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PROOF SYSTEMS FOR EXACT ENTAILMENT

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Abstract. We present a series of proof systems for exact entailment (i.e., relevant truthmaker preservation from premises to conclusion) and prove soundness and completeness. Using the proof systems, we observe that exact entailment is hyperintensional not only in the sense of Cresswell, but also in the sense recently proposed by Odintsov and Wansing.

§1. Introduction. Recently, there has been a growing interest in so-called "exact truthmakers" in philosophical logic and semantics.[1] An *exact truthmaker* for ϕ is a state (of affairs, event, action, etc.) which necessitates ϕ's truth while being wholly relevant [14, p. 599].[2] For example, the ball being red is an exact truthmaker for "the ball is colored." The complex state of the ball being red *and round*, in contrast, is not an exact truthmaker, since the ball's shape is irrelevant to whether it's colored.

The concept of exact truthmaking gives rise to the non-classical consequence relation of *exact entailment*, i.e., guaranteed exact truthmaker preservation from premises to conclusion (cf. [11, p. 202], [12, p. 669], [17, pp. 536–537]). Understanding the logic of exact equivalence is of fundamental importance to the project of exact truthmaker semantics. Part of the reason for this is that *exact equivalence* (i.e., mutual exact entailment) amounts to sameness of truthmaker content.[3] Additionally, there is a close connection between exact entailment and the concept of *metaphysical ground* (see [9, pp. 71–74] and [13, pp. 685–687]).

But there is also a genuinely *logical* interest in the relation. Remember that a context is *hyperintensional* iff it does not respect (classical) logical equivalence [5, p. 25].[4] Recently, reasons have been discovered for taking a hyperintensional approach to logics that have traditionally been treated intensionally, such as conditional logics [7] or deontic logics [10]. It turns out that in many such cases, exact truthmakers provide a fruitful framework for developing hyperintensional logics for the relevant concepts (cf. [1, 2, 8, 15, 16]). In this setting, the logic of exact entailment becomes the appropriate logic for reasoning within hyperintensional contexts.

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1 See [14] for an introduction.
2 Dually, an exact *falsemaker* for ϕ is a state s which necessitates ϕ's *falsity* while being wholly relevant to it.
3 See [12, 13, 19] on the notion of truthmaker content.
4 See [3] for an introduction.

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It is the aim of this paper to develop a *proof theory* for the logic of exact entailment. To this end, we shall present a series of proof systems, each displaying different logical aspects of exact entailment. First, we shall directly axiomatize exact entailment as an asymmetric (many–one) consequence relation (§4). The resulting axiomatic calculus puts the focus on the logical laws governing exact entailment. We shall then describe a deductively equivalent Hilbert calculus (§5), which puts the emphasis on formula-to-formula inferences. Our third and final proof system is a symmetric (many–many) sequent calculus (§6). This calculus puts the focus on the way the logical connectives interact with exact entailment.

Our calculi overcome most limitations of the few existing calculi from the literature. Fine [11] and Correia [4] present Hilbert calculi for (what's essentially) exact equivalence. These calculi can double as calculi for exact entailment by exploiting that ϕ exactly entails ψ iff $\phi \vee \psi$ is exactly equivalent to ψ (cf. [11, p. 202] and [4, pp. 113–114]). But since these calculi are formulated using a *binary* operator for exact equivalence, they can only be used for *binary* exact entailment, i.e., single premise, single conclusion instances of the relation. As Fine and Jago [17] point out, however, there is an irreducibly asymmetric notion of exact entailment, i.e., a concept of exact entailment where multiple premise cases cannot always be reduced to single premise cases. As a result, the calculi of Fine [11] and Correia [4] cannot account for all forms of exact entailment. All our systems, instead, work for all relevant forms of exact entailment.

Fine and Jago [17, pp. 552–556] provide a sequent calculus for exact entailment. But this calculus has two important limitations. First, the Fine–Jago calculus only works for what Fine [11, p. 206] calls the "inclusive" variant of exact truthmaker semantics. In fact, Fine and Jago [17, p. 551] explicitly leave it open to develop a sequent calculus for exact entailment under what Fine [11, p. 210] calls the "replete" variant of exact truthmaker semantics. All our systems, in comparison, can easily be adjusted for the replete semantics, simply by adding axioms/rules. Indeed, these extended systems are, to the best of our knowledge, the first comprehensive proof systems for exact entailment on the replete semantics.

The second limitation is more proof-theoretic in nature. It turns out that even though the Fine–Jago calculus has the Cut-elimination property, it fails to have the subformula property (as acknowledged by authors; cf. [17, p. 560]). This is, of course, surprising since Cut-elimination typically implies the subformula property. The culprit is a structural rule, specific to exact entailment, which deletes formulas from derivations. As we'll show in §6, the problematic rule cannot be eliminated from the Fine–Jago calculus, and as a consequence, the subformula property fails. As a further consequence, the calculus doesn't allow for proof searches. Our symmetric sequent calculus, developed in §6, instead, absorbs all the structural rules, enjoys the subformula property, and allows for straightforward proof searches.

In addition to being technically well-behaved and fruitful, studying our calculi will also lead to interesting philosophical insights about the framework of exact truthmaker semantics. Odintsov and Wansing [23] argue for a notion of hyperintensionality according to which a logic only counts as hyperintensional if it is not *self-extensional*, where a logic is self-extensional just in case its operators respect logical equivalence (within the logic). They show that Leitgeb's system HYPE [21], an important proposal for a basic system of hyperintensional logic, does not qualify as hyperintensional according to this criterion. As we'll show in §4, exact entailment on the inclusive semantics, instead, does not fail the criterion. This observation can serve as further

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ammunition in the debate about which system to take as our base system for hyperintensional logic (cf. [6, 22]). Interestingly enough, the system for exact entailment on the *replete* semantics, however, turns out not to be hyperintensional in the sense of Odintsov and Wansing. Depending on one's perspective, one may take this as a reason to prefer the inclusive semantics over its replete alternative—or the other way around.

But the interest in the proof systems is not only theoretical in nature. From a purely pragmatic side, the systems we present can be used as a base system for developing proof systems for hyperintensional logics in an exact truthmaker setting. In the conclusion (§7), we sketch a quick example of how this can work. But before we get started, we quickly go through syntax (§2) and semantics (§3).

§2. Language. In the following, we'll work with a fixed propositional language $\mathcal{L}$, which has just the connectives $\neg, \wedge, \vee$ and is defined over a set $\mathcal{P} = \{p_i : i \in I\}$ of propositional variables. The Backus–Naur Form (BNF) of $\mathcal{L}$, correspondingly, is

$$\phi ::= p_i \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi).$$

We use $p, q, r, \ldots$ as meta-variables ranging over propositional variables and $\phi, \psi, \theta, \ldots$ as meta-variables ranging over formulas. Unless indicated otherwise, $\Gamma, \Delta, \Sigma, \ldots$ range over *finite* sets of formulas.[5] We'll follow the usual notational conventions with respect to parentheses etc.

We shall often find it convenient to rely on the following alternative syntax for $\mathcal{L}$. Remember that a *literal* is either a propositional variable or its negation. We denote the set of literals by Λ and use λ as a meta-variable ranging over literals. We can then define the $\mathcal{L}$-formulas using the following BNF:

$$\phi ::= \lambda \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg\neg\phi \mid \neg(\phi \wedge \phi) \mid \neg(\phi \vee \phi).$$

We shall refer to this as "the construction from literals."

We write $\bigwedge \Gamma$ or $\bigwedge_{\phi \in \Gamma} \phi$ for the conjunction of the Γ's, and $\bigvee \Gamma$ or $\bigvee_{\phi \in \Gamma} \phi$ for the disjunction of the Γ's. We'll justify this notation ex post by observing that both conjunction and disjunction are idempotent, commutative, and associative with respect to exact entailment. As a result, our notation is logically innocuous. We shall furthermore assume some background total order on $\mathcal{L}$, which allows us to choose $\bigwedge \Gamma$ and $\bigvee \Gamma$ uniquely for given Γ by respecting the order (relying on commutativity), ignoring repetitions (relying on idempotence), and parentheses (relying on associativity).

§3. Semantics. We sketch the necessary background on exact truthmaker semantics and exact entailment.[6] A *frame* (also known as a *state space* in the literature) is a structure $\mathcal{F} = (S, \sqsubseteq)$ such that $S = \{s, t, u, \ldots\}$ is a non-empty set ("states") and $\sqsubseteq$ is a partial order over S ("parthood"), such that for each $s, t \in S$ there exists a unique least upper bound $s \sqcup t \in S$ with respect to $\sqsubseteq$ ("fusion").[7]

[5] By the compactness of exact entailment (cf. [17, Theorem 5.2, p. 546]) the restriction to finite premise sets is *relatively* harmless. The restriction to finitary cases is proof-theoretic in spirit.

[6] For a more detailed exposition, see, e.g., [12, 13, 17].

[7] As Fine [14, p. 560] points out, the concept of a *state* in exact truthmaker semantics is a technical one, which encompasses anything that can reasonably be thought of as necessitating

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A *model* is a structure $\mathcal{M} = (S, \sqsubseteq, v)$, where $(S, \sqsubseteq)$ is a frame and $v = (v^+, v^-)$ is a pair of interpretation functions $v^+ : \mathcal{P} \to \wp(S)$ ("exact truthmaker assignment") and $v^- : \mathcal{P} \to \wp(S)$ ("exact falsemaker assignment"; cf. fn. 2), subject to the following condition for $\circ = +, -$:

$$\text{If } s, t \in v^\circ(p), \text{ then } s \sqcup t \in v^\circ(p). \qquad \text{(Atomic Closure)}$$

In a model $\mathcal{M}$, we define the exact truthmaker set $|\phi|^+_\mathcal{M}$ and the exact falsemaker set $|\phi|^-_\mathcal{M}$ by simultaneous recursion:

$$\begin{array}{llll} |p|^+_\mathcal{M} = v^+(p) & (\text{Sem-0}^+) & |\neg\phi|^-_\mathcal{M} = |\phi|^+_\mathcal{M}, & (\text{Sem-}\neg^-) \\ |p|^-_\mathcal{M} = v^-(p) & (\text{Sem-0}^-) & |\neg\phi|^+_\mathcal{M} = |\phi|^-_\mathcal{M}, & (\text{Sem-}\neg^+) \end{array}$$

$$\begin{array}{ll} |\phi \wedge \psi|^+_\mathcal{M} = \{s \sqcup t : s \in |\phi|^+_\mathcal{M}, t \in |\psi|^+_\mathcal{M}\}, & (\text{Sem-}\wedge^+) \\ |\phi \wedge \psi|^-_\mathcal{M} = |\phi|^-_\mathcal{M} \cup |\psi|^-_\mathcal{M} \cup |\phi \vee \psi|^-_\mathcal{M}, & (\text{Sem-}\wedge^-) \\ & \\ |\phi \vee \psi|^+_\mathcal{M} = |\phi|^+_\mathcal{M} \cup |\psi|^+_\mathcal{M} \cup |\phi \wedge \psi|^+_\mathcal{M}, & (\text{Sem-}\vee^+) \\ |\phi \vee \psi|^-_\mathcal{M} = \{s \sqcup t : s \in |\phi|^-_\mathcal{M}, t \in |\psi|^-_\mathcal{M}\}. & (\text{Sem-}\vee^-) \end{array}$$

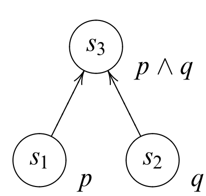
We further set $|\Gamma|^\circ_\mathcal{M} = \{|\phi|^\circ_\mathcal{M} : \phi \in \Gamma\}$ for $\circ = +, -$.

The following *canonical model* is of fundamental importance to the study of exact entailment.[8] The canonical model $\mathfrak{M} = (\mathfrak{S}, \sqsubseteq_\mathfrak{M}, v_\mathfrak{M})$ is defined by $\mathfrak{S} = \wp(\Lambda)$, $\sqsubseteq_\mathfrak{M} = \subseteq\restriction_\mathfrak{S}$, and $v_\mathfrak{M} = (v^+_\mathfrak{M}, v^-_\mathfrak{M})$, where $v^+_\mathfrak{M}(p) = \{\{p\}\}$ and $v^-_\mathfrak{M}(p) = \{\{\neg p\}\}$.

Fine and Jago [17, p. 536] show that there are *two* natural ways of explicating guaranteed exact truthmaker preservation from premises to conclusion:

- $\Gamma \vDash^c \phi$ iff for all $\mathcal{M}$, $|\bigwedge\Gamma|^+_\mathcal{M} \subseteq |\phi|^+_\mathcal{M}$ ("conjunctive exact entailment"),
- $\Gamma \vDash^d \phi$ iff for all $\mathcal{M}$, $\bigcap|\Gamma|^+_\mathcal{M} \subseteq |\phi|^+_\mathcal{M}$ ("distributive exact entailment").

As Fine and Jago [17] point out, $\vDash^c$ and $\vDash^d$ are indeed *different* consequence relations: while we (trivially) have $p, q \vDash^d p$, we have $p, q \nvDash^c p$. An instructive countermodel can be found at [17, pp. 536–537]:



What's depicted here is the Hasse diagram of the underlying frame, where the subscripts indicate the exact truthmaking relation in the natural way. Since in this model, $|p \wedge q|^+_\mathcal{M} = \{s_3\}$ and $|p|^+_\mathcal{M} = \{s_1\}$, we have a countermodel which shows that $p, q \nvDash^c p$. Note that the model shows that the familiar rule of conjunction elimination fails with respect to exact entailment—both in its conjunctive and in its disjunctive flavor. More generally, it's easily seen that by definition, distributive exact entailment is a monotonic

the truth of statements and can enter in mereological relations. This includes states of affairs, events, actions, norms, etc.

8 Indeed, it appears in [4, p. 116], [11, pp. 215–216] and [17, p. 540].

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consequence relation: if $\Gamma \vDash^d \phi$, then $\Gamma, \psi \vDash^d \phi$. Conjunctive exact entailment, in contrast, is not monotonic: while we trivially have $p \vDash^c p$, we don't have $p, q \vDash^c p$.

Note, however, that conjunctive exact entailment reduces to a special case of distributive exact entailment:

PROPOSITION 3.1 (Semantic reduction). $\Gamma \vDash^c \phi$ *iff* $\bigwedge \Gamma \vDash^d \phi$.

Proof. First note that, by definition, we immediately have $\Gamma \vDash^c \phi$ iff $\bigwedge \Gamma \vDash^c \phi$. Then note that the definitions of distributive and conjunctive exact entailment coincide in the single premise case: $\phi \vDash^c \psi$ iff $\phi \vDash^d \psi$. □

Observe that this reduction is not in conflict with the different structural properties of distributive and conjunctive exact entailment. On this reduction, the failure of monotonicity for conjunctive exact entailment is preserved via the failure of conjunction elimination for distributive exact entailment.

The reduction allows us to focus our attention on distributive exact entailment for the purposes of this paper. Hence in the following, we shall simply speak of "exact entailment," intending distributive exact entailment, and use the symbol $\vDash$ to represent the relation.[9] We shall write $\phi \dashv\vDash \psi$ as an abbreviation for the conjunction of $\phi \vDash \psi$ and $\psi \vDash \phi$.

We'd like to remark two facts about exact entailment that heavily influence the design of our proof systems.

First, we note that $\vDash$ is *irreducibly asymmetric* in the sense that we can't reduce all premise sets to single formulas with the same consequences. To make this more precise, for $\phi(p_1, \dots, p_n)$ a formula in the propositional variables $p_1, \dots, p_n$, let $\phi(\psi_1, \dots, \psi_n)$ denote the result of uniformly substituting ψ_i for p_i in ϕ. Let $\Gamma = \{\psi_1, \dots, \psi_n\}$. Then a *premise reduction* for Γ would then be a formula ϕ_Γ such that $\Gamma \vDash \theta$ iff $\phi_\Gamma \vDash \theta$. But it's easily shown that there is no premise reduction for $\{p, q\}$ already. For suppose that there were such a $\phi_{p,q}$. Since $p, q \vDash p$ and $p, q, \vDash q$, we'd get $\phi_{p,q} \vDash p$ and $\phi_{p,q} \vDash q$. Since the canonical model is a model, this entails by the definition of $\vDash$ that $|\phi_{\{p,q\}}|^+_{\mathfrak{M}} \subseteq |p|^+_{\mathfrak{M}} = \{\{p\}\}$ and $|\phi_{\{p,q\}}|^+_{\mathfrak{M}} \subseteq |q|^+_{\mathfrak{M}} = \{\{q\}\}$. Since it's easily checked (by induction) that for all ψ, we have $|\psi|^+_{\mathfrak{M}} \neq \emptyset$, it follows that $|\phi_{\{p,q\}}|^+_{\mathfrak{M}} = \{\{p\}\}$ and that $|\phi_{\{p,q\}}|^+_{\mathfrak{M}} = \{\{q\}\}$, which is impossible since $p \neq q$.[10]

The second remark we'd like to make is that exact entailment has no theorems in the sense of formulas that are exactly made true by all states in all models (cf. [17, p. 539]). More precisely, if we define $\vDash \phi$ to mean that $|\phi|^+_{\mathcal{M}} = \mathcal{S}$ for all $\mathcal{M}$, we get:

PROPOSITION 3.2 (No theorems). $\nvDash \phi$, *for all* $\phi \in \mathcal{L}$.

Proof. We may assume without loss of generality that $\mathcal{P}$ is infinite. To see this, note that if $\mathcal{L}$ is not already defined over an infinite set of propositional variables, extending the variables of $\mathcal{L}$ is not going to change the theorems of $\mathcal{L}$. Next, note that by a straightforward inductive argument (left to the reader), if $\{p\} \in |\phi|^+_{\mathfrak{M}}$, then p is a subformula of ϕ. Since $\mathcal{P}$ is infinite, for every formula ϕ we can find a propositional variable p that doesn't occur in ϕ. We get for this p that $\{p\} \notin |\phi|^+_{\mathfrak{M}}$ by contrapositive reasoning. □

9 For a discussion of concrete philosophical reasons to be interested in distributive exact entailment, see [17, p. 537].

10 Note that in the case of *conjunctive* exact entailment, the desired formula $\phi(p_1, \dots, p_n)$ *does* exists and is, in fact, simply $p_1 \wedge \dots \wedge p_n$.

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This means that the logic of exact entailment is "purely inferential."

Let's briefly talk about alternative frameworks for exact truthmaker semantics. Note that by a simple inductive argument, we can establish that Closure (p. 4) extends to all formulas, i.e., exact truthmakers and exact falsemakers are closed under (finitary) fusions:[11]

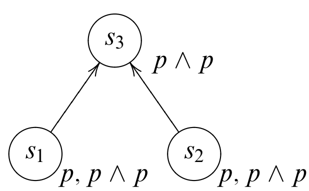
$$\text{If } s, t \in |\phi|^{\circ}_{\mathcal{M}}, \text{ then } s \sqcup t \in |\phi|^{\circ}_{\mathcal{M}}, \text{ for } \circ = +, -. \qquad \text{(Full Closure)}$$

This is why Fine [11, p. 206] calls the present version of exact truthmaker semantics the "inclusive" semantics. There are two prominent alternatives discussed in the literature: the *non-inclusive* semantics and the *replete* semantics.

On the non-inclusive semantics, essentially due to Fraassen [25], we drop Closure and change (Sem-$\vee^+$) and (Sem-$\wedge^-$) to

$$|\phi \vee \psi|^+_{\mathcal{M}} = |\phi|^+_{\mathcal{M}} \cup |\psi|^+_{\mathcal{M}} \quad (\text{Sem-}\vee^+_{\text{nc}}) \quad |\phi \wedge \psi|^-_{\mathcal{M}} = |\phi|^-_{\mathcal{M}} \cup |\psi|^-_{\mathcal{M}}. \quad (\text{Sem-}\wedge^-_{\text{nc}})$$

The most important difference between the inclusive and the non-inclusive semantics is that on the latter, the idempotence of conjunction fails. This can be seen by adjusting our countermodel for $p, q \nvDash^c p$ from before (see p. 1263):



We have $|p|^+_{\mathcal{M}} = \{s_1, s_2\}$ but $|p \wedge p|^+_{\mathcal{M}} = \{s_1, s_2, s_3\}$, and so, since $|p \wedge p|^+_{\mathcal{M}} \not\subseteq |p|^+_{\mathcal{M}}$, we have that $p \wedge p$ does not exactly entail p on the non-inclusive semantics. This works, of course, because on the non-inclusive semantics, the interpretation of p no longer needs to closed under fusions, and with $s_3 = s_1 \sqcup s_2$ not being an exact truthmaker of p in $\mathcal{M}$ while s_1, s_2 are, this is exactly what we're exploiting. Note, however, that the converse direction of idempotence—that ϕ exactly entails $\phi \wedge \phi$—is still valid on the non-inclusive semantics.

