin \mathfrak{B}\} - \mathbb{P}\{\phi \in \mathfrak{A}\} \times \mathbb{P}\{\phi \in \mathfrak{B}\} + \mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\}. \end{aligned}$$

Thus,  $\mathbb{P}\{\phi \notin \mathfrak{A} \cup \mathfrak{B}\} - \mathbb{P}\{\phi \notin \mathfrak{A}\} \times \mathbb{P}\{\phi \notin \mathfrak{B}\} - \mathbb{P}\{\phi \in \mathfrak{A} \cap \mathfrak{B}\} + \mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{B}\} \leq 0$  if and only if  $\mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{B}\} - \mathbb{P}\{\phi \in \mathfrak{A}\} \times \mathbb{P}\{\phi \in \mathfrak{B}\} \leq 0$ .  $\square$

Suppose now that  $\mathfrak{A} = \mathfrak{B}$ . The inequality in (13) becomes

$$\mathbb{P}\{\phi \notin \mathfrak{A}\} \leq \mathbb{P}\{\phi \notin \mathfrak{A}\}^2 + \mathbb{P}\{\phi \in \mathfrak{A}\} - \mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{A}\}. \tag{14}$$

If the sets of  $S(\mathfrak{A}) = \{A_1, \dots, A_n\}$  are disjoint, that is if  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ , then  $\{\phi \in \mathfrak{A} \setminus (\mathfrak{A} \circ \mathfrak{A})\} = \bigcup_{i=1}^n \{A_i \subset \phi, A_j \not\subset \phi, \text{ for all } j \neq i\}$ . Therefore,

$$\begin{aligned} \mathbb{P}\{\phi \in \mathfrak{A}\} - \mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{A}\} &= \mathbb{P}\{\mathfrak{A} \setminus (\mathfrak{A} \circ \mathfrak{A})\} \\ &= \sum_{i=1}^n \mathbb{P}\{A_i \subset \phi, A_j \not\subset \phi, \text{ for all } j \neq i\} \\ &= \sum_{i=1}^n [\mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\} - \mathbb{P}\{A_i \not\subset \phi, \text{ for all } i = 1, \dots, n\}] \\ &= \sum_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\} - n\mathbb{P}\{\phi \notin \mathfrak{A}\}, \end{aligned}$$

and (14) takes the form

$$(n + 1)\mathbb{P}\{\phi \notin \mathfrak{A}\} \leq \mathbb{P}\{\phi \notin \mathfrak{A}\}^2 + \sum_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}. \tag{15}$$

Now fix  $n_0 \geq 2$ , and suppose that Conjecture 1 is fulfilled for all  $2 \leq n \leq n_0$ .

**Lemma 1.** *Under this hypothesis, for all  $A_i, i = 1, \dots, n$ , disjoint subsets of  $\{1, \dots, N\}$  with  $2 \leq n \leq n_0$  and such that  $\mathbb{P}\{A_i \not\subset \phi, \text{ for all } i = 1, \dots, n\} > 0$ , we have*

$$\mathbb{P}\{A_i \not\subset \phi, \text{ for all } i = 1, \dots, n\}^{n-1} \leq \prod_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}. \tag{16}$$

*Proof.* For  $n = 2$  the inequality (16) is the well-known correlation inequality. For  $n > 2$ , applying (2) we get  $\prod_{k=2}^n \mathbb{P}\{A_k \not\subset \phi \mid A_j \not\subset \phi, \text{ for all } j \neq k\} \leq \prod_{k=2}^n \mathbb{P}\{A_k \not\subset \phi \mid A_j \not\subset \phi, \text{ for all } j \neq 1, k\}$  if and only if

$$\frac{\mathbb{P}\{A_i \not\subset \phi, \text{ for all } i = 1, \dots, n\}^{n-1}}{\prod_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}} \leq \frac{\mathbb{P}\{A_i \not\subset \phi, \text{ for all } i \neq 1\}^{n-2}}{\prod_{k=2}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq 1, k\}},$$

and thus Lemma 1 follows by induction. □

We will need the following elementary lemma. Its proof being trivial, we omit it.

