# ON THE INFLUENCE OF NONLINEARITIES ON THE EIGENFREQUENCIES OF FIVE-MINUTE OSCILLATIONS OF THE SUN*

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**Abstract.** Fitting the results of linear normal-mode analysis of the solar five-minute oscillations to the observed $k - \omega$ diagram selects a class of models of the Sun's envelope. It is a property of all the models in this class that their convection zones are too deep to permit substantial transmission of internal $g$ modes of degree 20 or more. This is in apparent conflict with Hill and Caudell's (1979) claim to have detected such modes in the photosphere.

A proposal to resolve the conflict was made by Rosenwald and Hill (1980). They pointed out that despite the impressive agreement between linearized theory and observation, nonlinear phenomena in the solar atmosphere might influence the eigenfrequencies considerably. In particular, they suggested that a correct nonlinear analysis could predict a shallow convection zone. This paper is an enquiry into whether their hypothesis is plausible.

We construct $k - \omega$ diagrams assuming that the modes suffer local nonlinear distortions in the atmosphere that are insensitive to the amplitude of oscillation over the range of amplitudes that are observed. The effect of the nonlinearities on the eigenfrequencies is parameterized in a simple way. Taking a class of simple analytical models of the Sun's envelope, we compute the linear eigenfrequencies of one model and show that no other model can be found whose nonlinear eigenfrequencies agree with them. We show also that the nonlinear eigenfrequencies of a particular solar model with a shallow convective zone, computed with more realistic physics, cannot be made to agree with observation. We conclude, therefore, that the hypothesis of Rosenwald and Hill is unlikely to be correct.

## 1. Introduction

In the computation of solar five-minute acoustic oscillations it is usual to apply linearized theory out to some level high in the atmosphere. There boundary conditions are applied. The dynamical boundary condition normally chosen is either that the Lagrangian pressure perturbation vanishes or that the solution matches onto a causal linear eigenfunction of a plane parallel atmosphere. The causal eigenfunctions are those that

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correspond to forcing from below; they exclude the possibility of incoming waves from infinity or evanescent motion produced by pressure perturbations from above.

The eigenfrequencies of the five-minute oscillations, excepting the chromospheric modes, hardly depend on which of those boundary conditions is chosen. This is so even if the boundary condition is applied no higher in the atmosphere than at the temperature minimum, as has been demonstrated for the modes of high degree by Berthomieu *et al.* (1980) and for the modes of low degree by Christensen-Dalsgaard and Gough (1981a). The reason is clear, and is almost taken for granted in modern discussions of stellar pulsation (e.g., Unno *et al.*, 1979; Cox, 1980): provided its eigenfrequency is substantially below the value of Lamb's acoustical cut-off frequency characteristic of the photospheric regions, a mode cannot propagate through the atmosphere, and the motion in the subphotospheric zone of propagation, where the oscillation is mainly controlled, is essentially decoupled from conditions high in the atmosphere. Of course, any mode whose frequency is great enough to permit trapping in the chromosphere is liable to be influenced by the atmospheric boundary conditions. We exclude these from our discussion.

An alternative argument is the following: owing to the rapid decrease of density and pressure with height, the normal-mode equations resemble equations that have a singularity just above the level at which the boundary conditions are applied. The adiabatic wave equation, for example, admits only one regular solution at that singularity, as can easily be seen if one approximates the undisturbed solar envelope by a polytrope (Lamb, 1932). Even though pressure and density do not actually vanish in the true atmosphere, the eigenfunctions are qualitatively similar to their polytropic counterparts, and any boudary condition of the type mentioned above combines comparable amounts of the singular and regular solutions. In the propagation zone beneath the photosphere the singular solution has declined almost to zero. Thus in the region where the dynamics is controlled, the eigenfunction is essentially independent of the amount of singular solution admitted by the boundary conditions, and is therefore indistinguishable from the regular solution. For this reason it is common practice in stellar pulsation theory to dispense with the upper atmosphere, and simply select what is apparently the regular solution.