Fine and Jago [17] don't discuss the non-inclusive semantics and, for ease of exposition, we shall follow suit. The main complication is that the study of exact entailment under the non-inclusive semantics requires us to change the notion of our canonical model. To see this first note that the canonical model defined above has the (desirable) property that whenever $\phi \nvDash \psi$, then the canonical model $\mathfrak{M}$ provides a countermodel, i.e., $|\phi|^+_{\mathfrak{M}} \not\subseteq |\psi|^+_{\mathfrak{M}}$ (we'll prove this rigorously in Corollary 3.5). Now, while the canonical model is a model also in the sense of the non-inclusive semantics, it no longer has this important property. We can illustrate the issue with our pathological case of idempotence. When we consider $\mathfrak{M}$ as a model on the non-inclusive semantics, we have that $|p|^+_{\mathfrak{M}} = |p \wedge p|^+_{\mathfrak{M}} = \{\{p\}\}$ (note that the clause for conjunction are the same on the inclusive and non-inclusive semantics). In other words, $\mathfrak{M}$ doesn't provide a countermodel to the failure of the exact entailment form $p \wedge p$ to p.

[11] For a proof see [17, Lemma 3.3, p. 541].

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The point is that in order to obtain a counterexample to the inference from $p \wedge p$ to p, we need at least two separate states s, t which individually exactly truthmake p but who's fusion fails to be an exact truthmaker for p—just like in our modified countermodel above. Now it *is* possible to define a canonical model $\mathfrak{M}^\dagger$, which has the desired property for the non-inclusive semantics. To achieve this, we set $\mathfrak{S}_\dagger = \wp(\Lambda \times \mathbb{N})$, $\sqsubseteq_{\mathfrak{M}^\dagger} = \subseteq\restriction_{\mathfrak{S}_\dagger}$, and $v^+_{\mathfrak{M}^\dagger}(p) = \{\{(p, i)\} : i \in \mathbb{N}\}$ as well as $v^-_{\mathfrak{M}^\dagger}(p) = \{\{(\neg p, i)\} : i \in \mathbb{N}\}$. It's easily verified that in this model, $|p \wedge p|^+_{\mathfrak{M}^\dagger} = \{\{(p, i), (p, j)\} : i, j \in \mathbb{N}\}$, where each state $\{(p, i), (p, j)\}$ for $i \neq j$ provides an example of an exact truthmaker for $p \wedge p$ that fails to exactly truthmake p.[12]

It is, in fact, possible to show that the canonical $\mathfrak{M}^\dagger$ can play the same role for exact entailment on the non-inclusive semantics as $\mathfrak{M}$ plays for the inclusive (and replete) semantics (see below). However, the semantic theory of non-inclusive exact truthmaking is comparatively underdeveloped in the literature. In particular, the semantic characterization results due to Fine and Jago [17], which play an central role in our completeness results, are not extended to the non-inclusive system. While it's likely that we can generalize these results to the non-inclusive setting, going through the details and re-proving the relevant semantic theory is unfortunately beyond the scope of this paper. We shall therefore restrict our attention to the inclusive and replete semantics (which *are* covered by Fine and Jago).[13]

On the "replete semantics," discovered by Fine [11], instead, we make two additional philosophical assumptions:

- Every statement has at least one exact truthmaker and at least one exact falsemaker ("non-vacuity").
- If a state lies between two exact truthmakers/falsemakers (in the sense of parthood), it is itself an exact truthmaker/falsemaker ("convexity").[14]

There are different ways in which we can formally implement the previous assumptions (cf. [17, pp. 547–551]). We shall follow the method described by Fine and Jago [17, pp 550–551]. We say that a model $\mathcal{M} = (S, \sqsubseteq, v)$ is *non-vacuous* iff both $v^+(p) \neq \emptyset$ and $v^-(p) \neq \emptyset$. A straightforward induction on complexity establishes that in non-vacuous models, the non-emptiness property extends to all exact truthmaker sets and exact falsemaker sets:

$$\text{If } \mathcal{M} \text{ is a non-vacuous model, then } |\phi|^\circ_\mathcal{M} \neq \emptyset, \text{ for } \circ = +, -. \qquad \text{(Non-Vacuity)}$$

Note that the canonical model, in particular, is non-vacuous. In a frame $\mathcal{F} = (S, \sqsubseteq)$, we define the *convex closure* X_* of a set of states $X \subseteq S$ as $\{s \in S : \exists t, u \in X(t \sqsubseteq s \text{ and } s \sqsubseteq u)\}$. For an $\mathcal{X} \subseteq S$, we set $(\mathcal{X})_* = \{(X)_* : X \in \mathcal{X}\}$.

With these notions at hand, we can now define exact entailment on the replete semantics as follows (cf. [17, definition 7.1, p. 550]):

- $\Gamma \vDash^c_{nv*} \phi$ iff for all non-vacuous models $\mathcal{M}$, $(|\bigwedge \Gamma|^+_\mathcal{M})_* \subseteq (|\phi|^+_\mathcal{M})_*$.
- $\Gamma \vDash^d_{nv*} \phi$ iff for all non-vacuous models $\mathcal{M}$, $\bigcap(|\Gamma|^+_\mathcal{M})_* \subseteq (|\phi|^+_\mathcal{M})_*$.

12 Note that we also get counterexamples to the inferences from $p \wedge p \wedge p$ to $p \wedge p$, from $p \wedge p \wedge p \wedge p$ to $p \wedge p \wedge p$, and so on and so forth.

13 We shall, however, sketch the necessary steps throughout a series of footnotes in our paper.

14 Fine and Jago [17] prove that postulating either of these two assumptions by themselves (over the inclusive semantics) doesn't change the resulting logic for exact entailment (theorem 6.3, p. 547, and theorem 6.12, p. 550).

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Note that Propositions 3.1 and 3.2 immediately carry over to $\vDash_{nv*}$ (see the so-called "Convex Selection Lemma" [17, Lemma 6.4]), and just like in the case of $\vDash$, there is no premise reduction for $\vDash_{nv*}$ either. As a result, we can focus on the distributive variant, which we'll denote by $\vDash_{nv*}$ using $\dashv\vDash_{nv*}$ for equivalence.

PROPOSITION 3.3 (Inclusion of $\vDash$ in $\vDash_{nv*}$). *If* $\Gamma \vDash \phi$, *then* $\Gamma \vDash_{nv*} \phi$.

Proof. Since $\cdot_*$ is a closure operator in the technical sense, it enjoys the Monotonicity Property: if $X \subseteq Y$, then $X_* \subseteq Y_*$. The claim follows immediately from Monotonicity and some basic set-theory. □

As a consequence, $\vDash_{nv*}$ inherits the logical laws of $\vDash$. The main difference between $\vDash$ and $\vDash_{nv*}$ concerns distributivity, which we'll discuss in the following section.

Before we move to formulating our proof systems, we would like to point out the characterization theorems provided by Fine and Jago [17] for both $\vDash$ and $\vDash_{nv*}$, since we'll rely heavily on them for our completeness proofs.

The theorems are stated in terms of *selections*. A *canonical selection* for a set Γ is a function $f : \Gamma \to \wp(\Lambda)$ such that $f(\phi) \in |\phi|^+_{\mathfrak{M}}$ for all $\phi \in \Gamma$. Intuitively, these selections give us a syntactic representation of the different exact truthmakers for the members of Γ. For each $\phi \in \Gamma$, the value $f(\phi)$ under a given selection function is a member of $|\phi|^+_{\mathfrak{M}}$, so a set of literals $\{\lambda_1, \dots, \lambda_n\} \subseteq \Lambda$. Now a core lemma of [17], the so-called "Selection Lemma," states that for any model $\mathcal{M}$ and state s therein,

$$s \in |\phi|^+_{\mathcal{M}} \text{ iff for some selection } f \text{ for } \{\phi\}, s \in |\bigwedge f(\phi)|^+_{\mathcal{M}}.$$

In words: for a state to be an exact truthmaker for ϕ in some model is for the state to be an exact truthmaker for the conjunction of some exact truthmaker of ϕ in the canonical model (which, crucially, is just a set of literals). It is in terms of these syntactic representations that Fine and Jago characterize exact entailment:[15]

THEOREM 3.4 [17, Theorem 4.12, p. 546]. *The following are equivalent*:

1. $\Gamma \vDash \phi$.
2. *For all canonical selections* f *for* Γ, *there exists a* $\Delta \in |\phi|^+_{\mathfrak{M}}$, *such that for some* $\psi \in \Gamma$,

$$f(\psi) \subseteq \Delta \subseteq \bigcup_{\xi \in \Gamma} f(\xi).$$

COROLLARY 3.5. $\phi \vDash \psi$ *iff* $|\phi|^+_{\mathfrak{M}} \subseteq |\psi|^+_{\mathfrak{M}}$.

Proof. Note that any canonical selection for the premise set $\{\phi\}$ simply picks a member of $|\phi|^+_{\mathfrak{M}}$. So the condition of the theorem applied to the situation at hand boils down to saying that for each member $\Gamma \in |\phi|^+_{\mathfrak{M}}$ there exists a member of $\Delta \in |\psi|^+_{\mathfrak{M}}$ such that $\Gamma \subseteq \Delta \subseteq \Gamma$—which immediately gives us $|\phi|^+_{\mathfrak{M}} \subseteq |\psi|^+_{\mathfrak{M}}$. □

The notion of a selection function can straightforwardly be generalized to the replete semantics, while preserving the underlying motivation sketched above. A *convex*

15 If we're working on the non-inclusive semantics, our aim would be to prove a comparable theorem to the following with respect to $\mathfrak{M}^\dagger$ (see above). However, this would require re-proving a significant chunk of semantic theory, specifically [17, lemmas 4.4–10] as well as the Selection Lemma. Most of the work is relatively straightforward, though occasionally new arguments are required.

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selection for Γ is a function $f_* : \Gamma \to \wp(\Lambda)$, such that $f(\phi) \in (|\phi|^*_{\mathfrak{M}})_*$ for all $\phi \in \Gamma$. Using this concept, we can prove:

THEOREM 3.6 [17, theorem 7.4, p. 551]. *The following are equivalent*:

1. $\Gamma \vDash_{nv*} \phi$.
2. *For all convex selections f_* for Γ,*[16] *there exist a $\Delta \in (|\phi|^+_{\mathfrak{M}})_*$ and a $\psi \in \Gamma$, such that*

$$f_*(\psi) \subseteq \Delta \subseteq \bigcup_{\xi \in \Gamma} f_*(\xi).$$

COROLLARY 3.7. $\phi \vDash_{nv*} \psi$ *iff* $(|\phi|^+_{\mathfrak{M}})_* \subseteq (|\psi|^+_{\mathfrak{M}})_*$.

Proof. Analogous to the proof of Corollary 3.5. □

§4. Axiomatizing exact entailment. In this section, we directly axiomatize exact entailment as an asymmetric consequence relation. That is, we view the relation as a set of *consequence pairs* of the form (Γ, ϕ) and we axiomatize membership in this set. Correspondingly, our system, A, operates on consequence pairs, allowing us to derive such pairs via inference rules from distinguished axiom pairs. We write $\Gamma \vdash_{\mathsf{A}} \phi$ to say that the pair (Γ, ϕ) is derivable in our system and we write $\phi \dashv\vdash_{\mathsf{A}} \psi$ as an abbreviation for the conjunction of $\phi \vdash_{\mathsf{A}} \psi$ and $\psi \vdash_{\mathsf{A}} \phi$.

The axioms and rules of our system are as follows:

$$\phi \vdash_{\mathsf{A}} \phi, \qquad (\text{Reflexivity}_{\mathsf{A}})$$

$$\frac{\Gamma \vdash_{\mathsf{A}} \phi}{\Gamma, \Delta \vdash_{\mathsf{A}} \phi}, \qquad (\text{Weakening}_{\mathsf{A}})$$

$$\frac{\Gamma \vdash_{\mathsf{A}} \phi \quad \Sigma, \phi \vdash_{\mathsf{A}} \psi}{\Gamma, \Delta \vdash_{\mathsf{A}} \psi}, \qquad (\text{Cut}_{\mathsf{A}})$$

$$\phi, \phi \wedge \psi \wedge \theta \vdash_{\mathsf{A}} \phi \wedge \psi, \qquad (\wedge - \text{Convexity}_{\mathsf{A}})$$

$$\frac{\Gamma, \phi_1 \vdash_{\mathsf{A}} \psi_1 \quad \Gamma, \phi_2 \vdash_{\mathsf{A}} \psi_2}{\Gamma, \phi_1 \wedge \phi_2 \vdash_{\mathsf{A}} \psi_1 \wedge \psi_2}, \qquad (\wedge - \text{Monotonicity}_{\mathsf{A}})$$

$$\phi \wedge \phi \dashv\vdash_{\mathsf{A}} \phi, \qquad (\wedge - \text{Idempotence}_{\mathsf{A}})$$

$$\phi \wedge \psi \dashv\vdash_{\mathsf{A}} \psi \wedge \phi, \qquad (\wedge - \text{Commutativity}_{\mathsf{A}})$$

$$\phi \wedge (\psi \wedge \theta) \dashv\vdash_{\mathsf{A}} (\phi \wedge \psi) \wedge \theta, \qquad (\wedge - \text{Associativity}_{\mathsf{A}})$$

$$\phi \vdash_{\mathsf{A}} \phi \vee \psi \qquad \psi \vdash_{\mathsf{A}} \phi \vee \psi, \qquad (\vee - \text{Introduction}_{\mathsf{A}})$$

$$\frac{\Gamma, \phi_1 \vdash_{\mathsf{A}} \theta \quad \Gamma, \phi_2 \vdash_{\mathsf{A}} \theta}{\Gamma, \phi_1 \vee \phi_2 \vdash_{\mathsf{A}} \theta}, \qquad (\vee - \text{Elimination}_{\mathsf{A}})$$

[16] There appears to be a typo in [17, p. 551], where in the statement of the theorem, the closure operator for the premise set Γ is missing. The proof provided by Fine and Jago [17], however, covers the result stated here.

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$$\phi \wedge (\psi \vee \theta) \dashv\vdash_{\mathsf{A}} (\phi \wedge \psi) \vee (\phi \wedge \theta), \qquad (\wedge/\vee - \text{Distribution}_{\mathsf{A}})$$

$$\phi \dashv\vdash_{\mathsf{A}} \neg\neg\phi, \qquad (\text{Double Negation}_{\mathsf{A}})$$

$$\neg(\phi \vee \psi) \dashv\vdash_{\mathsf{A}} \neg\phi \wedge \neg\psi \quad \neg(\phi \wedge \psi) \dashv\vdash_{\mathsf{A}} \neg\phi \vee \neg\psi. \qquad (\text{De Morgan Laws}_{\mathsf{A}})$$

First, a word on notation and implicit reasoning. As we noted in §2, we write $\bigwedge \Gamma$ or $\bigwedge_{\phi\in\Gamma} \phi$ for the conjunction of the Γ's, and $\bigvee \Gamma$ or $\bigvee_{\phi\in\Gamma} \phi$ for the disjunction of the Γs. In the context of this notation, we often rely on implicit applications of $\wedge$-Idempotence, $\wedge$-Commutativity, and $\wedge$-Associativity, as well as corresponding reasoning using $\vee$-Introduction and $\vee$-Elimination in treating certain expressions are "notationally equivalent." For example, we shall treat $(\bigwedge \Gamma) \wedge (\bigwedge \Delta)$ and $\bigwedge(\Gamma \cup \Delta)$ as notational variants of each other, though, strictly speaking, we're just relying on implicit reasoning in which we're eliminating duplicates (using $\wedge$-Idempotence) and re-arranging conjuncts (using $\wedge$-Commutativity and $\wedge$-Associativity). We also refer to this reasoning as "notational reasoning."[17]

Working towards a soundness result, we say that a pair (Γ, ϕ) is *valid* iff $\Gamma \vDash \phi$. The validity of the axioms follows directly from the corresponding laws for exact entailment observed by Fine and Jago [17, p. 539] and since the proofs are almost immediate by definition, we omit them here. We only cover $\wedge$-Convexity, since this law has not been observed by Fine and Jago [17].

It turns out that $\wedge$-Convexity is of central importance to the logic of exact entailment and closely related to the Fine–Jago characterization theorems. Note that $\wedge$-Convexity is the only genuine multi-premise axiom of our system, and other multi-premise laws, like $\phi, \psi \vdash_{\mathsf{A}} \phi \wedge \psi$, are derived (cf. Proposition 4.7). We shall find it convenient to prove its validity in the following more general form:

PROPOSITION 4.1 ($\wedge$-Convexity). *For $\Gamma \subseteq \Delta \subseteq \Sigma$, we have that $\bigwedge \Gamma, \bigwedge \Sigma \vDash \bigwedge \Delta$.*

Proof. It's straightforward to show this using the Fine–Jago characterization (Theorem 3.4), but it's instructive to prove it directly instead. So, suppose that $s \in |\bigwedge \Gamma|^+_{\mathcal{M}}$ and $s \in |\bigwedge \Sigma|^+_{\mathcal{M}}$ and $\Gamma \subseteq \Delta \subseteq \Sigma$. By application of (Sem-$\wedge^+$), we get that for each $\phi \in \Sigma$, there exists a state $s_\phi \in S$ such that $s_\phi \in |\phi|^+_{\mathcal{M}}$ and $s = \bigsqcup_{\phi\in\Sigma} s_\phi$. We can infer that for each $\phi \in \Sigma$, $s_\phi \sqsubseteq s$. Since $\Delta \subseteq \Sigma$, we get that for each $\phi \in \Delta$, there exists an $s_\phi \in |\phi|^+_{\mathcal{M}}$ with $s_\phi \sqsubseteq s$. By (Sem-$\wedge^+$), $\bigsqcup_{\phi\in\Delta} s_\phi \in |\bigwedge \Delta|^+_{\mathcal{M}}$. Since $s \in |\bigwedge \Gamma|^+_{\mathcal{M}}$, again by (Sem-$\wedge^+$), we get that $s \sqcup \bigsqcup_{\phi\in\Delta} s_\phi \in |\bigwedge \Gamma \wedge \bigwedge \Delta|^+_{\mathcal{M}}$. But since for each $\phi \in \Delta$, $s_\phi \sqsubseteq s$, $s \sqcup \bigsqcup_{\phi\in\Delta} s_\phi = s$. It's easily checked that $|\bigwedge \Gamma \wedge \bigwedge \Delta|^+_{\mathcal{M}} = |\bigwedge(\Gamma \cup \Delta)|^+_{\mathcal{M}}$. But since $\Gamma \subseteq \Delta$, we have that $\Gamma \cup \Delta = \Delta$, and so $s \in |\bigwedge \Delta|^+_{\mathcal{M}}$. □

The corresponding generalized axiom for our system, $\bigwedge \Gamma, \bigwedge \Sigma \vdash_{\mathsf{A}} \bigwedge \Delta$ where $\Gamma \subseteq \Delta \subseteq \Sigma$, is easily derived from the official axiom of $\wedge$-Convexity and repeated applications of $\wedge$-Monotonicity (see Proposition 4.7).

In a rule, we call the expressions on top of the inference line the "premises" of the rule and the expression below the line its "conclusion." We say that a rule is *sound* iff its conclusion is valid whenever its premises are. The structural rules of Weakening

[17] Note that if we're trying to develop a system for the non-inclusive semantics, we would use *multi*sets instead of sets in our sequents and drop the axiom $\phi \wedge \phi \vdash_A \phi$ from ($\wedge$-Idempotence$_{\mathsf{A}}$). Consequently, much of the following arguments would need to be adjusted, especially when they involve notational reasoning.

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and Cut are straightforwardly seen to be valid from the definition of $\vDash$. Fine and Jago [17, p. 554, proof of theorem 9.2] effectively prove the soundness of $\wedge$-Monotonicity and $\vee$-Elimination by proving the soundness of corresponding rules for their sequent calculus. Since the corresponding proofs for our system are essentially notational variants of the proofs for the Fine–Jago system, we shall only provide the argument for $\wedge$-Monotonicity and leave the case of $\vee$-Elimination for the reader to work out.