**Lemma 2.** *For all  $0 < a \leq 1$  and  $n > 0$ ,  $(n + 1) - a - na^{-1/n} \leq 0$ .*

**Theorem 1.** *Let  $\mathfrak{A}$  be an increasing event generated by disjoint sets  $A_1, \dots, A_n$ . Suppose that Conjecture 1 holds. Then*

$$\mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{A}\} \leq \mathbb{P}\{\phi \in \mathfrak{A}\}^2. \tag{17}$$

*Proof.* We have to prove (15). By Lemma 2 we obtain  $(n + 1)\mathbb{P}\{\phi \notin \mathfrak{A}\} \leq \mathbb{P}\{\phi \notin \mathfrak{A}\}^2 + n\mathbb{P}\{\phi \notin \mathfrak{A}\}^{(n-1)/n}$ . Lemma 1 implies that  $\mathbb{P}\{\phi \notin \mathfrak{A}\}^{n-1} = \mathbb{P}\{A_i \not\subset \phi, \text{ for all } i = 1, \dots, n\}^{n-1} \leq \prod_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}$ , so it remains to apply the arithmetic–geometric mean inequality,  $n \prod_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}^{\frac{1}{n}} \leq \sum_{i=1}^n \mathbb{P}\{A_j \not\subset \phi, \text{ for all } j \neq i\}$ , to obtain (15) as desired. □

**Remark 4.** Consider an event  $\tilde{S} = \{D_1, \dots, D_{n_0}\} \subset 2^{\mathcal{N}}$  of disjoint sets such that  $\mathbb{P}\{D \not\subset \phi, \text{ for all } D \in \tilde{S}\} > 0$ . Write  $\psi = \{\phi \mid D \not\subset \phi, \text{ for all } D \in \tilde{S}\}$ . If Conjecture 1 holds, then it is obvious that the inequality (2) is also satisfied for the conditioned process  $\psi$  provided that the sets occurring in (2) are disjoint from those in  $\tilde{S}$ . Consequently, if  $\mathfrak{A}$  is an increasing event generated by disjoint sets  $A_1, \dots, A_n$  such that  $A_i \cap D = \emptyset$  for all  $i = 1, \dots, n$  and  $D \in \tilde{S}$ , then we obtain

$$\mathbb{P}\{\psi \in \mathfrak{A} \circ \mathfrak{A}\} \leq \mathbb{P}\{\psi \in \mathfrak{A}\}^2. \tag{18}$$

Let  $S_1 = \{A_i, i = 1, \dots, n_1\}, S_2 = \{B_i, i = 1, \dots, n_2\}$ , and  $S = \{C_i, i = 1, \dots, n_3\}$  be events such that all sets in  $S_1 \cup S_2 \cup S \subset 2^{\mathcal{N}}$  are pairwise disjoint.

**Theorem 2.** Suppose that Conjecture 1 holds. Then, for increasing events  $\mathfrak{A}$  and  $\mathfrak{B}$  such that  $S(\mathfrak{A}) = S_1 \cup S$  and  $S(\mathfrak{B}) = S_2 \cup S$ , we have

$$\mathbb{P}\{\psi \in \mathfrak{A} \circ \mathfrak{B}\} \leq \mathbb{P}\{\psi \in \mathfrak{A}\} \times \mathbb{P}\{\psi \in \mathfrak{B}\}, \tag{19}$$

where  $\psi = \{\phi \mid D \not\subset \phi, \text{ for all } D \in \tilde{S}\}$  and all sets in  $S_1 \cup S_2 \cup S \cup \tilde{S} \subset 2^{\mathcal{N}}$  are pairwise disjoint.

*Proof.* The proof proceeds by induction using Theorem 1 and, starting from (18), applying Lemma 3 step by step. □

**Lemma 3.** Fix  $S_1, S_2$ , and  $S$ , and suppose that the BK inequality (19) is fulfilled for all conditioned processes  $\psi$  subjected to the conditions of Theorem 2. Fix  $A \subset \mathcal{N}, A \neq \emptyset$ , such that  $A \cap A' = \emptyset$  for all  $A' \in S_1 \cup S_2 \cup S$ . Denote by  $\tilde{\mathfrak{A}} = \sigma\{A, \mathfrak{A}\}$  the increasing event generated by  $A$  and  $\mathfrak{A}$ . Then, the BK inequality

$$\mathbb{P}\{\psi \in \tilde{\mathfrak{A}} \circ \mathfrak{B}\} \leq \mathbb{P}\{\psi \in \tilde{\mathfrak{A}}\} \times \mathbb{P}\{\psi \in \mathfrak{B}\} \tag{20}$$

is satisfied for all conditioned processes  $\psi = \{\phi \mid D \not\subset \phi \text{ for all } D \in \tilde{S}\}$  such that all sets of  $S(\tilde{\mathfrak{A}}) \cup S_2 \cup S \cup \tilde{S} \subset 2^{\mathcal{N}}$  are pairwise disjoint.