LEMMA 4.2. *$\wedge$-Monotonicity$_{\mathsf{A}}$ is sound on the inclusive semantics.*

Proof. We proceed using the Fine–Jago characterization theorem (Theorem 3.4). So assume that $\Gamma, \phi_1 \vDash \psi_1$ and $\Gamma, \phi_2 \vDash \psi_2$ We wish to show that $\Gamma, \phi_1 \wedge \phi_2 \vDash \psi_1 \wedge \psi_2$, which by Theorem 3.4 means that for each selection f for $\Gamma \cup \{\phi_1 \wedge \phi_2\}$, we find a $\Delta \in |\psi_1 \wedge \psi_2|^+_{\mathfrak{M}}$ and a $\theta \in \Gamma \cup \{\phi_1 \wedge \phi_2\}$ with $f(\theta) \subseteq \Delta \subseteq \bigcup_{\xi \in \Gamma} f(\xi)$.

So consider an arbitrary selection function f for $\Gamma \cup \{\phi_1 \wedge \phi_2\}$. Since $|\phi_1 \wedge \phi_2|^+_{\mathfrak{M}} = \{\Sigma_1 \cup \Sigma_2 : \Sigma_i \in |\phi_i|^+_{\mathfrak{M}}\}$, we can write $f(\phi_1 \wedge \phi_2) = \Sigma_1 \cup \Sigma_2$ for some $\Sigma_i \in |\phi_i|^+_{\mathfrak{M}}$. Based on this observation, we define selections f_i for $\Gamma \cup \{\phi_i\}$, $i = 1, 2$, by simply setting $(f_i)_{\restriction \Gamma \setminus \{\phi_i\}} = f$ and $f_i(\phi_i) = \Sigma_i$. Our assumption gives us Δ_i's in $|\psi_i|^+_{\mathfrak{M}}$ and θ_i's in $\Gamma \cup \{\phi_i\}$ with: $f_i(\theta_i) \subseteq \Delta_i \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_i\}} f_i(\xi)$ via Theorem 3.4. We shall simply set our desired Δ to be $\Delta_1 \cup \Delta_2$. We then get that $\Delta \in |\psi_1 \wedge \psi_2|^+_{\mathfrak{M}}$, since $\Delta_1 \in |\psi_1|^+_{\mathfrak{M}}$ and $\Delta_2 \in |\psi_2|^+_{\mathfrak{M}}$ and so $\Delta_1 \cup \Delta_2 \in |\psi_1 \wedge \psi_2|^+_{\mathfrak{M}}$ by (Sem-$\wedge^+$).

We distinguish two cases: either (a) $\theta_1 = \phi_1, \theta_2 = \phi_2$ or (b) some $\theta_i \in \Gamma$. Either way, we claim that there's a $\theta \in \Gamma \cup \{\phi_1 \wedge \phi_2\}$ with $f(\theta) \subseteq \Delta_1 \cup \Delta_2$. In case (a), we note again that $f_1(\phi_1) \cup f_2(\phi_2) = \Sigma_1 \cup \Sigma_2 = f(\phi_1 \wedge \phi_2)$ and so $f(\phi_1 \wedge \phi_2) \subseteq \Delta_1 \cup \Delta_2$. So we can set $\theta = \phi_1 \wedge \phi_2$. In case (b), we can simply set $\theta = \theta_i$ for the $\theta_i \in \Gamma$ since for both θ_i's we have by simple set-theory that $f(\theta_i) \subseteq \Delta_1 \cup \Delta_2$. Next, note that by set theory we have $\Delta_1 \cup \Delta_2 \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_2\}} f_1(\xi) \cup \bigcup_{\xi \in \Gamma \cup \{\phi_2\}} f_2(\xi)$. Since $(f_i)_{\restriction \Gamma \setminus \{\phi_i\}} = f$, we get $\Delta_1 \cup \Delta_2 \subseteq \bigcup_{\xi \in \Gamma} f(\xi) \cup f_1(\phi_1) \cup f_2(\phi_2)$. But we have $f_1(\phi_1) \cup f_2(\phi_2) = \Sigma_1 \cup \Sigma_2 = f(\psi_1 \wedge \psi_2)$, so $\Delta_1 \cup \Delta_2 \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1 \wedge \phi_2\}} f(\xi)$, as desired. $\square$

Next, we extend our system to a system for exact entailment on the replete semantics. By Proposition 3.3, all axioms that are valid with respect to $\vDash$ are also valid with respect to $\vDash_{nv*}$. The main difference concerns the distributivity of $\vee$ over $\wedge$. As the reader will easily confirm (perhaps using the derived rules given in Proposition 4.7), we can derive

$$\phi \vee (\psi \wedge \theta) \vdash_{\mathsf{A}} (\phi \vee \psi) \wedge (\phi \vee \theta).$$

The inverse direction, however, is not derivable—in fact, it is invalid on the inclusive semantics. Just observe that

$$|p \vee (q \wedge r)|^+_{\mathfrak{M}} = \{\{p\}, \{q, r\}, \{p, q, r\}\},$$

$$|(p \vee q) \wedge (p \vee r)|^+_{\mathfrak{M}} = \{\{p\}, \{p, r\}, \{p, q\}, \{q, r\}, \{p, q, r\}\}.$$

Since $|(p \vee q) \wedge (p \vee r)|^+_{\mathfrak{M}} \nsubseteq |p \vee (q \wedge r)|^+_{\mathfrak{M}}$, we get that $(p \vee q) \wedge (p \vee r) \nvDash p \vee (q \wedge r)$.

On the replete semantics, instead, the inference becomes valid. Just observe that

$$(|p \vee (q \wedge r)|^+_{\mathfrak{M}})_* = \{\{p\}, \{p, r\}, \{p, q\}, \{q, r\}, \{p, q, r\}\},$$

$$(|(p \vee q) \wedge (p \vee r)|^+_{\mathfrak{M}})_* = \{\{p\}, \{p, r\}, \{p, q\}, \{q, r\}, \{p, q, r\}\}.$$

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Using Corollary 3.7 from Theorem 3.6, we can infer that $(p \vee q) \wedge (p \vee r) \dashv\vDash_{nv*} p \vee (q \wedge r)$. Indeed, the law holds in its general form, i.e., $(\phi \vee \psi) \wedge (\phi \vee \theta) \vDash_{nv*} \phi \vee (\psi \wedge \theta)$, as is easily (but perhaps tediously) seen via Theorem 3.4 (left to the reader).

We denote derivability in our system for $\vDash_{nv*}$ correspondingly by $\vdash_{\mathsf{A}^{nv*}}$. The rules and axioms for this system are the same as for $\vdash_{\mathsf{A}}$ except that we add the missing distributivity law as an axiom:

$$(\phi \vee \psi) \wedge (\phi \vee \theta) \vdash_{\mathsf{A}^{nv*}} \phi \vee (\psi \wedge \theta).$$

We note that the arguments for the soundness of the "structural" rules, Weakening and Cut, are straightforward. The arguments for the soundness of $\wedge$-Monotonicity$_{\mathsf{A}}$ and $\vee$-Elimination$_{\mathsf{A}}$ use Theorem 3.6 in a similar way as the proof of Lemma 4.2 uses Theorem 3.4. We provide the proof for the soundness of $\vee$-Elimination$_{\mathsf{A}}$ to illustrate the idea and leave the (easier) case of $\wedge$-Monotonicity$_{\mathsf{A}}$ to the interested reader:

LEMMA 4.3. *The rule $\vee$-Elimination$_{\mathsf{A}}$ is sound on the replete semantics.*

Proof. Assume that $\Gamma, \phi_1 \vDash_{nv*} \psi$ and $\Gamma, \phi_2 \vDash_{nv*} \psi$. We wish to derive that $\Gamma, \phi_1 \vee \phi_2 \vDash_{nv*} \psi$. First, note that by $\wedge$-Monotonicity and $\wedge$-Idempotence, we can infer that $\Gamma, \phi_1 \wedge \phi_2 \vDash_{nv*} \psi$. We proceed using the Fine–Jago characterization theorem for the replete semantics (Theorem 3.6).

We want to show that for each convex selection f_* for $\Gamma \cup \{\phi_1 \vee \phi_2\}$, we find a $\Delta \in (|\theta|^+_{\mathfrak{M}})_*$ such that the conditions of Theorem 3.6.(a and b) are satisfied, i.e., there's a $\theta \in \Gamma \cup \{\phi_1 \vee \phi_2\}$ with $f_*(\theta) \subseteq \Delta \subseteq \bigcup_{\theta \in (\Gamma \cup \{\phi_1 \vee \phi_2\})} f_*(\theta)$. So take an arbitrary convex selection f_* for $\Gamma \cup \{\phi_1 \vee \phi_2\}$. Consider $f_*(\phi_1 \vee \phi_2) = \Sigma$. Note that $\Sigma \in (|\phi_1 \vee \phi_2|^+_{\mathfrak{M}})_*$ iff there exists $\Sigma_\uparrow \in |\phi_1|^+_{\mathfrak{M}} \cup |\phi_2|^+_{\mathfrak{M}}$ and a $\Sigma_\downarrow \in |\phi_1 \wedge \phi_2|^+_{\mathfrak{M}}$ such that $\Sigma_\uparrow \subseteq \Sigma \subseteq \Sigma_\downarrow$. Without loss of generality, we can assume that for our Σ, we have $\Sigma_\uparrow \in |\phi_1|^+_{\mathfrak{M}}$ since the case for $\Sigma_\uparrow \in |\phi_2|^+_{\mathfrak{M}}$ is analogous.

We define a convex selection $f^\uparrow_*$ for $\Gamma \cup \{\phi_1\}$ by setting $(f^\uparrow_*)\restriction_{\Gamma \setminus \{\phi_1\}} = f_*$ and $f^\uparrow_*(\phi_1) = \Sigma^\uparrow$. By our assumption that $\Gamma, \phi_1 \vDash_{nv*} \psi$ and Theorem 3.6, we get that there exists a $\theta^\uparrow \in \Gamma \cup \{\phi_1\}$ and a $\Delta^\uparrow \in |\psi|^+_{\mathfrak{M}}$ with $f^\uparrow_*(\gamma) \subseteq \Delta^\uparrow \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1\}} f^\uparrow_*(\xi)$. We distinguish two cases: (a) $\theta^\uparrow \in \Gamma \setminus \{\phi_1\}$ and (b) $\theta^\uparrow = \phi_1$. In case (a), we can simply let $\Delta^\uparrow$ be our Δ and $\theta^\uparrow$ our θ since by definition of $f^\uparrow_*$ we have

$$f^\uparrow_*(\theta^\uparrow) = f_*(\theta^\uparrow) \subseteq \Delta^\uparrow \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1\}} f^\downarrow_*(\xi) \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1 \vee \phi_2\}} f_*(\xi).$$

In case (b), we have $f^\uparrow_*(\phi_1) \subseteq \Delta^\uparrow \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1\}} f^\uparrow_*(\xi)$. We define another convex selection $f^\downarrow_*$, this time for $\Gamma \cup \{\phi_1 \wedge \phi_2\}$ by setting $(f^\downarrow_*)\restriction_{\Gamma \setminus \{\phi_1 \wedge \phi_2\}} = f_*$ and $f^\downarrow_*(\phi_1 \wedge \phi_2) = \Sigma^\downarrow$. Since we've already seen that $\Gamma, \phi_1 \wedge \phi_2 \vDash \theta$, we can apply Theorem 3.6 to infer that there exists a $\theta^\downarrow \in \Gamma \cup \{\phi_1 \wedge \phi_2\}$ and a $\Delta^\downarrow \in (|\psi|^+_{\mathfrak{M}})$ where $f^\downarrow_*(\theta^\downarrow) \subseteq \Delta^\downarrow \subseteq \bigcup_{\xi \in \Gamma \cup \{\phi_1 \wedge \phi_2\}} f^\downarrow_*(\xi)$. We again distinguish two cases: (a.1) $\theta^\downarrow \notin \Gamma \setminus \{\phi_1 \wedge \phi_2\}$ and (a.2) $\theta^\downarrow = \phi_1 \wedge \phi_2$.

In case (a.1), we can set $\theta = \theta^\downarrow$ and $\Delta = \Delta^\uparrow \cup f_*(\theta^\downarrow)$. To establish this, we first note that since $\theta^\downarrow \in \Gamma$, we have $f^\downarrow_*(\theta^\downarrow) = f^\uparrow_*(\theta^\downarrow) = f_*(\theta^\downarrow)$ by the definition of $f^\downarrow_*$. We then infer using basic set-theory that $f^\downarrow_*(\theta^\downarrow) \subseteq \Delta^\uparrow \cup f^\downarrow_*(\theta^\downarrow)$. Furthermore,

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since $\Delta^{\uparrow} \subseteq \bigcup_{\xi\in\Gamma\cup\{\phi_1\}} f_*^{\uparrow}(\xi)$, we can infer that $\Delta^{\uparrow} \cup f_*^{\downarrow}(\theta^{\downarrow}) \subseteq \bigcup_{\xi\in\Gamma\cup\{\phi_1\}} f_*^{\uparrow}(\xi) \cup f_*^{\downarrow}(\theta^{\downarrow}) = \bigcup_{\xi\in\Gamma\cup\{\phi_1\}} f_*^{\uparrow}(\xi)$. We now have that

$$f_*(\theta^{\downarrow}) \subseteq \Delta^{\uparrow} \cup f_*(\theta^{\downarrow}) \subseteq \Delta^{\uparrow} = \Delta \subseteq \bigcup_{\xi\in\Gamma\cup\{\phi_1\}} f_*^{\downarrow}(\xi) \subseteq \bigcup_{\xi\in\Gamma\cup\{\phi_1\vee\phi_2\}} f_*(\xi).$$

What remains to be shown is that $\Delta^{\uparrow} \cup f_*(\theta^{\downarrow}) \in (|\psi|^+_{\mathfrak{M}})_*$. To see this note that since $\Delta^{\uparrow}, \Delta^{\downarrow} \in (|\psi|^+_{\mathfrak{M}})_*$, by Full Closure we have $\Delta^{\uparrow} \cup \Delta^{\downarrow} \in (|\psi|^+_{\mathfrak{M}})_*$. We already know that $f_*(\theta^{\downarrow}) = f_*^{\downarrow}(\theta^{\downarrow}) \subseteq \Delta^{\downarrow}$, so it follows that $\Delta^{\uparrow} \subseteq \Delta^{\uparrow} \cup f_*(\theta^{\downarrow}) \subseteq \Delta^{\uparrow} \cup \Delta^{\downarrow}$. By the convexity of $(|\psi|^+_{\mathfrak{M}})_*$, we can infer that $\Delta^{\uparrow} \cup f_*(\theta^{\downarrow}) \in (|\psi|^+_{\mathfrak{M}})_*$ as desired.

We've arrived at our final case, viz. $\theta^{\uparrow} = \phi_1$ and $\theta^{\downarrow} = \phi_1 \wedge \phi_2$. We summarize

$$\begin{aligned} f_*^{\downarrow}(\phi_1 \wedge \phi_2) \subseteq \Delta^{\downarrow} \subseteq &\bigcup_{\xi\in\Gamma\cup\{\phi_1\wedge\phi_2\}} f_*^{\downarrow}(\xi), \\ f_*^{\uparrow}(\phi_1) \subseteq \Delta^{\uparrow} \subseteq &\bigcup_{\xi\in\Gamma\cup\{\phi_1\}} f_*^{\uparrow}(\xi) = \bigcup_{\xi\in\Gamma} f_*(\xi) \cup f_*^{\uparrow}(\phi_1). \end{aligned}$$

Set $\Sigma^* = \Sigma \setminus \Sigma^{\uparrow}$. Since $f_*^{\uparrow}(\phi_1) = \Sigma^{\uparrow}$, we get that

$$\underbrace{\Sigma^{\uparrow} \cup \Sigma^*}_{=\Sigma = f_*(\phi_1\vee\phi_2)} \subseteq \Delta^{\uparrow} \cup \Sigma^* \subseteq \bigcup_{\xi\in\Gamma} f_*(\xi) \cup \underbrace{\Sigma^{\uparrow} \cup \Sigma^*}_{=f_*(\phi_1\vee\phi_2)} = \bigcup_{\xi\in\Gamma\cup\{\phi_1\vee\phi_2\}} f_*(\xi).$$

Since $f_*^{\downarrow}(\phi_1 \wedge \phi_2) = \Sigma_2$ and $\Sigma^* \subseteq \Sigma_2$, we get that $\Delta^{\uparrow} \subseteq \Delta^{\uparrow} \cup \Sigma^* \subseteq \Delta^{\uparrow} \cup \Delta^{\downarrow}$. And just like in case (a.1), $\theta^{\downarrow} \in \Gamma \setminus \{\phi_1 \wedge \phi_2\}$, we can infer that $\Delta^{\uparrow} \cup \Sigma^* \in (|\psi|^+_{\mathfrak{M}})_*$. So in our final case, we can set $\theta = \phi_1 \vee \phi_2$ and $\Delta = \Delta^{\uparrow} \cup \Sigma^*$, establishing our claim. □

Summarily, we get:

THEOREM 4.4 (Soundness for A and A_{nv*}). *We have*:

1. *If* $\Gamma \vdash_{\mathsf{A}} \phi$, *then* $\Gamma \vDash \phi$.
2. *If* $\Gamma \vdash_{\mathsf{A}^{nv*}} \phi$, *then* $\Gamma \vDash_{nv*} \phi$.

Before turning to completeness, we observe some facts about the logic of exact entailment via our proof system. As Odintsov and Wansing [23, p. 51] point out, thinking about Cresswell's definition of hyperintensional contexts as ones that do not respect logical equivalence, naturally leads us to the notion of *self-extensionality* of a logic. A logic is called fully self-extensional or *congruential* iff all the operators respect logical equivalence in that logic. In our case, that means that exact entailment is congruential (on the inclusive semantics) iff

$$\text{If } \phi \dashv\vdash_{\mathsf{A}} \psi, \text{ then } \neg\phi \dashv\vdash_{\mathsf{A}} \neg\psi. \qquad (\neg\text{-Congruency})$$

$$\text{If } \phi_1 \dashv\vdash_{\mathsf{A}} \psi_1 \text{ and } \phi_2 \dashv\vdash_{\mathsf{A}} \psi_2, \text{ then } \phi_1 \wedge \phi_2 \dashv\vdash_{\mathsf{A}} \psi_1 \wedge \psi_2. \qquad (\wedge\text{-Congruency})$$

$$\text{If } \phi_1 \dashv\vdash_{\mathsf{A}} \psi_1 \text{ and } \phi_2 \dashv\vdash_{\mathsf{A}} \psi_2, \text{ then } \phi_1 \vee \phi_2 \dashv\vdash_{\mathsf{A}} \psi_1 \vee \psi_2. \qquad (\vee\text{-Congruency})$$

We observe that exact entailment on the inclusive semantics is not fully congruential since it is not $\neg$-congruential. To see this, note first that by $\wedge/\vee$-Distributivity, we have that $\neg p \wedge (\neg q \vee \neg r) \dashv\vdash_{\mathsf{A}} (\neg p \wedge \neg q) \vee (\neg p \vee \neg r)$. But observe that

$$|\neg(\neg p \vee (\neg q \wedge \neg r))|^+_{\mathfrak{M}} = \{\{\neg p\}, \{\neg q, \neg r\}, \{\neg p, \neg q, \neg r\}\},$$

$$|\neg((\neg p \vee \neg q) \wedge (\neg p \vee \neg r))|^+_{\mathfrak{M}} = \{\{\neg p\}, \{\neg p, \neg r\}, \{\neg p, \neg q\}, \{\neg q, \neg r\}, \{\neg p, \neg q, \neg r\}\}.$$

It follows that $\neg((p \wedge q) \vee (p \vee r)) \not\models \neg(p \wedge (q \vee r))$ and so $\neg((p \wedge q) \vee (p \vee r)) \nvdash_{\mathsf{A}} \neg(p \wedge (q \vee r))$ by Soundness.[18]

An immediate consequence of this observation is that the following pair of rules is not sound:

$$\frac{\phi \dashv\vdash_{\mathsf{A}} \psi}{\theta(\phi) \vdash_{\mathsf{A}} \theta(\psi)} \quad \frac{\phi \dashv\vdash_{\mathsf{A}} \psi}{\theta(\psi) \vdash_{\mathsf{A}} \theta(\phi)}, \qquad \text{(Replacement)}$$

where $\theta(p)$ is any formula in the propositional variable p. That is, on the inclusive semantics, we cannot replace exactly equivalent formulas in all contexts while preserving exact entailment—in contexts involving negation, things might break down.

It is, however, easy to establish that our logic is what we might call *positively* congruential:

PROPOSITION 4.5 (Positive congruence). *The logic of exact entailment on the inclusive semantics is both $\wedge$-Congruential and $\vee$-Congruential.*

Proof. $\wedge$-Congruency is easily derived using $\wedge$-Monotonicity and $\vee$-Congruency using $\vee$-Introduction and $\vee$-Elimination. □

We can use this observation to show that *positive* replacement rules are *admissible* in our system. By the "admissibility" of a rule we mean that whenever premises of the rule are derivable, so is the conclusion. Remember that an occurrence of a subformula within a formula is *positive* iff the occurrence is not within the scope of an odd number of negations. The rules we shall prove admissible, then, are

$$\frac{\phi \dashv\vdash_{\mathsf{A}} \psi}{\theta(\phi) \vdash_{\mathsf{A}} \theta(\psi)} \quad \frac{\phi \dashv\vdash_{\mathsf{A}} \psi}{\theta(\psi) \vdash_{\mathsf{A}} \theta(\phi)}, \qquad \text{(Positive Replacement)}$$

where p occurs only positively in $\theta(p)$. We have:

PROPOSITION 4.6. *Positive Replacement is admissible in the system for the inclusive semantics.*

Proof. By a straightforward induction on θ following the construction from literals. We only sketch the argument. There are two base cases: p and $\neg p$. But both are trivial: in the case of p, $\theta(\phi) \dashv\vdash_{\mathsf{A}} \theta(\psi)$ is just $\phi \dashv\vdash_{\mathsf{A}} \psi$; and in the case of $\neg p$, p does not occur positively in θ, so the claim holds vacuously. The inductive steps for $\theta_1 \wedge \theta_2$ and $\theta_1 \vee \theta_2$ are immediate using $\wedge$-Congruency and $\vee$-Congruency and the fact that $(\theta_1 \circ \theta_2)(\phi) = \theta_1(\phi) \circ \theta_2(\phi)$ for $\circ = \wedge, \vee$. The remaining three cases, $\neg\neg\theta'$, $\neg(\theta_1 \wedge \theta_2)$, and $\neg(\theta_1 \vee \theta_2)$ are covered by the de Morgan laws, $\wedge$-Congruency and $\vee$-Congruency, and the observation that if p occurs positively in $\neg\neg\theta'$ or $\neg(\theta_1 \circ \theta_2)$, then p occurs positively in θ', $\neg\theta_1$, and $\neg\theta_2$. □

18 This example was discovered and brought to me by Simone Picenni. It plays an important role in future joint work.

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This means that we can freely use Positive Replacement in our system. It shall be convenient to abbreviate a certain pattern of reasoning involving Positive Replacement. It's easily seen that using Positive Replacement and Cut, we can replace exactly equivalent formulas for one another *anywhere* within a derivation given that they occur positively. To see this, consider the following example:

$$\dfrac{\dfrac{\dfrac{}{\phi \wedge \phi \dashv\vdash_{\mathsf{A}} \phi}\ \wedge\text{-Idempotence}}{\phi \vee \psi \vdash_{\mathsf{A}} (\phi \wedge \phi) \vee \psi}\ \text{Pos. Repl.} \qquad \begin{array}{c}\vdots\ \text{Some reasoning}\\ (\phi \wedge \phi) \vee \psi, \theta \vdash_{\mathsf{A}} ((\phi \wedge \phi) \vee \psi) \wedge \theta\end{array}}{\phi \vee \psi, \theta \vdash_{\mathsf{A}} ((\phi \wedge \phi) \vee \psi) \wedge \theta}\ \text{Cut.}$$

To see that the application of Positive Replacement is indeed valid, just note that p occurs positively in $p \vee \psi$, as well as $(p \vee \psi)(\phi \wedge \phi) = (\phi \wedge \phi) \vee \theta$ and $(p \vee \psi)(\phi) = \phi \vee \theta$. For conciseness' sake, we shall abbreviate the reasoning pattern as in the following example:

$$\dfrac{(\phi \wedge \phi) \vee \psi, \theta \vdash_{\mathsf{A}} ((\phi \wedge \phi) \vee \psi) \wedge \theta}{\phi \vee \psi, \theta \vdash_{\mathsf{A}} ((\phi \wedge \phi) \vee \psi) \wedge \theta}\ \text{Pos. Repl.} + \wedge\text{-Idem.}$$

The fact that the logic of exact entailment on the inclusive semantics is not $\neg$-congruential means that the logic is not only hyperintensional in the usual sense of distinguishing *classically* equivalent formulas,[19] but *also* hyperintensional in the sense of Odintsov and Wansing [23]: it is hyperintensional by its own logical standards. Odintsov and Wansing [23, p. 53] argue for a conception of hyperintensionality where (a) a logic is only hyperintensional if it is not congruential and (b) a connective is hyperintensional within a logic only if it is not congruential in the logic. So, one way to summarize our observations so far is that exact entailment on the inclusive semantics is hyperintensional in the sense of Odintsov and Wansing because negation is hyperintensional in the logic. This result has the potential for philosophical application when one tries to determine the most appropriate system for hyperintensional logic.

There is a debate in the literature on which concept(s) to build a basic system of hyperintensional logic. Fine [14, p. 565], for example, argues in favor of using exact truthmaking over an alternative *in*exact notion, which doesn't require complete relevance but only partial relevance. Deigan [6], instead, argues for taking the inexact notion as our starting point and Leitgeb [22] provides further arguments to support this position. Odintsov and Wansing [23] show that Leitgeb's HYPE, a system of hyperintensional logic based on an inexact conception of truthmaking, does not qualify as hyperintensional by their standards. Together with our previous observation, we can use this result to argue in favor of exact truthmaker semantics over HYPE: if Odintsov–Wansing hyperintensionality is what you're after (for all the reasons given by them), you should go with exact truthmaking rather than HYPE. We leave further philosophical exploration of the result, e.g., of how the behavior of negation can be used to capture certain philosophical phenomena, for future work.

If, instead, a fully congruential is what we're after, there are some options. One way to go would be to impose additional constraints on exact entailment that ensure its self-

[19] Take p and $p \vee (p \wedge q)$ as a particularly instructive example. We have $|p|^{+}_{\mathfrak{M}} = \{\{p\}\}$ and $|p \vee (p \wedge q)|^{+}_{\mathfrak{M}} = \{\{p\}, \{p, q\}\}$. Using Corollary 3.5, we get $p \vee (p \wedge q) \nvDash p$.

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extensionality, such as exact falsemaker anti-inclusion, or formally, $|\phi|^- \subseteq \bigcup |\Gamma|^-$.[20] As it turns out, however, we've already described a fully congruential logic for exact entailment, viz. the logic on the replete semantics. We shall prove this next.

First, we note for later use:

PROPOSITION 4.7. *The following are derivable*:

1. $\Gamma \vdash_{\mathsf{A}} \bigwedge \Gamma$. ($\wedge$-*Introduction*)
2. $\bigwedge \Gamma \vdash_{\mathsf{A}} \bigvee \Gamma$. (*Closure*)
3. *If* $\Gamma \subseteq \Delta \subseteq \Sigma$, *then* $\bigwedge \Gamma, \bigwedge \Sigma \vdash_{\mathsf{A}} \bigwedge \Delta$. ($\wedge$-*Convexity*)
4. *If* $\Gamma \subseteq \Delta \subseteq \Sigma$, *then* $\bigwedge \Delta \vdash_{\mathsf{A}^{nv*}} \bigwedge \Gamma \vee \bigwedge \Sigma$. ($\vee$-*Convexity*)

Proof. Most arguments are standard and/or straightforward. Since the general arguments are somewhat opaque, we give derivations of simplified cases that are easily seen to generalize. 1. is derived using $\wedge$-Monotonicity and $\wedge$-Idempotence as the following simplified example:

$$\dfrac{\dfrac{\dfrac{\phi \vdash_{\mathsf{A}} \phi}{\phi, \psi \vdash_{\mathsf{A}} \phi}\text{ Weakening} \quad \dfrac{\psi \vdash_{\mathsf{A}} \psi}{\phi, \psi \vdash_{\mathsf{A}} \psi}\text{ Weakening}}{\phi, \psi \wedge \psi \vdash_{\mathsf{A}} \phi \wedge \psi}\ \wedge\text{-Monotonicity}}{\phi, \psi \vdash_{\mathsf{A}} \phi \wedge \psi}\text{ Cut} + \wedge\text{-Idem.}$$

The derivation for 2. is a generalization of the following:

$$\dfrac{\dfrac{\phi \vdash_{\mathsf{A}} \phi \vee \psi \quad \psi \vdash_{\mathsf{A}} \phi \vee \psi}{\phi \wedge \psi \vdash_{\mathsf{A}} (\phi \vee \psi) \wedge (\phi \vee \psi)}\ \wedge\text{-Mon.}}{\phi \wedge \psi \vdash_{\mathsf{A}} \phi \vee \psi}\text{ Cut} + \wedge\text{-Idem.}$$

We call 2. "Closure," since it's another syntactic expression of the semantic fact that truthmakers are closed under fusions next to $\wedge$-Idempotence. The derivation for 3. is simply repeated applications of $\wedge$-Monotonicity to $\wedge$-Convexity. Finally, the derivation of 4. is a generalization of the following sketch:

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi \vdash_{\mathsf{A}^{nv*}} \phi \vee \phi \quad \psi \vdash_{\mathsf{A}^{nv*}} \phi \vee \psi}{\phi \wedge \psi \vdash_{\mathsf{A}^{nv*}} (\phi \vee \psi) \wedge (\phi \vee \psi)}\ \wedge\text{-Mon.} \quad \phi \vdash_{\mathsf{A}^{nv*}} \phi \vee \theta}{\phi \wedge \psi \wedge \phi \vdash_{\mathsf{A}^{nv*}} (\phi \vee \phi) \wedge (\phi \vee \psi) \wedge (\phi \vee \theta)}\ \wedge\text{-Mon.}}{\phi \wedge \psi \wedge \phi \vdash_{\mathsf{A}^{nv*}} \phi \vee (\phi \wedge \psi \wedge \theta)}\ \vee/\wedge\text{-Distr., Cut}}{\phi \wedge \psi \vdash_{\mathsf{A}^{nv*}} \phi \vee (\phi \wedge \psi \wedge \theta)}\text{ Pos. Repl. } \wedge\text{-Idem., } \wedge\text{-Com.}$$

□

The proof that exact entailment on the replete semantics is fully congruential relies on one of the core theorems for exact entailment, viz. its *Disjunctive Normal Form (DNF)* theorem. We shall now state and prove this theorem, which plays a central role in our completeness argument.

A *conjunctive clause* is a formula of the form $\bigwedge \Gamma$ for $\Gamma \subseteq \Lambda$. A formula ϕ is in DNF iff it is of the form $\bigvee \phi_i$, where the ϕ_i's are conjunctive clauses.

[20] We leave it to the interested reader to verify that this constraint indeed makes exact entailment self-extensional.

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THEOREM 4.8 (DNF theorem). *We have*:

1. $\phi \dashv\vdash_{\mathsf{A}} \bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma$.
2. (a) $\phi \dashv\vdash_{\mathsf{A}^{nv*}} \bigvee_{\Gamma \in (|\phi|^+_{\mathfrak{M}})_*} \bigwedge \Gamma$.
   (b) $\neg\phi \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma \in (|\phi|^+_{\mathfrak{M}})_*} \bigvee \neg\Gamma$, *where* $\neg\Gamma = \{\neg\psi : \psi \in \Gamma\}$.

Proof. The proof of 1. is by induction on ϕ following the construction from literals. The proof is more or less the same as the standard proof of the DNF theorem for classical logic with an additional use of the fact that $\phi \vee \psi \dashv\vdash_{\mathsf{A}} (\phi \vee \psi) \vee (\phi \wedge \psi)$, which is quickly derived using Proposition 4.7. We cover only one case besides the base case to illustrate the relevant reasoning.

The base cases are straightforward, since $|p|^+_{\mathfrak{M}} = \{\{p\}\}$ and $|\neg p|^+_{\mathfrak{M}} = |p|^-_{\mathfrak{M}} = \{\{\neg p\}\}$. Next, we cover the case for $\phi_1 \vee \phi_2$. By the induction hypothesis, we have $\phi_1 \dashv\vdash_{\mathsf{A}} \bigvee_{\Gamma \in |\phi_1|^+_{\mathfrak{M}}} \bigwedge \Gamma$ and $\phi_2 \dashv\vdash_{\mathsf{A}} \bigvee_{\Gamma \in |\phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma$. By $\vee$-Elimination and $\vee$-Introduction, we get

$$\phi_1 \vee \phi_2 \dashv\vdash_{\mathsf{A}} \left(\bigvee_{\Gamma \in |\phi_1|^+_{\mathfrak{M}}} \bigwedge \Gamma \right) \vee \left(\bigvee_{\Gamma \in |\phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma \right).$$

The right-hand side of this equivalence is easily seen to be notationally equivalent to $\bigvee_{\Gamma \in |\phi_1|^+_{\mathfrak{M}} \cup |\phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma$. Using $\phi \vee \psi \dashv\vdash_{\mathsf{A}} (\phi \vee \psi) \vee (\phi \wedge \psi)$ repeatedly, we get

$$\phi_1 \vee \phi_2 \dashv\vdash_{\mathsf{A}} \left(\bigvee_{\Gamma \in |\phi_1|^+_{\mathfrak{M}} \cup |\phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma \right) \vee \left(\bigvee_{\Gamma_i \in |\phi_i|^+_{\mathfrak{M}}} \left(\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \right) \right).$$

Since $\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2$ is notationally equivalent to $\bigwedge(\Gamma_1 \cup \Gamma_2)$ and $|\phi_1 \wedge \phi_2|^+_{\mathfrak{M}} = \{\Gamma_1 \cup \Gamma_2 : \Gamma_i \in |\phi_i|^+_{\mathfrak{M}}\}$ (by Sem-$\wedge^+$), we get that $\bigvee_{\Gamma_i \in |\phi_i|^+_{\mathfrak{M}}} (\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2)$ is notationally equivalent to $\bigvee_{\Gamma \in |\phi_1 \wedge \phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma$. This gives us, again via notational equivalence, that

$$\phi_1 \vee \phi_2 \dashv\vdash_{\mathsf{A}} \bigvee_{\Gamma \in |\phi_1|^+_{\mathfrak{M}} \cup |\phi_2|^+_{\mathfrak{M}} \cup |\phi_1 \wedge \phi_2|^+_{\mathfrak{M}}} \bigwedge \Gamma.$$

Since $|\phi_1 \vee \phi_2|^+_{\mathfrak{M}} = |\phi_1|^+_{\mathfrak{M}} \cup |\phi_2|^+_{\mathfrak{M}} \cup |\phi_1 \wedge \phi_2|^+_{\mathfrak{M}}$ by Sem-$\vee^+$ the case is complete.

We leave the remaining cases to the interested reader and turn our attention to 2. First, we establish (a). Since $(|\phi|^+_{\mathfrak{M}})_* = |\phi|^+_{\mathfrak{M}} \cup \{\Delta : \exists \Gamma_1, \Gamma_2 \in |\phi|^+_{\mathfrak{M}} \text{ with } \Gamma_1 \subseteq \Delta \subseteq \Gamma_2\}$, we know that $\bigvee_{\Gamma \in (|\phi|^+_{\mathfrak{M}})_*} \bigwedge \Gamma$ is notationally equivalent to

$$\left(\bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma \right) \vee \left(\bigwedge_{\exists \Gamma_1, \Gamma_2 \in |\phi|^+_{\mathfrak{M}}, \Gamma_1 \subseteq \Delta \subseteq \Gamma_2} \bigwedge \Delta \right).$$

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By 1., we know that $\phi \dashv\vdash_{\mathsf{A}^{nv*}} \bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma$. Repeatedly using $\vee$-Convexity (Proposition 4.7), we can derive

$$\bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma \dashv\vdash_{\mathsf{A}^{nv*}} \left(\bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma \right) \vee \left(\bigwedge_{\exists \Gamma_1, \Gamma_2 \in |\phi|^+_{\mathfrak{M}}, \Gamma_1 \subseteq \Delta \subseteq \Gamma_2} \bigwedge \Delta \right).$$

From this, our claim quickly follows via Cut and notational reasoning.

Finally, we establish 2.(b) by induction on ϕ following the construction from literals. We only sketch the argument since it's essentially a dual version of the proof for 1. The base cases are again trivially since $(|p|^\circ_{\mathfrak{M}})_* = |p|^\circ_{\mathfrak{M}}$ for $\circ = +, -$. The most interesting case is for $\phi_1 \vee \phi_2$, since it involves an argument via $\vee/\wedge$-Distribution, which is not available for $\vdash_{\mathsf{A}}$. Hence this case shows why we can't prove a comparable theorem for $\vdash_{\mathsf{A}}$.

By the induction hypothesis, we have $\phi_1 \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma_1 \in |\phi_1|^+_{\mathfrak{M}}} \bigvee \neg\Gamma_1$ and $\phi_2 \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma_1 \in |\phi_2|^+_{\mathfrak{M}}} \bigvee \neg\Gamma_2$. Using $\vee$-Congruence, we can derive

$$\phi_1 \vee \phi_2 \dashv\vdash_{\mathsf{A}^{nv*}} \left(\bigwedge_{\Gamma_1 \in (|\phi_1|^+_{\mathfrak{M}})_*} \bigvee \neg\Gamma_1 \right) \vee \left(\bigwedge_{\Gamma_1 \in (|\phi_2|^+_{\mathfrak{M}})_*} \bigvee \neg\Gamma_2 \right).$$

Using $\vee/\wedge$-Distribution repeatedly as well as notational reasoning, we can derive

$$\left(\bigwedge_{\Gamma_1 \in (|\phi_1|^+_{\mathfrak{M}})_*} \bigvee \neg\Gamma_1 \right) \vee \left(\bigwedge_{\Gamma_1 \in (|\phi_2|^+_{\mathfrak{M}})_*} \bigvee \neg\Gamma_2 \right) \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma \in \left((|\phi_1|^+_{\mathfrak{M}})_* \cup (|\phi_2|^+_{\mathfrak{M}})_*\right)} \bigvee \neg\Gamma.$$

From here, we can reason as in 1. using $\phi \vee \psi \dashv\vdash_{\mathsf{A}} (\phi \vee \psi) \vee (\phi \wedge \psi)$, as in 2.(a) using $\vee$-Convexity, and using the (rather convoluted) semantic fact that $(|\phi_1 \vee \phi_2|^+_{\mathfrak{M}})_* = (|\phi_1|^+_{\mathfrak{M}})_* \cup (|\phi_2|^+_{\mathfrak{M}})_* \cup (|\phi_1 \wedge \phi_2|^+_{\mathfrak{M}})_* \cup \{\Delta : \exists \Gamma_1, \Gamma_2 \in (|\phi_1|^+_{\mathfrak{M}})_* \cup (|\phi_2|^+_{\mathfrak{M}})_* \cup (|\phi_1 \wedge \phi_2|^+_{\mathfrak{M}})_*, \Gamma_1 \subseteq \Delta \subseteq \Gamma_2\}$, to derive

$$\bigwedge_{\Gamma \in \left((|\phi_1|^+_{\mathfrak{M}})_* \cup (|\phi_2|^+_{\mathfrak{M}})_*\right)} \bigvee \neg\Gamma \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma \in \left(|\phi_1 \vee \phi_2|^+_{\mathfrak{M}}\right)_*} \bigvee \neg\Gamma.$$

From this our claim follows via a series of $\mathrm{Cut}_{\mathsf{A}}$'s. $\square$

Note that the DNFs in our theorem are indeed *canonical* DNFs: $\bigvee_{\Gamma \in |\phi|^+_{\mathfrak{M}}} \bigwedge \Gamma$ is what's known as the "standard" DNF of ϕ, and $\bigvee_{\Gamma \in (|\phi|^+_{\mathfrak{M}})_*} \bigwedge \Gamma$ is what Fine [11, p. 215] calls "maximally standard" DNFs. Note also that these DNFs are unique up to logical equivalence since by Soundness (Theorem 4.4), we get: if $\phi \dashv\vdash_{\mathsf{A}} \psi$, then $|\phi|^+_{\mathfrak{M}} = |\psi|^+_{\mathfrak{M}}$; and if $\phi \dashv\vdash_{\mathsf{A}^{nv*}} \psi$, then $(|\phi|^+_{\mathfrak{M}})_* = (|\psi|^+_{\mathfrak{M}})_*$.