*Proof.* By (13), we may suppose that  $\mathbb{P}\{A' \not\subset \psi, \text{ for all } A' \in S(\tilde{\mathfrak{A}}) \cup S_2 \cup S \cup \tilde{S}\} > 0$ . We have  $\{\psi \in \mathfrak{A} \cap \mathfrak{B} \setminus \mathfrak{A} \circ \mathfrak{B}\} = \cup_{C \in S} \{C \subset \psi, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S, A' \neq C\}$  and

$$\begin{aligned} &\{\psi \in \tilde{\mathfrak{A}} \cap \mathfrak{B} \setminus \tilde{\mathfrak{A}} \circ \mathfrak{B}\} \\ &= \cup_{C \in S} \{C \subset \psi, A \not\subset \psi, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S, A' \neq C\}. \end{aligned} \tag{21}$$

Formulas (13) and (21) imply that the BK inequality (20) can be written as

$$\begin{aligned} &\mathbb{P}\{A \not\subset \psi, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S\} \\ &\leq \mathbb{P}\{A \not\subset \psi, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S\} \times \mathbb{P}\{A' \not\subset \psi, \text{ for all } A' \in S_2 \cup S\} \\ &\quad + \sum_{C \in S} \mathbb{P}\{C \subset \psi, A \not\subset \psi, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S, A' \neq C\} \end{aligned}$$

or, introducing the process  $\psi_0 = \{\phi \mid A \not\subset \phi, A' \not\subset \phi \text{ for all } A' \in \tilde{S}\}$ , as

$$\begin{aligned} &\mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_1 \cup S_2 \cup S\} \\ &\leq \mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_1 \cup S\} \times \mathbb{P}\{A' \not\subset \psi, \text{ for all } A' \in S_2 \cup S\} \\ &\quad + \sum_{C \in S} \mathbb{P}\{C \subset \psi_0, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S, A' \neq C\}. \end{aligned} \tag{22}$$

The stated hypotheses imply that

$$\begin{aligned} & \mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_1 \cup S_2 \cup S\} \\ & \leq \mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_1 \cup S\} \times \mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_2 \cup S\} \\ & \quad + \sum_{C \in S} \mathbb{P}\{C \subset \psi_0, A' \not\subset \psi, \text{ for all } A' \in S_1 \cup S_2 \cup S, A' \neq C\}. \end{aligned} \tag{23}$$

It is easy to see that Conjecture 1 implies the inequality  $\mathbb{P}\{A' \not\subset \psi_0, \text{ for all } A' \in S_2 \cup S\} \leq \mathbb{P}\{A' \not\subset \psi, \text{ for all } A' \in S_2 \cup S\}$ , and by this and (23) we obtain (22), which finishes the proof of Lemma 3.  $\square$

#### 4. The BK inequality for increasing events generated by simple points

As mentioned in the introduction, the inequality (2) is satisfied when the occurring sets are reduced to being simple points. This follows easily, for example, from Proposition 1. Therefore, Theorem 2 implies the following result.

**Theorem 3.** *Let  $\mathfrak{A}, \mathfrak{B}$  be increasing events generated by simple points. The BK inequality  $\mathbb{P}\{\phi \in \mathfrak{A} \circ \mathfrak{B}\} \leq \mathbb{P}\{\phi \in \mathfrak{A}\} \times \mathbb{P}\{\phi \in \mathfrak{B}\}$  is then satisfied for all determinantal discrete processes  $\phi$  associated with sets of orthonormal vectors of  $\mathbb{C}^N$ .*

**Remark 5.** For sets reduced to being simple points, the key inequality (16) can be seen from the point of view given by the CSD. Indeed, consider the CSD in case I applied to  $J = \{x_1, \dots, x_n\}$  and, accordingly, let  $v^j = (v_1^j, \dots, v_n^j)^t, v_i^j = (\sin \theta_j)u_i^j, i, j = 1, \dots, n$  be the vectors such that  $\mathbb{P}\{\{x_{i_1}, \dots, x_{i_k}\} \subset \phi^c\} = \|\bigwedge_{j=1}^k v_{i_j}\|^2$  for all  $\{x_{i_1}, \dots, x_{i_k}\} \subset J$ . Write  $\tilde{v}_i = \bigwedge_{j \neq i} v_j = (\tilde{v}_i^1, \dots, \tilde{v}_i^n) \in \mathbb{C}^n, i = 1, \dots, n$ , where  $\tilde{v}_i^j = \prod_{k \neq j} \sin \theta_k \times \tilde{u}_i^j$  and  $\tilde{u}_i^j$  is the  $(i, n - j + 1)$ -minor of the unitary matrix  $U = (u_i^j)_{i,j=1,\dots,n}$ . By (5), we obtain