We get now as straightforward corollaries:

Corollary 4.9 (Full congruence). *The logic of exact entailment on the replete semantics is fully congruential.*

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Proof. It suffices to show that the logic is $\neg$-Congruential since the arguments for $\wedge$-Congruentiality and $\vee$-Congruentiality go through as for $\vdash_{\mathsf{A}}$. So, suppose that $\phi \dashv\vdash_{\mathsf{A}^{nv*}} \psi$. By Theorem 4.8, we have

$$\neg\phi \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma\in(|\phi|^{+}_{\mathfrak{M}})_*} \bigvee \neg\Gamma \qquad \neg\psi \dashv\vdash_{\mathsf{A}^{nv*}} \bigwedge_{\Gamma\in(|\psi|^{+}_{\mathfrak{M}})_*} \bigvee \neg\Gamma.$$

But since DNFs are unique up to logical equivalence, we get that $\bigwedge_{\Gamma\in(|\phi|^{+}_{\mathfrak{M}})_*} \bigvee_{\theta\in\Gamma} \neg\theta$ and $\bigwedge_{\Gamma\in(|\psi|^{+}_{\mathfrak{M}})_*} \bigvee_{\theta\in\Gamma} \neg\theta$ are identical. So we can derive $\neg\phi \dashv\vdash_{\mathsf{A}^{nv*}} \neg\psi$ by a single application of $\mathrm{Cut}_{\mathsf{A}}$. □

Corollary 4.10 (Full). *Replacement is admissible in the system for the replete semantics.*

Proof. By induction using the Congruence laws. □

We conclude the section by proving completeness.

Lemma 4.11. *We have*:

1. *If* $\Delta \in |\phi|^{+}_{\mathfrak{M}}$, *then* $\bigwedge \Delta \vdash_{\mathsf{A}} \phi$.
2. *If* $\Delta \in (|\phi|^{+}_{\mathfrak{M}})_*$, *then* $\bigwedge \Delta \vdash_{\mathsf{A}^{nv*}} \phi$.

Proof. Since $\phi \dashv\vdash_{\mathsf{A}} \bigvee_{\Gamma\in|\phi|^{+}_{\mathfrak{M}}} \bigwedge \Gamma$ by Theorem 4.8, 1. follows using $\vee$-Intro and Cut. 2. follows analogously from Theorem 4.8. □

Theorem 4.12 (Completeness for A and A_{nv*}). *We have*:

1. *If* $\Gamma \vDash \phi$, *then* $\Gamma \vdash_{\mathsf{A}} \phi$.
2. *If* $\Gamma \vDash_{nv*} \phi$, *then* $\Gamma \vdash_{\mathsf{A}^{nv*}} \phi$.

Proof. For 1. take $\Gamma = \{\psi_1, \dots, \psi_n\}$ with $\Gamma \vdash_{\mathsf{A}} \phi$. By the Fine–Jago theorem for the inclusive semantics (Theorem 3.4), we get that for each selection function f for $|\Gamma|^{+}_{\mathfrak{M}}$, there exists a $\Delta \in |\phi|^{+}_{\mathfrak{M}}$, such that for some $\psi_i \in \Gamma$, $f(|\psi_i|^{+}_{\mathfrak{M}}) \subseteq \Delta \subseteq \bigcup_{\psi_i\in\Gamma} f(|\psi_i|^{+}_{\mathfrak{M}})$. For $\psi_i \in \Gamma$, we can write $|\psi_i|^{+}_{\mathfrak{M}} = \{\Gamma^1_i, \dots, \Gamma^{j(i)}_i\}$, where j maps i to the number of elements in $|\psi_i|^{+}_{\mathfrak{M}}$. Now pick a selection function such that $f(|\psi_i|^{+}_{\mathfrak{M}}) = \Gamma^1_i$ for $1 \le i \le n$. We get that there exists a $\Delta \in |\phi|^{+}_{\mathfrak{M}}$ such that for some Γ^1_i, $\Gamma^1_i \subseteq \Delta \subseteq \bigcup_{1\le i\le n} \Gamma^1_i$. Using $\wedge$-Convexity, Proposition 4.7, we get that $\bigwedge \Gamma^1_i, \bigwedge_{1\le i\le n} \Gamma^1_i \vdash_{\mathsf{A}} \bigwedge \Delta$. Using $\wedge$-Introduction, Proposition 4.7, together with Cut, we can infer that $\bigwedge \Gamma^1_1, \dots, \bigwedge \Gamma^1_n \vdash_{\mathsf{A}} \bigwedge \Delta$. Using Lemma 4.11 and Cut, we get that $\bigwedge \Gamma^1_1, \dots, \bigwedge \Gamma^1_n \vdash_{\mathsf{A}} \phi$. Completely analogously, just by choosing $f(|\psi_1|^{+}_{\mathfrak{M}}) = \Gamma^2_1$ and $f(|\psi_i|^{+}_{\mathfrak{M}}) = \Gamma^1_i$ for $1 < i \le n$, we get $\bigwedge \Gamma^2_1, \bigwedge^1_2, \dots, \bigwedge \Gamma^1_n \vdash_{\mathsf{A}} \phi$. And so on, giving us

$$\begin{array}{ccc} \bigwedge \Gamma^1_1 & ,\Gamma^1_2, \dots, & \Gamma^1_n \vdash_{\mathsf{A}} \phi \\ \vdots & \vdots & \vdots \\ \bigwedge \Gamma^{j(1)}_1 & ,\Gamma^1_2, \dots, & \Gamma^1_n \vdash_{\mathsf{A}} \phi. \end{array}$$

Repeated application of $\vee$-Elimination gives us $\bigvee_{\Gamma\in|\psi_1|^{+}_{\mathfrak{M}}} \bigwedge \Gamma, \Gamma^1_2, \dots, \Gamma^1_n \vdash_{\mathsf{A}} \phi$. By the DNF theorem (Theorem 4.8), we have $\bigvee_{\Gamma\in|\psi_1|^{+}_{\mathfrak{M}}} \bigwedge \Gamma \dashv\vdash_{\mathsf{A}} \psi_1$, so by Cut, we get

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$\psi_1, \Gamma_2^1, \dots, \Gamma_n^1 \vdash_{\mathsf{A}} \phi$. We repeat this reasoning with suitable selection functions to obtain

$$\begin{array}{ccc} \psi_1, \Gamma_2^1, & \Gamma_3^1, \dots, \Gamma_n^1 & \vdash_{\mathsf{A}} \phi \\ \vdots & \vdots & \vdots \\ \psi_1, \Gamma_2^{j(2)}, & \Gamma_3^1, \dots, \Gamma_n^1 & \vdash_{\mathsf{A}} \phi. \end{array}$$

From this we get $\psi_1, \psi_2, \Gamma_3^1, \dots, \Gamma_n^1 \vdash_{\mathsf{A}} \phi$ using $\vee$-Elimination, the DNF theorem and Cut. By repeating this reasoning, we finally obtain $\psi_1, \dots, \psi_n \vdash_{\mathsf{A}} \phi$.

The proof for 2. proceeds exactly analogously just that it relies on Theorem 3.6, Lemma 4.11 and Theorem 4.8. □

Note that while the proof of completeness is direct (i.e., not via contrapositive reasoning), it is not constructive (i.e., it doesn't generate a proof, it just shows that one exists). This is because of the non-constructive application of the Fine–Jago characterization theorem in our proof.

§5. Hilbert calculus.

In this section, we present two Hilbert calculi for exact entailment, one for the inclusive semantics and one for the replete semantics. The calculi are inspired by the Hilbert calculus for FDE described by Font [18], which in turn relies on ideas used by Rebagliato and Ventura [24] to obtain a calculus for the implicationless fragment of intuitionistic logic. In these calculi, certain logical inferences are "nested" within disjunctive contexts, as for example in the inference from $\neg\neg\phi \vee \psi$ to $\phi \vee \psi$, where ψ provides a "disjunctive context" for the logical inference from $\neg\neg\phi$ to ϕ. The use of disjunctive contexts essentially allows us to absorb disjunction-elimination-style reasoning—inferences from $\Gamma, \phi \vdash \theta$ and $\Gamma, \psi \vdash \theta$ to $\Gamma, \phi \vee \psi \vdash \theta$—as a meta-rule (see Proposition 5.5). Without the disjunctive contexts, this meta-rule would need to become an explicit rule of our calculus. This would change the nature of our calculus from a Hilbert calculus for formula-to-formula inferences to something more akin to the direct axiomatization from the previous section.

It turns out that in order to accommodate exact entailment on the inclusive semantics, in particular in light of the failure of $\wedge$-Elimination, we need an *additional* conjunctive context, nested within the disjunctive context as in the inference from $(\neg\neg\phi \wedge \psi) \vee \xi$ to $(\phi \wedge \psi) \vee \xi$. Just like the disjunctive contexts allow us to absorb the disjunction elimination as a meta-rule, the conjunction ultimately allow us to prove $\wedge$-Monotonicity as a meta-rule (see Lemma 5.3 and Proposition 5.5). The use of disjunctive and conjunctive contexts together is what allows us to formulate a proper formula-to-formula Hilbert calculus for exact entailment.

Since exact entailment has no theorems (Proposition 3.2), there are no axioms. The calculus, H, consists entirely of the following rules:

$$\frac{\phi \vee \xi \quad \psi \vee \xi}{(\phi \wedge \psi) \vee \xi}, \quad (\mathrm{R}_1) \qquad \frac{(\phi_1 \wedge \dots \wedge \phi_n) \vee \xi}{(\phi_{\sigma(1)} \wedge \dots \wedge \phi_{\sigma(n)}) \vee \xi}, \quad (\mathrm{R}_2)$$

σ a permutation of $(1, \dots, n)$

$$\frac{\bigwedge(\Gamma_1 \wedge \dots \wedge \Gamma_n) \vee \xi}{\bigwedge(\Gamma_1 \cup \dots \cup \Gamma_n) \vee \xi}, \quad (\mathrm{R}_3) \qquad \frac{\phi \vee \xi \quad (\phi \wedge \psi \wedge \theta) \vee \xi}{(\phi \wedge \psi) \vee \xi}, \quad (\mathrm{R}_4)$$

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$$\frac{\phi}{\phi \vee \psi}, \qquad (\mathrm{R}_5) \qquad\qquad \frac{\phi \vee \phi}{\phi}, \qquad (\mathrm{R}_6)$$

$$\frac{\psi \vee \phi}{\phi \vee \psi}, \qquad (\mathrm{R}_7) \qquad\qquad \frac{\phi \vee (\psi \vee \theta)}{(\phi \vee \psi) \vee \theta}, \qquad (\mathrm{R}_8)$$

$$\frac{\bigwedge(\Gamma \cup \{\phi \wedge (\psi \vee \theta)\}) \vee \xi}{\bigwedge(\Gamma \cup \{(\phi \wedge \theta) \vee (\psi \wedge \theta)\}) \vee \xi}, \qquad (\mathrm{R}_9) \qquad \frac{\bigwedge(\Gamma \cup \{(\phi \wedge \theta) \vee (\psi \wedge \theta)\}) \vee \xi}{\bigwedge(\Gamma \cup \{\phi \wedge (\psi \vee \theta)\}) \vee \xi}, \qquad (\mathrm{R}_{10})$$

$$\frac{\bigwedge(\Gamma \cup \{\neg\neg\phi\}) \vee \xi}{\bigwedge(\Gamma \cup \{\phi\}) \vee \xi}, \qquad (\mathrm{R}_{11}) \qquad \frac{\bigwedge(\Gamma \cup \{\phi\}) \vee \xi}{\bigwedge(\Gamma \cup \{\neg\neg\phi\}) \vee \xi}, \qquad (\mathrm{R}_{12})$$

$$\frac{\bigwedge\left(\Gamma \cup \{\neg(\phi \wedge \psi)\}\right) \vee \xi}{\bigwedge(\Gamma \cup \{\neg\phi \vee \neg\psi\}) \vee \xi}, \qquad (\mathrm{R}_{13}) \qquad \frac{\bigwedge(\Gamma \cup \{\neg\phi \vee \neg\psi\}) \vee \xi}{\bigwedge\left(\Gamma \cup \{\neg(\phi \wedge \psi)\}\right) \vee \xi}, \qquad (\mathrm{R}_{14})$$

$$\frac{\bigwedge\left(\Gamma \cup \{\neg(\phi \vee \psi)\}\right) \vee \xi}{\bigwedge(\Gamma \cup \{\neg\phi \wedge \neg\psi\}) \vee \xi}, \qquad (\mathrm{R}_{15}) \qquad \frac{\bigwedge(\Gamma \cup \{\neg\phi \wedge \neg\psi\}) \vee \xi}{\bigwedge\left(\Gamma \cup \{\neg(\phi \vee \psi)\}\right) \vee \xi}. \qquad (\mathrm{R}_{16})$$

First, a quick remark on notation. Observe that we've absorbed the idempotence, associativity, and commutativity of conjunction in the single rule R_3.[21] The rule R_2 is still necessary since in §2, we've decided on a canonical background ordering which the $\bigwedge$ and $\bigvee$ notation respects.

We write $\Gamma \vdash_{\mathsf{H}} \phi$ to say that there is a derivation of ϕ using the above rules from assumptions exclusively in Γ. Just like before, we write $\phi \dashv\vdash_{\mathsf{H}} \psi$ as an abbreviation for both $\phi \vdash_{\mathsf{H}} \psi$ and $\psi \vdash_{\mathsf{H}} \phi$.

Rather than proving soundness and completeness of the system directly, we shall show that the system is deductively equivalent to A.

Proposition 5.1. *We have the following*:

1. $\phi, \psi \vdash_{\mathsf{H}} \phi \wedge \psi$. ($\wedge$-*Introduction*$_{\mathsf{H}}$)
2. $\phi \wedge \phi \dashv\vdash_{\mathsf{H}} \phi$. ($\wedge$-*Idempotence*$_{\mathsf{H}}$)
3. $\phi \wedge \psi \dashv\vdash_{\mathsf{H}} \psi \wedge \phi$. ($\wedge$-*Commutativity*$_{\mathsf{H}}$)
4. $(\phi \wedge \psi) \wedge \theta \dashv\vdash_{\mathsf{H}} \phi \wedge (\psi \wedge \theta)$. ($\wedge$-*Associativity*$_{\mathsf{H}}$)
5. $\phi \vdash_{\mathsf{H}} \phi \vee \psi \quad \psi \vdash_{\mathsf{H}} \phi \vee \psi$. ($\vee$-*Introduction*$_{\mathsf{H}}$)
6. $\phi \wedge (\psi \vee \theta) \dashv\vdash_{\mathsf{H}} (\phi \wedge \psi) \vee (\phi \wedge \theta)$. ($\wedge/\vee$-*Distribution*$_{\mathsf{H}}$)
7. $\phi, \phi \wedge \psi \wedge \theta \vdash_{\mathsf{H}} \phi \wedge \theta$. ($\wedge$-*Convexity*$_{\mathsf{H}}$)
8. $\neg\neg\phi \dashv\vdash_{\mathsf{H}} \phi$. (*Double Negation*$_{\mathsf{H}}$)
9. $\neg(\phi \vee \psi) \dashv\vdash_{\mathsf{H}} \neg\phi \wedge \neg\psi \quad \neg(\phi \wedge \psi) \dashv\vdash_{\mathsf{H}} \neg\phi \vee \neg\psi$. (*De Morgan*$_{\mathsf{H}}$)

Proof. The arguments are all analogous: in each case, the idea is to use R_5 to introduce the desired conclusion as a disjunctive context, apply the relevant rule, and then use R_6 to infer the conclusion. We provide the derivation for 1. as an example:

$$\dfrac{\dfrac{\dfrac{\phi}{\phi \vee (\phi \wedge \psi)}\, R_5 \quad \dfrac{\psi}{\psi \vee (\phi \wedge \psi)}\, R_5}{(\phi \wedge \psi) \vee (\phi \wedge \psi)}\, R_1}{\phi \wedge \psi}\, R_6.$$

[21] Note that if we're interested in developing a calculus for exact entailment on the non-inclusive semantics, we'd be formulating R_3 as well as the rules $\mathrm{R}_{9\text{–}16}$ using multisets instead of sets.

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We leave verifying the remaining cases to the interested reader. $\square$

Next, we establish that the rules of our previous calculus hold as meta-theorems for our Hilbert calculus. First, note that Reflexivity, Weakening, and Cut are covered using standard structural reasoning about Hilbert calculi.

PROPOSITION 5.2. *We have*:

1. $\phi \vdash_{\mathsf{H}} \phi$. $\hfill$ (*Reflexivity*$_{\mathsf{H}}$)
2. *If* $\Gamma \vdash_{\mathsf{H}} \phi$, *then* $\Gamma, \Delta \vdash_{\mathsf{H}} \phi$. $\hfill$ (*Weakening*$_{\mathsf{H}}$)
3. *If* $\Gamma \vdash_{\mathsf{H}} \phi$ *and* $\Sigma, \phi \vdash_{\mathsf{H}} \psi$, *then* $\Gamma, \Sigma \vdash_{\mathsf{H}} \psi$. $\hfill$ (*Cut*$_{\mathsf{H}}$)

The following lemma is where the additional conjunctive contexts are really put to work.

LEMMA 5.3. *We have*:

1. (a) *For* $i = 1, 4$, *if* $\dfrac{\phi_1 \quad \phi_2}{\psi}$ *is an instance of* R_i, *then* $\phi_1 \wedge \xi, \phi_2 \wedge \xi \vdash_{\mathsf{H}} \psi \wedge \xi$.
   (b) *For* $i \neq 1, 4$, *if* $\dfrac{\phi}{\psi}$ *is an instance of* R_i, *then* $\phi \wedge \xi \vdash_{\mathsf{H}} \psi \wedge \xi$.
2. (a) *For* $i = 1, 4$, *if* $\dfrac{\phi_1 \quad \phi_2}{\psi}$ *is an instance of* R_i, *then* $\phi_1 \vee \xi, \phi_2 \vee \xi \vdash_{\mathsf{H}} \psi \vee \xi$.
   (b) *For* $i \neq 1, 4$, *if* $\dfrac{\phi}{\psi}$ *is an instance of* R_i, *then* $\phi \vee \xi \vdash_{\mathsf{H}} \psi \vee \xi$.

Proof. The arguments for 1. are all analogous in that they essentially rely on permuting the relevant rules with $\vee/\wedge$-Distribution$_{\mathsf{H}}$. We shall show $(\phi \vee \xi), (\psi \vee \xi) \wedge \gamma \vdash_{\mathsf{H}} ((\phi \wedge \psi) \vee \xi) \wedge \gamma$ as an example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\phi \vee \theta) \wedge \xi}{\xi \wedge (\phi \vee \theta)}\, R_2}{(\phi \wedge \xi) \vee (\theta \wedge \xi)}\, \vee/\wedge\text{-Distribution} \qquad \dfrac{\dfrac{(\psi \vee \theta) \wedge \xi}{\xi \wedge (\psi \vee \theta)}\, R_2}{(\psi \wedge \xi) \vee (\theta \wedge \xi)}\, \vee/\wedge\text{-Distribution}}{((\phi \wedge \xi) \wedge (\psi \wedge \xi)) \vee (\theta \wedge \xi)}\, R_1}{((\phi \wedge \psi) \wedge \xi) \vee (\theta \wedge \xi)}\, R_2, R_3}{\xi \wedge ((\phi \wedge \psi) \vee \theta)}\, \vee/\wedge\text{-Distribution}}{((\phi \wedge \psi) \vee \theta) \wedge \xi}\, R_2.$$

For 1.(b), we show R_3 as an example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\bigwedge(\Gamma_1 \wedge \cdots \wedge \Gamma_n) \vee \phi) \wedge \psi}{\psi \wedge (\bigwedge(\Gamma_1 \wedge \cdots \wedge \Gamma_n) \vee \phi)}\, R_2}{\bigwedge(\Gamma_1 \wedge \cdots \wedge \Gamma_n \wedge \psi) \vee (\phi \wedge \psi)}\, \wedge/\vee\text{-Distribution}}{\bigwedge(\Gamma_1 \cup \cdots \cup \Gamma_n \cup \{\psi\}) \vee (\phi \wedge \psi)}\, R_3}{(\bigwedge(\Gamma_1 \cup \cdots \cup \Gamma_n) \wedge \psi) \vee (\phi \wedge \psi)}\, R_2}{\psi \wedge (\bigwedge(\Gamma_1 \cup \cdots \cup \Gamma_n) \vee \phi)}\, \vee/\wedge\text{-Distribution}}{(\bigwedge(\Gamma_1 \cup \cdots \cup \Gamma_n) \vee \phi) \wedge \psi}\, R_2.$$

The cases 2. are all straightforward given the disjunctive contexts in the premises. $\square$

Note the crucial role played by the conditional contexts in the derivation for R_3 in 1.(b).

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Using the previous lemma, we prove:

LEMMA 5.4. *We have*:

1. *If* $\Gamma, \phi \vdash_{\mathsf{H}} \psi$, *then* $\Gamma, \phi \wedge \xi \vdash_{\mathsf{H}} \psi \wedge \xi$.
2. *If* $\Gamma, \phi \vdash_{\mathsf{H}} \psi$, *then* $\Gamma, \phi \vee \xi \vdash_{\mathsf{H}} \psi \vee \xi$.

Proof. 1. Assume that $\Gamma, \phi \vdash_{\mathsf{H}} \psi$ and suppose that $\Gamma = \{\phi_1, \dots, \phi_n\}$. Using Lemma 5.3, a straightforward induction on the length of derivations establishes that $\phi_1 \wedge \xi, \dots, \phi_n \wedge \xi, \phi \wedge \xi \vdash \psi \wedge \xi$. Observe that for each ϕ_i, $1 \leq i \leq n$, we can deduce as follows:

$$\dfrac{\phi_i \qquad \dfrac{\dfrac{\phi \wedge \xi \quad \phi_i}{\phi \wedge \xi \wedge \phi_i}\ \wedge\text{-Introduction}_{\mathsf{H}}}{(\phi_i \wedge \xi) \wedge \phi}\ R_2}{\phi_i \wedge \xi}\ \wedge\text{-Convexity}_{\mathsf{H}}.$$

By repeated applications $\mathrm{Cut}_{\mathsf{H}}$, we get $\phi_1, \dots, \phi_n, \phi \wedge \xi \vdash_{\mathsf{H}} \psi \wedge \xi$ as desired.

2. Assume that $\Gamma, \phi \vdash_{\mathsf{H}} \psi$ and suppose that $\Gamma = \{\phi_1, \dots, \phi_n\}$. Using Lemma 5.3, a straightforward induction on the length of derivations establishes that $\phi_1 \vee \xi, \dots, \phi_n \vee \xi, \phi \vee \xi \vdash \psi \vee \xi$. Applying $\mathrm{Cut}_{\mathsf{H}}$ and $\phi_i \vdash_{\mathsf{H}} \phi_i \vee \xi$ ($\vee$-Introduction$_{\mathsf{H}}$), we get $\phi_1, \dots, \phi_n, \phi \vee \xi \vdash_{\mathsf{H}} \psi \vee \xi$. □

We're now in a position to prove that the rules of our axiomatic system from §4 hold as meta-theorems for our Hilbert calculus.

PROPOSITION 5.5. *We have*:

1. *If* $\Gamma, \phi_1 \vdash_{\mathsf{H}} \psi_1$ *and* $\Gamma, \phi_2 \vdash_{\mathsf{H}} \psi_2$, *then* $\Gamma, \phi_1 \wedge \phi_2 \vdash_{\mathsf{H}} \psi_1 \wedge \psi_2$.
2. *If* $\Gamma, \phi_1 \vdash_{\mathsf{H}} \psi$ *and* $\Gamma, \phi_2 \vdash_{\mathsf{H}} \psi$, *then* $\Gamma, \phi_1 \vee \phi_2 \vdash_{\mathsf{H}} \psi$.

Proof. For 1., assume that $\Gamma, \phi_1 \vdash_{\mathsf{H}} \psi_1$ and $\Gamma, \phi_2 \vdash_{\mathsf{H}} \psi_2$. Using Lemma 5.4, we get $\Gamma, \phi_1 \wedge \phi_2 \vdash_{\mathsf{H}} \psi_1 \wedge \phi_2$ and, additionally using $\wedge$-Commutativity$_{\mathsf{H}}$, $\Gamma, \psi_1 \wedge \phi_2 \vdash_{\mathsf{H}} \psi_1 \wedge \psi_2$. The claim follows by one application of $\mathrm{Cut}_{\mathsf{H}}$.

For 2., $\Gamma, \phi_1 \vdash_{\mathsf{H}} \psi$ and $\Gamma, \phi_2 \vdash_{\mathsf{H}} \psi$. Using Lemma 5.4 and R_7, we get $\Gamma, \phi_1 \vee \phi_2 \vdash_{\mathsf{H}} \phi_2 \vee \theta$ from the first assumption. The second assumption similarly gives us $\Gamma, \phi_2 \vee \theta \vdash_{\mathsf{H}} \theta \vee \theta$. Reasoning with $\mathrm{Cut}_{\mathsf{H}}$, we get $\Gamma, \phi_1 \vee \phi_2 \vdash_{\mathsf{H}} \theta \vee \theta$, from which we get the our claim via R_6. □

Putting Propositions 5.1–5.5 together in a straightforward induction on the length of derivations, we get:

THEOREM 5.6. *If* $\Gamma \vdash_{\mathsf{A}} \phi$, *then* $\Gamma \vdash_{\mathsf{H}} \phi$.

In order to establish the converse of the previous theorem, thereby giving us that $\vdash_{\mathsf{A}}$ and $\vdash_{\mathsf{H}}$ are deductively equivalent, we first establish:

LEMMA 5.7. *We have*:

1. *For* $i = 1, 4$, *if* $\dfrac{\phi_1 \quad \phi_2}{\psi}$ *is an instance of* R_i, *then* $\phi_1, \phi_2 \vdash_{\mathsf{A}} \psi$.
2. *For* $i \neq 1, 4$, *if* $\dfrac{\phi}{\psi}$ *is an instance of* R_i, *then* $\phi \vdash_{\mathsf{A}} \psi$.

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Proof. For 1., we sketch the derivation for R_1 and leave the analogous derivation for R_2 to the interested reader:

$$\dfrac{\dfrac{\dfrac{}{\phi, \psi \vdash \phi \wedge \psi}\,\textit{Prop. 7.1}}{\phi, \psi \vdash (\phi \wedge \psi) \vee \xi}\,\vdots\ \vee\text{-Intro., Cut} \qquad \dfrac{\dfrac{}{\xi \vdash (\phi \wedge \psi) \vee \xi}\,\vee\text{-Intro.}}{\psi, \xi \vdash (\phi \wedge \psi) \vee \xi}\,\text{Weak.}}{\phi, \psi \vee \xi \vdash (\phi \wedge \psi) \vee \xi}\,\vee\text{-Elim.} \qquad \dfrac{\dfrac{}{\xi \vdash (\phi \wedge \psi) \vee \xi}\,\vee\text{-Intro.}}{\psi \vee \xi, \xi \vdash (\phi \wedge \psi) \vee \xi}\,\text{Weak.}$$
$$\overline{\phi \vee \xi, \psi \vee \xi \vdash (\phi \wedge \psi) \vee \xi}\ \vee\text{-Elim.}$$

For 2., first note that $R_{5\text{–}8}$ are standard using $\vee$-Introduction and $\vee$-Introduction. The arguments for the remaining rules are all analogous applications of Positive Replacement with the formula $\bigwedge(\Gamma \cup \{p\}) \vee \xi$ and the corresponding axioms for $\vdash_{\mathsf{A}}$ as in the following example:

$$\dfrac{\dfrac{}{\phi \wedge (\psi \vee \theta) \dashv\vdash_{\mathsf{A}} (\phi \wedge \psi) \vee (\phi \wedge \theta)}\ \wedge/\vee\text{-Distr.}}{\bigwedge(\Gamma \cup \{(\phi \wedge \psi) \vee (\phi \wedge \theta)\}) \vee \xi \vdash_{\mathsf{A}} \bigwedge(\Gamma \cup \{\phi \wedge (\psi \vee \theta)\}) \vee \xi}\ \text{Pos. Repl.}$$

□

A straightforward inductive argument using the previous lemma then gives us:

Theorem 5.8. *If* $\Gamma \vdash_{\mathsf{H}} \phi$, *then* $\Gamma \vdash_{\mathsf{A}} \phi$.

So, we've seen that H is deductively equivalent to A. To obtain a calculus for exact entailment on the replete semantics, we simply add the following two rules to H:

$$\dfrac{\bigwedge(\Gamma \cup \{\phi \vee (\psi \wedge \theta)\}) \vee \xi}{\bigwedge(\Gamma \cup \{(\phi \vee \theta) \wedge (\psi \vee \theta)\}) \vee \xi,}\ (R_{17}) \qquad \dfrac{\bigwedge(\Gamma \cup \{(\phi \vee \theta) \wedge (\psi \vee \theta)\}) \vee \xi}{\bigwedge(\Gamma \cup \{\phi \vee (\psi \wedge \theta)\}) \vee \xi}\ .\ (R_{18})$$

We write $\Gamma \vdash_{\mathsf{H}^{nv*}} \phi$ for derivability in the resulting calculus and $\phi \dashv\vdash_{\mathsf{H}^{nv*}} \psi$ as an abbreviation for $\phi \vdash_{\mathsf{H}^{nv*}} \psi$ and $\psi \vdash_{\mathsf{H}^{nv*}} \phi$.

For soundness and completeness, we can be quick: Proposition 5.2 carries over to $\vdash_{\mathsf{H}^{nv*}}$ without any adjustments:

Proposition 5.9. *We have*:

1. $\phi \vdash_{\mathsf{H}^{nv*}} \phi$. (*Reflexivity*$_{\mathsf{H}^{nv*}}$)
2. *If* $\Gamma \vdash_{\mathsf{H}^{nv*}} \phi$, *then* $\Gamma, \Delta \vdash_{\mathsf{H}^{nv*}} \phi$. (*Weakening*$_{\mathsf{H}^{nv*}}$)
3. *If* $\Gamma \vdash_{\mathsf{H}^{nv*}} \phi$ *and* $\Sigma, \phi \vdash_{\mathsf{H}^{nv*}} \psi$, *then* $\Gamma, \Sigma \vdash_{\mathsf{H}^{nv*}} \psi$. (*Cut*$_{\mathsf{H}^{nv*}}$)

For Proposition 5.5, just note that $R_{17,18}$ are of the same form as $R_{9,10}$ and thus we can carry over all the arguments building up to the relevant proof:

Proposition 5.10. *We have*:

1. *If* $\Gamma, \phi_1 \vdash_{\mathsf{H}^{nv*}} \psi_1$ *and* $\Gamma, \phi_2 \vdash_{\mathsf{H}^{nv*}} \psi_2$, *then* $\Gamma, \phi_1 \wedge \phi_2 \vdash_{\mathsf{H}^{nv*}} \psi_1 \wedge \psi_2$.
2. *If* $\Gamma, \phi_1 \vdash_{\mathsf{H}^{nv*}} \psi$ *and* $\Gamma, \phi_2 \vdash_{\mathsf{H}} \psi$, *then* $\Gamma, \phi_1 \vee \phi_2 \vdash_{\mathsf{H}^{nv*}} \psi$.

And finally, again using the fact that $R_{17,18}$ are of the same form as $R_{9,10}$, we can easily adjust the proof of Lemma 5.7 to include $R_{17,18}$:

Lemma 5.11. *We have*:

1. *For* $i = 1, 4$, *if* $\dfrac{\phi_1 \quad \phi_2}{\psi}$ *is an instance of* R_i, *then* $\phi_1, \phi_2 \vdash_{\mathsf{A}^{nv*}} \psi$.
2. *For* $i \neq 1, 4$, *if* $\dfrac{\phi}{\psi}$ *is an instance of* R_i, *then* $\phi \vdash_{\mathsf{A}^{nv*}} \psi$.

We summarily conclude:

THEOREM 5.12. $\Gamma \vdash_{\mathsf{A}^{nv*}} \phi$ *iff* $\Gamma \vdash_{\mathsf{H}^{nv*}} \phi$.

We conclude the section by noting that since A and H are deductively equivalent, we shall simply write $\Gamma \vdash \phi$ and $\phi \dashv\vdash \psi$ (and similarly for the $nv*$ variants).

§6. Sequent calculus. In this section, we present a sequent calculus for exact entailment. We begin by reviewing the calculus presented by Fine and Jago [17, pp. 551–556].

What sets the calculus apart from ordinary sequent calculi is that it operates on sequents of the form $\{\Gamma_1, \dots, \Gamma_n\} \Rightarrow \Delta$. That is, a sequent in the Fine–Jago calculus has a (finite) set of sets of formulas on the left and a single set of formulas on the right. The intended reading of a sequent is $\bigwedge\Gamma_1, \dots, \bigwedge\Gamma_n \vDash \bigwedge\Delta$. That is, the Γ_i's and Δ are read conjunctively, while $\{\Gamma_1, \dots, \Gamma_n\}$ is read distributively. This makes the Fine–Jago calculus akin to a single-conclusion sequent calculus. In the following, we'll use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \dots$ as variables for finite sets of sets of formulas. So, we can represent the form of a sequent as $\mathcal{X} \Rightarrow \Delta$.[22] To cut down on set-braces, we use ";" to separate the members of $\mathcal{X}$ and we use "," to separate the members of Δ and of the $\Gamma \in \mathcal{X}$. So, for example, $\phi, \psi; \theta \Rightarrow \gamma, \delta$ is shorthand for $\{\{\phi, \psi\}, \{\theta\}\} \Rightarrow \{\gamma, \delta\}$.

Note that the following structural rules are absorbed in the notation:

$$\frac{\mathcal{X}; \Gamma, \phi, \phi \Rightarrow \Delta}{\mathcal{X}; \Gamma, \phi \Rightarrow \Delta}\ (W, L) \qquad \frac{\mathcal{X}; \Gamma \Rightarrow \Delta, \phi, \phi}{\mathcal{X}; \Gamma \Rightarrow \Delta, \phi}\ (W, R) \qquad \frac{\mathcal{X}; \Gamma; \Gamma \Rightarrow \Delta}{\mathcal{X}; \Gamma \Rightarrow \Delta}\ (W; L)$$

$$\frac{\mathcal{X} \Rightarrow \phi, \psi, \Delta}{\mathcal{X} \Rightarrow \psi, \phi, \Delta}\ (Ex, L) \qquad \frac{\mathcal{X}; \phi, \psi, \Gamma \Rightarrow \Delta}{\mathcal{X}; \psi, \phi, \Gamma \Rightarrow \Delta}\ (Ex, R) \qquad \frac{\mathcal{Z}; \mathcal{X}_1; \mathcal{X}_2 \Rightarrow \Delta}{\mathcal{Z}; \mathcal{X}_2; \mathcal{X}_1 \Rightarrow \Delta}\ (Ex; L).$$

From a proof-theoretic perspective, this is slightly unsatisfactory since it gives us less control over the structural aspects of the calculus. Semantically, however, the issue is immaterial: the validity of (W, L), (W, R), (Ex, L), and (Ex, R) follows immediately from the idempotence and commutativity of conjunction and the validity of $(W; L)$ and $(Ex; L)$ follows from the definition of exact entailment.

The Fine–Jago calculus, G_{FJ}, has the following axioms and rules:

Logical Axioms

$$\Gamma \Rightarrow \Gamma$$

Structural Rules

$$\frac{\mathcal{X} \Rightarrow \Delta}{\mathcal{X}; \Gamma \Rightarrow \Delta}\ (Weak) \qquad \frac{\mathcal{X} \Rightarrow \Gamma \quad \mathcal{X}; \Gamma \Rightarrow \Delta}{\mathcal{X} \Rightarrow \Delta}\ (Cut_{\mathsf{G}_{FJ}})$$

$$\frac{\mathcal{X}; \Gamma \Rightarrow \Sigma \quad \mathcal{X}; \Delta \Rightarrow \Pi}{\mathcal{X}; \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (,1) \qquad \frac{\mathcal{X} \Rightarrow \Gamma \quad \mathcal{X} \Rightarrow \Delta, \Sigma}{\mathcal{X} \Rightarrow \Gamma, \Delta}\ (,2)$$

Logical Rules

$$\frac{\mathcal{X}; \Gamma, \phi, \psi \Rightarrow \Delta}{\mathcal{X}; \Gamma, \phi \wedge \psi \Rightarrow \Delta}\ (\wedge L_{\mathsf{G}_{FJ}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \phi, \psi}{\mathcal{X} \Rightarrow \Delta, \phi \wedge \psi}\ (\wedge R_{\mathsf{G}_{FJ}})$$

[22] Note that if we're working on the non-inclusive semantics, however, we need to use a set (!) of multisets in the antecedent.

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$$\frac{\mathcal{X};\Gamma,\phi \Rightarrow \Delta \quad \mathcal{X};\Gamma,\psi \Rightarrow \Delta}{\mathcal{X};\Gamma,\phi \vee \psi \Rightarrow \Delta}\ (\vee L_{\mathsf{G}_{FJ}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta,\phi_i}{\mathcal{X} \Rightarrow \Delta,\phi_1 \vee \phi_2}\ (\vee R^i_{\mathsf{G}_{FJ}})$$

$$\frac{\mathcal{X};\Gamma,\phi \Rightarrow \Delta}{\mathcal{X};\Gamma,\neg\neg\phi \Rightarrow \Delta}\ (\neg\neg L_{\mathsf{G}_{FJ}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta,\phi}{\mathcal{X} \Rightarrow \Delta,\neg\neg\phi}\ (\neg\neg R_{\mathsf{G}_{FJ}})$$

$$\frac{\mathcal{X};\Gamma,\neg\phi \Rightarrow \Delta \quad \mathcal{X};\Gamma,\neg\psi \Rightarrow \Delta}{\mathcal{X};\Gamma,\neg(\phi \wedge \psi) \Rightarrow \Delta}\ (\neg\wedge L_{\mathsf{G}_{FJ}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta,\neg\phi_i}{\mathcal{X} \Rightarrow \Delta,\neg(\phi_1 \wedge \phi_2)}\ (\neg\wedge R^i_{\mathsf{G}_{FJ}})$$

$$\frac{\mathcal{X};\Gamma,\neg\phi,\neg\psi \Rightarrow \Delta}{\mathcal{X};\Gamma,\neg(\phi \vee \psi) \Rightarrow \Delta}\ (\neg\vee L_{\mathsf{G}_{FJ}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta,\neg\phi,\neg\psi}{\mathcal{X} \Rightarrow \Delta,\neg(\phi \vee \psi)}\ (\neg\vee R_{\mathsf{G}_{FJ}}).$$

We write $\mathcal{X} \vdash_{\mathsf{G}_{FJ}} \Delta$ to say that the sequent $\mathcal{X} \Rightarrow \Delta$ is derivable in the calculus.

We say that a sequent $\mathcal{X} \Rightarrow \Delta$ is *valid*, symbolically $\mathcal{X} \vDash \Delta$, iff $\{\bigwedge\Gamma : \Gamma \in \mathcal{X}\} \vDash \bigwedge\Delta$. In a rule, we call the sequents above the inference line "upper sequents" and the one below the "lower sequent." A rule is *sound* iff its lower sequent is valid whenever its upper sequents are. Fine and Jago [17, theorem 9.2, p. 554, and theorem 9.6, p. 555] establish:

THEOREM 6.1 (Soundness and completeness for G_{FJ}). $\mathcal{X} \vDash \Delta$ *iff* $\mathcal{X} \vdash_{\mathsf{G}_{FJ}} \Delta$.

Since their proof doesn't make use of $Cut_{\mathsf{G}_{FJ}}$, Fine and Jago [17, theorem 9.7, p. 556] infer as a corollary that their calculus has the Cut-elimination property, i.e., if $\mathcal{X} \vdash_{\mathsf{G}_{FJ}} \Delta$, then the sequent is derivable without any applications of Cut. We shall now investigate the calculus in more detail from a proof-theoretic perspective.

First, note that the rule Weak cannot be eliminated from the calculus: without the rule we already couldn't derive $p; q \Rightarrow p$.[23] Having a weakening rule around in a sequent calculus is not ideal since it complicates proof searches (though it is, of course, strictly speaking not problematic). But this is an easy fix: just take as axioms all sequents of the form $\mathcal{X}, \Gamma \Rightarrow \Gamma$. Call the resulting calculus $\mathsf{G}_{FJ'}$. Then it's straightforward to see:

PROPOSITION 6.2 (Weak-eliminability). *If* $\mathcal{X} \vdash_{\mathsf{G}_{FJ'}} \Gamma$, *then there's a derivation without applications of Weak.*

Proof. By a straight-forward induction on derivations using that all rules are context preserving in $\mathcal{X}$. □

Since all axioms of G_{FJ} are also axioms of $\mathsf{G}_{FJ'}$ (just let $\mathcal{X} = \emptyset$), the two calculi are clearly deductively equivalent.