$$\begin{aligned} \mathbb{P}\{x_i \in \phi^c, i = 1, \dots, n\}^{n-1} &= \prod_{i=1}^n (\sin \theta_i)^{2(n-1)} \\ &= \left\| \bigwedge_{i=1}^n v_i \right\|^{2(n-1)} \\ &= \left\| \bigwedge_{i=1}^n \tilde{v}_i \right\|^2 \\ &\leq \prod_{i=1}^n \|\tilde{v}_i\|^2 = \prod_{i=1}^n \mathbb{P}\{x_j \in \phi^c, \text{ for all } j \neq i\}. \end{aligned}$$

**Remark 6.** It was pointed out to us that for an increasing event  $\mathfrak{A}$  generated by simple points  $S = \{x_1, \dots, x_n\}$ , the inequality (17), which can be read as

$$\mathbb{P}\{|S \cap \phi| \geq 2\} \leq \mathbb{P}\{|S \cap \phi| \geq 1\}^2, \tag{24}$$

can also be obtained by a direct computation from (4) of Proposition 1 and, moreover, if we consider the product measure  $\mu = \otimes_{i=1}^n ((\cos^2 \theta_i)\delta_1 + (\sin^2 \theta_i)\delta_0)$  on the product space

$E = \{0, 1\}^n$  and increasing events  $\mathfrak{A}_i = \{a = (a_j) \in E \text{ such that } \sum_{j=1}^n a_j \geq i\}$ ,  $i = 0, \dots, n$ , then the formulas in (4) imply that  $P\{|\mathcal{S} \cap \phi| \geq i\} = \mu(\mathfrak{A}_i)$ . From [26, Theorem 3.3], we get

$$\mathbb{P}\{|\mathcal{S} \cap \phi| \geq i + j\} \leq \mathbb{P}\{|\mathcal{S} \cap \phi| \geq i\} \times \mathbb{P}\{|\mathcal{S} \cap \phi| \geq j\}, \quad 2 \leq i + j \leq n. \quad (25)$$

Furthermore, note that by Remark 3 the inequalities (24) and (25) are still valid for general determinantal processes (both discrete and continuous) taking for  $S$  a Borel set.

### 5. Extensions and concluding remarks

Theorem 3 can be easily extended in the setting of general discrete determinantal processes. From the construction given in [18, Paragraph 2.2], which starts from the basic processes, it follows at once that Theorems 1 and 2 are valid (the generated sets  $S(\mathfrak{A})$  and  $S(\mathfrak{B})$  being finite or infinite) for determinantal point processes defined on denumerable sets  $\mathcal{E}$  and associated with closed subspaces of  $l^2(\mathcal{E})$ . Now, let  $\phi$  be such a process on  $\mathcal{E}$ . Fix  $\mathcal{F} \subset \mathcal{E}$  and consider the process  $\psi = \phi \cap \mathcal{F}$ .

Let  $\mathfrak{A}, \mathfrak{B} \subset 2^{\mathcal{F}}$ ,  $\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}} \subset 2^{\mathcal{E}}$  be the increasing events generated respectively by  $S_1 = S(\mathfrak{A}) = S(\tilde{\mathfrak{A}}) \subset \mathcal{F}$  and  $S_2 = S(\mathfrak{B}) = S(\tilde{\mathfrak{B}}) \subset \mathcal{F}$ . The BK inequalities for  $\phi$ ,  $\tilde{\mathfrak{A}}$ ,  $\tilde{\mathfrak{B}}$  and  $\psi$ ,  $\mathfrak{A}$ ,  $\mathfrak{B}$  involve only the generating sets  $S_1$  and  $S_2$ . Consequently, Theorem 3 is valid for  $\psi$  as well. To finish, just note that, by [18, Paragraph 2.2], discrete determinantal processes associated with positive contractions (the general case) are of the form  $\psi = \phi \cap \mathcal{F}$ .

By the transference principle [18, Section 3.6], Theorem 3 could also be extended to the continuous case, but this is of little use because in the continuous setting the intensity measures related to determinantal processes of interest are of diffusive type, which implies that  $\mathbb{P}\{x \in \phi\} = 0$  for points  $x$  (however, as mentioned in Remark 6, inequalities (24) and (25) still hold).

### Acknowledgement

I thank the anonymous referees for their constructive comments, especially for a pertinent question about the validity of Conjecture 1 for the non-determinantal point processes, which led to looking at the process described in Remark 1.

### Funding information

There are no funding bodies to thank relating to this creation of this article.

### Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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