It turns out that once we've eliminated Weak, we can straightforwardly eliminate $(,1)$ as well!

PROPOSITION 6.3 ($(,1)$-eliminability). *If* $\mathcal{X} \vdash_{\mathsf{G}_{FJ'}} \Gamma$, *then there's a derivation without applications of* $(,1)$.

23 Note that the "other" weakening rules

$$\frac{\mathcal{X};\Gamma \Rightarrow \Delta}{\mathcal{X};\Gamma,\phi \Rightarrow \Delta}\ (Weak, L) \qquad \frac{\mathcal{X};\Gamma \Rightarrow \Delta}{\mathcal{X};\Gamma \Rightarrow \Delta,\phi}\ (Weak, L)$$

are not sound with respect to exact entailment and thus not included in the calculus.

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Proof. We show that for every derivation with exactly one application of $(,1)$, there exists a derivation without $(,1)$. The theorem then follows by a simple inductive argument.

Without loss of generality, we can focus on derivations without Weak where the application of $(,1)$ is the last step in the derivation:

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1 \\ \mathcal{X};\Gamma\Rightarrow\Sigma\end{matrix}\quad\begin{matrix}\vdots\ \mathcal{D}_2 \\ \mathcal{X};\Delta\Rightarrow\Pi\end{matrix}}{\mathcal{X};\Gamma,\Delta\Rightarrow\Sigma,\Pi}\ (,1).$$

We prove the claim by an induction on the height of this derivation, that is on $max(|\mathcal{D}_1|,|\mathcal{D}_2|)$.

For the base case, note that if $max(|\mathcal{D}_1|,|\mathcal{D}_2|)=0$, then both premises of $(,1)$ are axioms, that is, the derivation looks like this:

$$\dfrac{\mathcal{X};\Gamma\Rightarrow\Gamma\quad\mathcal{X};\Delta\Rightarrow\Delta}{\mathcal{X};\Gamma,\Delta\Rightarrow\Gamma,\Delta}\ (,1).$$

But then also the conclusion is an axiom.

We need to go through the possible last rules of $\mathcal{D}_1$ and of $\mathcal{D}_2$. Though there are many such possibilities, they reduce to a manageable amount of cases by relying on the duality of the $\neg$-rules and the positive rules. Here we only show the case where the left upper sequent was derived via $(,2)$ and the right upper sequent via $(\neg\neg R)$ to illustrate the idea. Consider:

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1^1 \\ \mathcal{X};\Gamma\Rightarrow\Sigma_1\end{matrix}\quad\begin{matrix}\vdots\ \mathcal{D}_1^2 \\ \mathcal{X};\Gamma\Rightarrow\Sigma_2,\Theta\end{matrix}}{\mathcal{X};\Gamma\Rightarrow\Sigma_1,\Sigma_2}\ (,2)\qquad\dfrac{\begin{matrix}\vdots\ \mathcal{D}_2' \\ \mathcal{X};\Delta,\phi\Rightarrow\Pi\end{matrix}}{\mathcal{X};\Delta,\neg\neg\phi\Rightarrow\Pi}\ \neg\neg R}{\mathcal{X};\Gamma,\Delta,\neg\neg\phi\Rightarrow\Sigma_1,\Sigma_2,\Pi}\ (,1).$$

This derivation can be transformed into:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1^1 \\ \mathcal{X};\Gamma\Rightarrow\Sigma_1\end{matrix}\quad\begin{matrix}\vdots\ \mathcal{D}_2' \\ \mathcal{X};\Delta,\phi\Rightarrow\Pi\end{matrix}}{\mathcal{X};\Gamma,\Delta,\phi\Rightarrow\Sigma_1,\Pi}\ (,1)\qquad\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1^2 \\ \mathcal{X};\Gamma\Rightarrow\Sigma_2,\Theta\end{matrix}\quad\begin{matrix}\vdots\ \mathcal{D}_2' \\ \mathcal{X};\Delta,\phi\Rightarrow\Pi\end{matrix}}{\mathcal{X};\Gamma,\Delta,\phi\Rightarrow\Sigma_2,\Pi,\Theta}\ (,1)}{\mathcal{X};\Gamma,\Delta,\phi\Rightarrow\Sigma_1,\Sigma_2,\Pi}\ (,2)}{\mathcal{X};\Gamma,\Delta,\neg\neg\phi\Rightarrow\Sigma_1,\Sigma_2,\Pi}\ (\neg\neg R).$$

Since $max(|\mathcal{D}_1^1|,|\mathcal{D}_2'|), max(|\mathcal{D}_1^2|,|\mathcal{D}_2'|) < max(|\mathcal{D}_1|,|\mathcal{D}_2|)$, we can derive the conclusions of the $(,1)$-applications without $(,1)$ by the induction hypothesis, giving us a $(,1)$-free derivation of the final sequent. We leave verifying the remaining cases to the interested reader. □

Note that if we hadn't absorbed $(Weak)$, the argument wouldn't straightforwardly go through, since $(,1)$ requires shared contexts in the premises and $(Weak)$ applied backwards might delete formulas from the context.

This leaves us with $(,2)$ as the last remaining structural rule. Unfortunately, however, it turns out that $(,2)$ cannot be eliminated from $\mathsf{G}_{FJ'}$. To see this, consider the following derivation:

$$\dfrac{p;p,q,r\Rightarrow p\quad p;p,q,r\Rightarrow p,q,r}{p;p,q,r\Rightarrow p,q}\ (,2).$$

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A very simple inductive argument on the height of derivations without $(,2)$ establishes that $p; p, q, r \Rightarrow p, q$ is not derivable without the rule. Given Propositions 6.2 and 6.3, we can focus our attention on derivations without $(Weak)$ and $(,1)$. Now, clearly $p; p, q, r \Rightarrow p, q$ is not a logical axiom. And since all the remaining rules of $\mathsf{G}_{FJ'}$ introduce connectives and $p; p, q, r \Rightarrow p, q$ is connective free, the rules couldn't have been used to derive the sequent. Since G_{FJ} and $\mathsf{G}_{FJ'}$ are deductively equivalent, the sequent is not derivable in the original calculus without $(,2)$ either. In short, $(,2)$ is not eliminable G_{FJ}.

In fact, we can use the above derivation to show that G_{FJ} doesn't enjoy the subformula property, i.e., it's not the case that every formula that occurs in a derivation is a subformula of a formula in the derived sequent. Consider:

$$\dfrac{p; p, q, r \Rightarrow p \qquad \dfrac{p; p, q, r \Rightarrow p, q, r}{p; p, q, r \Rightarrow p, q, r \vee s}\,(\vee R_1)}{p; p, q, r \Rightarrow p, q}\,(,2).$$

The formula $r \vee s$, which we first introduce via $(\vee R_1)$ only to immediately delete it via $(,2)$, does not occur as a subformula in the derived sequent $p; p, q, r \Rightarrow p, q$. Hence, the subformula property fails.

An unfortunate consequence of this observation is that the Fine–Jago calculus doesn't allow for proof searches. In general, the idea of a proof search algorithm is to consider all the possible ways in which a given sequent could have been derived. In a calculus that absorbs all the structural rules and enjoys the subformula property, the search-space is finite. To see this, note that by the subformula property, we can restrict our attention to derivations involving only subformulas of formulas in the sequent. Since the structural rules are absorbed, the only way in which a sequent can be derived is either as an axiom or by means of a left or right rule applied to the immediate subformulas of a formula in the sequent. Putting these two observations together, it follows that the search space is finite. In fact, the height of a derivation of a sequent is bounded by the maximum complexity of the formulas in the sequent.

In the presence of $(,2)$, however, this observation fails. Since $(,2)$ deletes formulas from a derivation, a derivation may involve formulas which don't occur in the final sequent. In fact, since we can always first introduce some formula and then delete it via a (superfluous) application of $(,2)$, the height of derivations of a sequent is not finitely bound. Since there are sequents that can only be derived using $(,2)$, this means that a search algorithm may fail to terminate.

It's worth thinking about the role that $(,2)$ plays in the Fine–Jago calculus. Essentially what it allows us to do is to derive instances of $\wedge$-Convexity. To see this, suppose that $\Gamma \subseteq \Delta \subseteq \Sigma$. We get the following derivation:

$$\dfrac{\dfrac{\Gamma; \Sigma \Rightarrow \Gamma \qquad \Gamma; \Sigma \Rightarrow \overbrace{\Delta \cup (\Sigma \setminus \Delta)}^{=\Sigma}}{\Gamma; \Sigma \Rightarrow \underbrace{\Gamma \cup \Delta}_{=\Delta}}\,(,2)}{\bigwedge \Gamma; \bigwedge \Sigma \Rightarrow \bigwedge \Delta}\;\vdots\text{ multiple applications of } \wedge L \text{ and } \wedge R.$$

Note especially that $\Gamma; \Sigma \Rightarrow \Gamma$ and $\Gamma; \Sigma \Rightarrow \Delta \cup (\Sigma \setminus \Delta)$ are axioms. Note further that no rules other than $(,2)$ and $(\wedge L)$ and $(\wedge R)$ are used in this derivation. Since the use of

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$(\wedge L)$ and $(\wedge R)$ in the derivation is simply to make the reading of the sequent $\Gamma, \Sigma \Rightarrow \Delta$ explicit, this indicates that the role of $(,2)$ is precisely to derive $\wedge$-Convexity.

At this point, we have a couple of options if we wish to develop a sequent calculus with the subformula property. The first would be to rely on a formulation of $(,2)$ that doesn't eliminate formulas. An example of such a rule would be

$$\frac{\mathcal{X}; \Gamma; \Gamma, \Sigma \Rightarrow \Pi}{\mathcal{X}; \Gamma; \Gamma, \Sigma, \Delta \Rightarrow \Pi}\ (,2').$$

The idea is that rather than *deleting* formulas on the right, we can just *introduce* formulas on the left. It's straightforward to see that $(,2)$ and $(,2')$ are inter-derivable:

- $(,2) \Rightarrow (,2')$

$$\dfrac{\dfrac{\mathcal{X}; \Gamma, \Sigma, \Delta; \Gamma \Rightarrow \Gamma \quad \mathcal{X}; \Gamma; \Gamma, \Sigma, \Delta \Rightarrow \Gamma, \Sigma, \Delta}{\mathcal{X}; \Gamma; \Gamma, \Sigma, \Delta \Rightarrow \Gamma, \Sigma}\ (,2) \quad \mathcal{X}; \Gamma; \Gamma, \Sigma \Rightarrow \Pi}{\mathcal{X}; \Gamma; \Gamma, \Sigma \Rightarrow \Pi}\ Cut_{\mathsf{G}_{FJ}}.$$

- $(,2') \Rightarrow (,2)$

$$\dfrac{\mathcal{X} \Rightarrow \Gamma \quad \dfrac{\dfrac{\mathcal{X} \Rightarrow \Gamma \quad \mathcal{X} \Rightarrow \Sigma, \Delta}{\mathcal{X} \Rightarrow \Gamma, \Sigma, \Delta}\ (,1) \quad \dfrac{\mathcal{X}; \Gamma; \Gamma, \Sigma \Rightarrow \Gamma, \Sigma}{\mathcal{X}; \Gamma; \Gamma, \Sigma, \Delta \Rightarrow \Gamma, \Sigma}\ (,2')}{\mathcal{X}; \Gamma \Rightarrow \Gamma, \Sigma}\ Cut_{\mathsf{G}_{FJ}}}{\mathcal{X} \Rightarrow \Gamma, \Sigma}\ Cut_{\mathsf{G}_{FJ}}.$$

Since $(,2')$ doesn't eliminate formulas, there would no longer be formula-deleting rules were we to replace $(,2)$ with $(,2')$. Consequently, the resulting calculus would indeed have the subformula property—as desired.

While we do obtain a calculus with the subformula property in this way, from a proof-theoretic perspective, the resulting calculus is still not ideal. It's easily seen, via an analogous argument as for $(,2)$, that the rule $(,2')$ would not be eliminable. Just consider the derivation

$$\frac{p; p, q \Rightarrow p, q}{p; p, q, r \Rightarrow p, q}\ (,2').$$

For the same reasons why this sequent wasn't derivable without $(,2)$, it is not derivable without $(,2')$. Though $(,2')$ is not problematic in itself, just like $(Weak)$, the rule is not ideal for proof-searches. We shall now develop a calculus that absorbs *all* structural rules. It turns out that we can use the same idea as in the case of $(Weak)$: we can absorb $(,2)$ on the axiomatic level.

To further facilitate proof searches, we shall move to a multi-conclusion sequent calculus, the benefits of which we shall reap in our completeness proof. Correspondingly, our sequents shall now be of the form $\mathcal{X} \Rightarrow \mathcal{Y}$. The notational conventions for sequents carry over to these sequents in a straightforward way. The intended reading of a sequent of the form $\Gamma_1; \dots; \Gamma_n \Rightarrow \Delta_1; \dots; \Delta_m$ is that $\bigwedge \Gamma_1, \dots, \bigwedge \Gamma_n \vDash \bigwedge \Delta_1 \vee \dots \vee \bigwedge \Delta_m$. That is, we read sets of formulas conjunctively as before, but the reading of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{X} \Rightarrow \mathcal{Y}$ is different: while the former is read distributively, the latter is read disjunctively.[24]

[24] Note that this works because of different behaviors of conjunction and disjunction with respect to exact entailment. While conjunction behaves "intensionally," as it were, disjunction behaves "extensionally" (cf. [20]), i.e., conjunction satisfies the usual rule of conjunction

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In our new setting, the following structural rules are absorbed in notation:

$$\frac{\mathcal{X};\Gamma,\phi,\phi\Rightarrow\mathcal{Y}}{\mathcal{X};\Gamma,\phi\Rightarrow\mathcal{Y}}\ (W,L)\qquad\frac{\mathcal{X}\Rightarrow\mathcal{Y};\Delta,\phi,\phi}{\mathcal{X};\Gamma\Rightarrow\mathcal{Y};\Delta,\phi}\ (W,R)$$

$$\frac{\mathcal{X};\Gamma;\Gamma\Rightarrow\mathcal{Y}}{\mathcal{X};\Gamma\Rightarrow\mathcal{Y}}\ (W;L)\qquad\frac{\mathcal{X}\Rightarrow\mathcal{Y};\Delta;\Delta}{\mathcal{X}\Rightarrow\mathcal{Y};\Delta}\ (W;R)$$

$$\frac{\mathcal{X}\Rightarrow\mathcal{Y};\Delta,\phi,\psi}{\mathcal{X}\Rightarrow\mathcal{Y};\Delta,\psi,\phi}\ (Ex,L)\qquad\frac{\mathcal{X};\phi,\psi,\Gamma\Rightarrow\mathcal{Y}}{\mathcal{X};\psi,\phi,\Gamma\Rightarrow\mathcal{Y}}\ (Ex,R)$$

$$\frac{\mathcal{Z};\mathcal{X}_1;\mathcal{X}_2\Rightarrow\mathcal{Y}}{\mathcal{Z};\mathcal{X}_2;\mathcal{X}_1\Rightarrow\mathcal{Y}}\ (Ex;L)\qquad\frac{\mathcal{X}\Rightarrow\mathcal{Z};\mathcal{Y}_1;\mathcal{Y}_2}{\mathcal{X}\Rightarrow\mathcal{Z};\mathcal{Y}_2;\mathcal{Y}_1}\ (Ex;R).$$

The validity of the new right rules is straightforwardly justified using the idempotence, associativity, and commutativity of disjunction.

In line with the intended reading of our sequents, we say that $\mathcal{X}\Rightarrow\mathcal{Y}$ is *valid* iff $\{\bigwedge\Gamma:\Gamma\in\mathcal{X}\}\vDash\bigvee\{\bigwedge\Delta:\Delta\in\mathcal{Y}\}$. What our calculus will do is to derive all valid sequences from valid sequences only involving literals, while essentially following the construction of the formulas from literals.

To motivate our choice of axioms, we shall begin with a couple of semantic observations. First, note that if $\Gamma\subseteq\Lambda$, then $|\bigwedge\Gamma|^+_{\mathfrak{M}}=\{\Gamma\}$, as is easily seen via (Sem-$\wedge^+$). Let's further define $cl(\mathcal{X})$, for $\mathcal{X}\subseteq\wp(\mathcal{L})$, to be the closure of $\mathcal{X}$ under $\cup$, i.e., $cl(\mathcal{X})=\bigcap\{\mathcal{Z}:\mathcal{X}\subseteq\mathcal{Z}$ and whenever $X,Y\in\mathcal{Z}$, then $X\cup Y\in\mathcal{Z}\}$. Relying on the previous observation and (Sem-$\vee^+$), we can show that if $\mathcal{Y}\subseteq\wp(\Lambda)$, then $|\bigvee_{\Delta\in\mathcal{Y}}\bigwedge\Delta|^+_{\mathfrak{M}}=cl(\mathcal{Y})$ using a straightforward induction on cardinality of $\mathcal{Y}$. Using Theorem 3.4, we're now in a position to prove:

LEMMA 6.4. *If $\mathcal{X},\mathcal{Y}\subseteq\wp(\Lambda)$, then $\mathcal{X}\vDash\mathcal{Y}$ iff there are $\Gamma\in\mathcal{X}$ and $\Delta\in cl(\mathcal{Y})$ with $\Gamma\subseteq\Delta\subseteq\bigcup\mathcal{X}$.*

Proof. Just note that by the first observation all the members of $\{|\bigwedge\Gamma|^+_{\mathfrak{M}}:\Gamma\in\mathcal{X}\}$ are singletons, so there is just one selection function for this set: the one with $f(|\Gamma|^+_{\mathfrak{M}})=\Gamma$. From here Theorem 3.4 gives the desired conclusion via the second semantic observation. □

Note that the condition that one might expect in light of the Fine–Jago characterization theorem (Theorem 3.4), viz. that there be $\Gamma\in\mathcal{X}$ and $\Delta\in\mathcal{Y}$ with $\Gamma\subseteq\Delta\subseteq\bigcup\mathcal{X}$, will not suffice to characterize all valid sequents. Take, for example, $\{p,q\}\Rightarrow\{p\},\{q\}$. This sequent is clearly valid since $p\wedge q\vDash p\vee q$ (cf. Proposition 4.7), but it doesn't satisfy the above constraint. The issue is, of course, that for an application of Theorem 3.4, we need to recruit our Δ from $|\bigvee\{\bigwedge\Delta:\Delta\in\mathcal{Y}\}|^+_{\mathfrak{M}}$, which is identical to $cl(\mathcal{Y})$ and not $\mathcal{Y}$. In our example, we can indeed pick $\{p,q\}\in cl(\{\{p\},\{q\}\})$ as our Δ, which witnesses the validity of $\{p,q\}\Rightarrow\{p\},\{q\}$ since trivially $\{p,q\}\subseteq\{p,q\}\subseteq\{p,q\}$.

introduction but not of conjunction elimination, while disjunction satisfies both the usual introduction and elimination rules. Correspondingly, in order to accommodate the conjunctive reading of the premises, we need take special precautions (viz. the multiset antecedents in the premises), while the disjunctive reading of the conclusions works "as usual."

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Now it's precisely the sequents that satisfy the constraint of Lemma 6.4 that will be the axioms of our calculus:

Logical Axioms

$$\mathcal{X} \Rightarrow \mathcal{Y},$$

where $\mathcal{X}, \mathcal{Y} \subseteq \wp(\Lambda)$ and $\mathcal{X} \vDash \mathcal{Y}$.

Note that by Lemma 6.4, the condition that characterizes our axioms, $\mathcal{X} \vDash \mathcal{Y}$ for $\mathcal{X}, \mathcal{Y} \subseteq \wp(\Lambda)$, is really just shorthand for the *syntactic* condition there are $\Gamma \in \mathcal{X}$ and $\Delta \in cl(\mathcal{Y})$ with $\Gamma \subseteq \Delta \subseteq \bigcup \mathcal{X}$.

Our calculus has no structural rules and the following logical rules:

Logical Rules

$$\frac{\mathcal{X}; \Gamma, \phi, \psi \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma, \phi \wedge \psi \Rightarrow \mathcal{Y}} \ (\wedge L_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \phi, \psi; \mathcal{Y}}{\mathcal{X} \Rightarrow \Delta, \phi \wedge \psi; \mathcal{Y}} \ (\wedge R_{\mathsf{G}})$$

$$\frac{\mathcal{X}; \Gamma, \phi \Rightarrow \mathcal{Y} \quad \mathcal{X}; \Gamma, \psi \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma, \phi \vee \psi \Rightarrow \mathcal{Y}} \ (\vee L_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \phi; \Delta, \psi; \mathcal{Y}}{\mathcal{X} \Rightarrow \Delta, \phi \vee \psi; \mathcal{Y}} \ (\vee R_{\mathsf{G}})$$

$$\frac{\mathcal{X}; \Gamma, \phi \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma, \neg\neg\phi \Rightarrow \mathcal{Y}} \ (\neg\neg L_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \phi; \mathcal{Y}}{\mathcal{X} \Rightarrow \Delta, \neg\neg\phi; \mathcal{Y}} \ (\neg\neg R_{\mathsf{G}})$$

$$\frac{\mathcal{X}; \Gamma, \neg\phi \Rightarrow \mathcal{Y} \quad \mathcal{X}; \Gamma, \neg\psi \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma, \neg(\phi \wedge \psi) \Rightarrow \mathcal{Y}} \ (\neg\wedge L_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \neg\phi; \Delta, \neg\psi; \mathcal{Y}}{\mathcal{X} \Rightarrow \Delta, \neg(\phi \wedge \psi); \mathcal{Y}} \ (\neg\wedge R_{\mathsf{G}})$$

$$\frac{\mathcal{X}; \Gamma, \neg\phi, \neg\psi \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma, \neg(\phi \vee \psi) \Rightarrow \mathcal{Y}} \ (\neg\vee L_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \Delta, \neg\phi, \neg\psi; \mathcal{Y}}{\mathcal{X} \Rightarrow \Delta, \neg(\phi \vee \psi); \mathcal{Y}} \ (\neg\vee R_{\mathsf{G}}).$$

We shall refer to this calculus as G and correspondingly denote derivability by $\mathcal{X} \vdash_{\mathsf{G}} \mathcal{Y}$.

First, soundness:

THEOREM 6.5 (Soundness for G). *If $\mathcal{X} \vdash_{\mathsf{G}} \mathcal{Y}$, then $\mathcal{X} \vDash \mathcal{Y}$.*

Proof. The soundness of the rules can, in most cases, be shown in the same way as for G_{FJ}, so we can rely on the proof provided by Fine and Jago [17, p. 554]. The only real exceptions are genuinely the new rules $\vee R_{\mathsf{G}}$ and $\neg\wedge R_{\mathsf{G}}$. The reasoning is straightforward relying on our previous proof systems (A and H). We cover $\vee R_{\mathsf{G}}$ as an example. Suppose that $\mathcal{X} \vDash \Delta, \phi; \Delta, \psi; \mathcal{Y}$. By definition, this means that $\bigwedge\{\bigwedge \Gamma : \Gamma \in \mathcal{X}\} \vDash \bigvee\{\bigwedge(\Delta \cup \{\phi\}), \bigwedge(\Delta \cup \{\psi\}), \bigwedge \Sigma : \Sigma \in \mathcal{Y}\}$. Using $\wedge/\vee$-Distribution, it's easily shown (in A or H) that $\bigwedge(\Delta \cup \{\phi\}) \vee \bigwedge(\Delta \cup \{\psi\}) \dashv\vDash \bigwedge(\Delta \cup \{\phi \vee \psi\})$. From here the claim follows quickly via Positive Replacement (applied semantically via completeness).

Since Lemma 6.4 says that all the axioms are valid and we've observed that all rules are sound, we can infer soundness of the calculus. □

There are several routes to completeness, but our calculus allows for a particularly pleasing proof, which relies on the following lemma:

LEMMA 6.6 (Invertibility of G). *For all rules of G, if the lower sequent of the rule is valid, then all its upper sequents are valid, too.*

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Proof. In most cases, the proof is completely analogous to the proof of soundness. Note, for example, that the argument we gave for the soundness of $\vee R_{\mathsf{G}}$ in the proof sketch for Theorem 6.5 also works the other way around (since we're just relying on exact equivalences and not entailments).

The only new cases are the two premise rules $\vee L_{\mathsf{G}}$ and $\neg\wedge L_{\mathsf{G}}$. The arguments are analogous, so we shall only give it for $\vee L_{\mathsf{G}}$ as an example. So assume that $\mathcal{X}; \Gamma, \phi \vee \psi \vDash \mathcal{Y}$, i.e., $\{\bigwedge\Sigma, \bigwedge(\Gamma \cup \{\phi \vee \psi\}) : \Sigma \in \mathcal{X}\} \vDash \bigvee\{\bigwedge\Delta : \Delta \in \mathcal{Y}\}$. Now note that it's straightforward to show in A using $\wedge$-Monotonicity and $\vee$-Introduction, that $\bigwedge(\Gamma \cup \{\phi\}) \vDash \bigwedge(\Gamma \cup \{\phi \vee \psi\})$ and $\bigwedge(\Gamma \cup \{\psi\}) \vDash \bigwedge(\Gamma \cup \{\phi \vee \psi\})$. Using Cut semantically, we get $\{\bigwedge\Sigma, \bigwedge(\Gamma \cup \{\phi \vee \psi\}) : \Sigma \in \mathcal{X}\} \vDash \bigvee\{\bigwedge\Delta : \Delta \in \mathcal{Y}\}$ and $\{\bigwedge\Sigma, \bigwedge(\Gamma \cup \{\phi \vee \psi\}) : \Sigma \in \mathcal{X}\} \vDash \bigvee\{\bigwedge\Delta : \Delta \in \mathcal{Y}\}$, i.e., $\mathcal{X}; \Gamma, \phi \vDash \mathcal{Y}$ and $\mathcal{X}; \Gamma, \psi \vDash \mathcal{Y}$, as desired. □

Using this insight, we are now in the position to show:

THEOREM 6.7 (Completeness for G). *If $\mathcal{X} \vDash \mathcal{Y}$, then $\mathcal{X} \vdash_{\mathsf{G}} \mathcal{Y}$.*

Proof. The proof makes use of a measure of complexity tracking the construction from literals:

- $c_\Lambda(\lambda) = 0$.
- $c_\Lambda(\phi \circ \psi) = c_\Lambda(\phi) + c_\Lambda(\psi) + 1$ for $\circ = \vee, \wedge$.
- $c_\Lambda(\neg\neg\phi) = c_\Lambda(\phi) + 1$.
- $c_\Lambda(\neg(\phi \circ \psi)) = c_\Lambda(\neg\phi) + c_\Lambda(\neg\psi) + 1$ for $\circ = \vee, \wedge$.

We proceed by induction on $c_\Lambda(\mathcal{X} \Rightarrow \mathcal{Y}) := \sum_{\phi \in \bigcup(\mathcal{X} \cup \mathcal{Y})} c_\Lambda(\phi)$.

For the base case, note that $\sum_{\phi \in \bigcup(\mathcal{X} \cup \mathcal{Y})} c_\Lambda(\phi)$ iff $c_\Lambda(\phi) = 0$ for all $\phi \in \bigcup(\mathcal{X} \cup \mathcal{Y})$, i.e., all these ϕ's are literals. But this just means that $\mathcal{X}, \mathcal{Y} \subseteq \Lambda$ and so, by Lemma 6.4, $\mathcal{X} \vDash \mathcal{Y}$ iff it's an initial sequent, and thus provable.

For the induction step, assume that $\sum_{\phi \in \bigcup(\mathcal{X} \cup \mathcal{Y})} c_\Lambda(\phi) = n$. Now pick any formula $\phi \in \cup(\mathcal{X} \cup \mathcal{Y})$ with $c_\Lambda(\phi) \neq 0$ (such a ϕ will exist by the assumption that $c_\Lambda(\mathcal{X} \Rightarrow \mathcal{Y}) = n > 0$). What we'll do is to backwards apply a suitable rule of our calculus to get valid (Lemma 6.6) sequents of a lower measure which by the induction hypothesis will be provable. Then we simply apply the rule forwards to get a prove of our sequent.

To illustrate, assume that $\phi = \neg(\phi_1 \wedge \phi_2)$ and $\Gamma \cup \{\phi\} \in \mathcal{X}$, i.e., our sequent is of the form $\mathcal{X}'; \Gamma, \neg(\phi_1 \vee \phi_2) \Rightarrow \mathcal{Y}$. Since $\mathcal{X}'; \Gamma, \neg(\phi_1 \vee \phi_2) \vDash \mathcal{Y}$, applying Lemma 6.6 with $(\neg\vee L_{\mathsf{G}})$, we get that $\mathcal{X}'; \Gamma, \neg\phi_1 \vDash \mathcal{Y}$ and $\mathcal{X}'; \Gamma, \neg\phi_2 \vDash \mathcal{Y}$. Since $c_\Lambda(\mathcal{X}'; \Gamma, \neg\phi_i \vDash \mathcal{Y}) = n - 1$, our induction hypothesis gives us $\mathcal{X}'; \Gamma, \neg\phi_i \vdash_{\mathsf{G}} \mathcal{Y}$ with witnessing derivations $\mathcal{D}_i$. We get

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ \mathcal{X}'; \Gamma, \neg\phi_1 \Rightarrow \mathcal{Y}\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ \mathcal{X}'; \Gamma, \neg\phi_2 \Rightarrow \mathcal{Y}\end{array}}{\mathcal{X}'; \Gamma, \neg(\phi_1 \vee \phi_2) \Rightarrow \mathcal{Y}}\ \neg\vee L_{\mathsf{G}}.$$

We leave verifying the remaining cases to the interested reader. □

Note that our completeness proof is essentially just a convenient proof-search method, based on invertability. Indeed, we can look at the method described in the proof as a concrete implementation (and slight simplification) of the decision procedure sketched by Fine and Jago [17, p. 547]: each leaf in our proof-search corresponds to a selection function for our initial sequent, and the leaves are axioms just in case they satisfy the condition of the Fine–Jago theorem.

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THEOREM 6.8. *The logic of exact entailment on the inclusive semantics is decidable.*

Proof. Just observe that the proof-search in the proof of Theorem 6.7 must terminate after $c_\Lambda(\mathcal{X} \Rightarrow \mathcal{Y})$-many steps. $\square$

Note also that our calculus absorbs the relevant structural rules:

THEOREM 6.9 (Admissibility of the structural rules). *The following rules are admissible in the sense that if their upper sequents are derivable, then their lower sequent is derivable, too*:

$$\frac{\mathcal{X} \Rightarrow \mathcal{Y}}{\mathcal{X}; \Gamma \Rightarrow \mathcal{Y}}\ (;\, WeakL_{\mathsf{G}}) \qquad \frac{\mathcal{X} \Rightarrow \mathcal{Y}}{\mathcal{X} \Rightarrow \Gamma; \mathcal{Y}}\ (;\, WeakR_{\mathsf{G}})$$

$$\frac{\mathcal{X}; \Gamma \Rightarrow \Sigma; \mathcal{Y} \quad \mathcal{X}; \Delta \Rightarrow \Pi; \mathcal{Y}}{\mathcal{X}; \Gamma, \Delta \Rightarrow \Sigma, \Pi; \mathcal{Y}}\ (,1_{\mathsf{G}}) \quad \frac{\mathcal{X} \Rightarrow \Gamma; \mathcal{Y} \quad \mathcal{X} \Rightarrow \Delta, \Sigma; \mathcal{Y}}{\mathcal{X} \Rightarrow \Gamma, \Delta; \mathcal{Y}}\ (,2_{\mathsf{G}})$$

$$\frac{\mathcal{X} \Rightarrow \Gamma; \mathcal{Y} \quad \mathcal{X}; \Gamma \Rightarrow \mathcal{Y}}{\mathcal{X} \Rightarrow \mathcal{Y}}\ (Cut_{\mathsf{G}}).$$

Proof. Simply note that the rules are validity preserving, and the claim then follows by completeness. $\square$

It is, in fact, also possible to prove the admissibility of the structural rules using genuinely proof-theoretic methods. The idea is, in each case, that applications of the structural rules can be pushed to the axioms (by means of an induction on the height of derivations) where they are very easily absorbed. Note, in particular, that this also works for $(,2_{\mathsf{G}})$, which wasn't the case with $(,2)$ in the original Fine–Jago calculus. Since the arguments are more or less standard (modulo our new notation), we shall leave verifying these claims to the interested reader.

We shall conclude this section by extending G to the replete semantics. Luckily, this is rather simple: we just need to add certain logical axioms. First, we straightforwardly extend the notion of validity to the replete semantics by setting $\mathcal{X} \vDash_{nv*} \mathcal{Y}$ iff $\{\bigwedge \Gamma : \Gamma \in \mathcal{X}\} \vDash_{nv*} \bigvee\{\bigwedge \Delta : \Delta \in \mathcal{Y}\}$.

LEMMA 6.10. *If $\mathcal{X}, \mathcal{Y} \subseteq \wp(\Lambda)$, then $\mathcal{X} \vDash_{nv*} \mathcal{Y}$ iff there are $\Gamma, \Delta \subseteq \mathcal{L}$, such that $\Gamma \in \mathcal{X}$ and there is a $\Sigma \in \mathcal{Y}$ with $\Sigma \subseteq \Delta \subseteq \bigcup \mathcal{Y}$ and $\Gamma \subseteq \Delta \subseteq \bigcup \mathcal{X}$.*

Proof. The proof is via the Fine–Jago theorem for the replete semantics (Theorem 3.6) using some semantic facts. Remember from the proof of Lemma 6.4, that if $\Gamma \subseteq \Lambda$, then $|\bigwedge \Gamma|^+_{\mathfrak{M}} = \{\Gamma\}$. Since $|\bigwedge \Gamma|^+_{\mathfrak{M}}$ is a singleton set, it follows that $(|\bigwedge \Gamma|^+_{\mathfrak{M}})_* = |\bigwedge \Gamma|^+_{\mathfrak{M}}$. It follows that there is only one selection function for $\{(|\bigwedge \Gamma|^+_{\mathfrak{M}})_* : \Gamma \in \mathcal{X}\}$, the one with $f((|\Gamma|^+_{\mathfrak{M}})_*) = \Gamma$. The main difference to before is that we need to consider the convex closure $(|\bigvee_{\Delta \in \mathcal{Y}} \bigwedge \Delta|^+_{\mathfrak{M}})_*$ of $|\bigvee_{\Delta \in \mathcal{Y}} \bigwedge \Delta|^+_{\mathfrak{M}} = \{\Delta_1 \cup \Delta_2 : \Delta_i \in \mathcal{Y}\}$. But it's easily checked that

$$\left(|\bigvee_{\Delta \in \mathcal{Y}} \bigwedge \Delta|^+_{\mathfrak{M}}\right)_* = \{\Delta : \exists \Sigma \in \mathcal{Y} \text{ with } \Sigma \subseteq \Delta \subseteq \bigcup \mathcal{Y}\}.$$

From these observations, our claim follows by Theorem 3.6. $\square$

We define the calculus G_{nv*} to be the calculus where the axioms are all the valid sequents $\mathcal{X} \Rightarrow \mathcal{Y}$ with $\mathcal{X}, \mathcal{Y} \subseteq \wp(\Lambda)$ such that $\mathcal{X} \vDash_{nv*} \mathcal{Y}$. The rules remain the same as in G.

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To obtain soundness and completeness, we observe that our arguments for the validity and invertability of our rules go through on the replete semantics, as well.

THEOREM 6.11 (Soundness for G_{nv*}). *If* $\mathcal{X} \vdash_{\mathsf{G}_{nv*}} \mathcal{Y}$, *then* $\mathcal{X} \vDash_{nv*} \mathcal{Y}$.

Proof. Analogous to the proof of Theorem 6.5. Note, in particular, that the argument using $\wedge/\vee$-Distribution we provided also goes through on the replete semantics. Note further that our the soundness of $\vee R_{\mathsf{G}}$ on the replete semantics can be established straightforwardly relying on the soundness of $\vee$-Elimination$_{\mathsf{A}}$ on the replete semantics (cf. Lemma 4.3). □

LEMMA 6.12 (Invertibility for G_{nv*}). *For all rules of* G_{nv*}, *if the lower sequent of the rule is valid* (*on the replete semantics*), *then all its upper sequents are valid* (*on the replete semantics*), *too.*

Proof. Analogous to the proof of Lemma 6.6. □

This means virtually the same completeness proof is available for G_{nv*} as for G:

THEOREM 6.13. *If* $\mathcal{X} \vDash_{nv*} \mathcal{Y}$, *then* $\mathcal{X} \vdash_{\mathsf{G}_{nv*}} \mathcal{Y}$.

We conclude with an example derivation of the crucial direction of $\vee/\wedge$-Distribution in G_{nv*} since it's both an instructive example for our proof-search method and for the axiom choice in G_{nv*}

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p; q, r \quad p, r \Rightarrow p; q, r}{p, p \vee r \Rightarrow p; q, r}\vee L_{\mathsf{G}} \quad \dfrac{q, p \Rightarrow p; q, r \quad q, r \Rightarrow p; q, r}{q, p \vee r \Rightarrow p; q, r}\vee L_{\mathsf{G}}}{p \vee q, p \vee r \Rightarrow p; q, r}\vee L_{\mathsf{G}}}{(p \vee q) \wedge (p \vee r) \Rightarrow p; q, r}\wedge L_{\mathsf{G}}}{(p \vee q) \wedge (p \vee r) \Rightarrow p; q \wedge r}\wedge R_{\mathsf{G}}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee (q \wedge r)}\vee R_{\mathsf{G}}.$$

To see that the sequents at the top are all axioms of G_{nv*} note from left to right that: $\{p\} \subseteq \{p\} \subseteq \{p, q, r\}$, $\{p\} \subseteq \{p, r\} \subseteq \{p, q, r\}$, $\{p\} \subseteq \{q, p\} \subseteq \{p, q, r\}$, and $\{q, r\} \subseteq \{q, r\} \subseteq \{p, q, r\}$.

§7. Conclusion. We have presented three proof systems for exact entailment on the inclusive semantics, all of which have natural extensions for the replete semantics:

- A is a direct axiomatization of exact entailment viewed as a relation between premise sets and conclusions. This proof system characterizes exact entailment in terms of its laws.
- H is an axiomless Hilbert system in the style Hilbert system for FDE provided by Font [18]. This system characterizes exact entailment in terms of inferences from formulas to formulas. The particularity of the system is that the inferences are essentially embedded within context formulas.
- G is a G_3-style sequent calculus, which absorbs all the structural rules and allows for proof searches. The system essentially builds valid exact entailments recursively from the entailments among sets of literals. In this way, the system displays how the connectives interact with exact entailment.

Note that the extensions for the replete semantics, A_{nv*}, H_{nv*}, and G_{nv*} are the first known proof systems for exact entailment on the replete semantics.

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In addition to the fundamental insights these systems provide into the nature of exact entailment, they have several use-cases. Here are some examples:

1. Still on the more theoretical side, the systems are promising starting points for determining the algebra of exact entailment—a project we wish to pursue in another paper.
2. By the well-known connection between exact entailment and metaphysical grounding (cf. [9]), our systems can help formulate proof systems for logics of ground.
3. The systems can be the starting point for proof systems for hyperintensional logics defined in an exact truthmaker setting. To illustrate consider the semantics for permission statements proposed by Fine [10, p. 335]. On this semantics, $P\phi$ ("ϕ is permitted") is true iff every exact truthmaker of ϕ lies within a distinguished set of *admissible* states. To obtain a proof system for the resulting logic of permission (with consequence defined as truth-preservation across all models), we combine two derivability relations, a classical relation $\vdash$ governed by the laws of classical logic and a relation $\vdash_e$ for exact entailment defined by one of our systems. We then connect these systems by the following rule:

$$\frac{\phi \vdash_e \psi}{P\psi \vdash P\phi}\ P\text{-Antitonicity}_e.$$

We leave proving completeness of the resulting system for future work, but soundness is easily seen: If $\phi \vdash_e \psi$, then for all models $\mathcal{M}$, $|\phi|^+_{\mathcal{M}} \subseteq |\psi|^+_{\mathcal{M}}$; so if $|\psi|^+_{\mathcal{M}}$ is a subset of the admissible states, so is $|\phi|^+_{\mathcal{M}}$.

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