These arguments rely on the validity of the linearization of the equations of motion. The amplitudes of the solar modes are low enough that linearized theory should be a good approximation beneath the photosphere. However, nonlinear processes are unlikely to be negligible in the atmosphere. Hill *et al.* (1978) have argued that nonlinear distortions of the eigenfunctions provide a substantial contribution to the oscillatory signal in the SCLERA diameter measurements, and Stebbins *et al.* (1980) have found that the height-dependence of the radial velocity amplitude deviates substantially from the prediction of linear theory (though in the opposite sense to that inferred by Hill *et al.*, 1978). What has not been established, however, is whether such distortions in the evanescent region are sufficient to influence the eigenfrequencies.

Comparisons of the eigenfrequencies of linear theory with observations of five-minute oscillations are leading us to a model of the solar interior (Berthomieu *et al.*, 1980;

Lubow *et al.*, 1980; Christensen-Dalsgaard and Gough, 1981a, b). An important feature of this model is that the convection zone extends through some thirty percent, by radius. This is roughly equal to, though somewhat deeper than, what had commonly been inferred from stellar evolution calculations. However, it presents an embarassment to an interpretation of some of the SCLERA diameter measurements, and to that we now turn our attention.

Hill (1978) and Hill and Caudell (1979) have claimed to have detected oscillation modes of degree between 20 and 40 with periods of 45 min and 66 min. At first sight the most obvious interpretation of such oscillations is that they are *g* modes trapped in the radiative interior. However, if the convection zone really is a deep as the five-minute oscillations indicate, it would attenuate high-degree *g* modes so severely that the observed amplitudes in the atmosphere would imply apparently implausible values in the radiative interior (Dziembowski and Pamjatnykh, 1978). If, on the other hand, one were to adopt a model with a shallow convection zone, one would not be faced with this difficulty. Moreover, one might even predict a neutrino flux in agreement with observation (Christensen-Dalsgaard *et al.*, 1979). However, it would still not be easy to explain the precise values of the frequencies of the oscillations detected by Hill and Caudell (1979) using linearized normal-mode analysis (Christensen-Dalsgaard *et al.*, 1980); and furthermore, the agreement between theory and observation would be destroyed for the five-minute oscillations.

It has been postulated by Rosenwald and Hill (1980) that the root of the problem lies in the linearization of the dynamics. If nonlinear processes are important in the upper part of the atmosphere it is not meaningful to apply boundary conditions to linear eigenfunctions there, however plausible the boundary conditions themselves might be. Consequently it is not out of the question that the ability of linearized theory to reproduce the observations of the five-minute oscillations is merely fortuitous. If that were the case we would have to rescind our conclusions about the structure of the Sun. It is therefore important to assess the credibility of the postulate. That is the purpose of the investigation reported in this paper.

## 2. Formulation of the Problem

We first assume that linearized theory is a good representation of the oscillations beneath the photosphere, as did Rosenwald and Hill (1980). We also adopt the adiabatic approximation; the computations of Berthomieu *et al.* (1980) provide adequate justification for that. We confine attention to modes of high degree $l$. These decay rapidly with depth, provided they are taken to be regular at the centre of the Sun. Therefore it suffices to consider just the outer part of the envelope of the Sun, extending to a depth no greater than half the solar radius. Moreover, perturbations to the gravitational potential can be neglected.

Subject to these approximations the linearized normal mode equations reduce to a second-order ordinary differential system (e.g. Unno *et al.*, 1979; Cox, 1980). Normally, one homogeneous boundary condition is applied at the base of the envelope, and another at the surface. Most commonly, the displacement eigenfunction is taken to vanish at the base of the envelope, though sometimes the solution is matched onto an asymptotic representation of the solution that decays with depth. We shall not address here the issue of whether the solution that is singular at the centre of the Sun can also be present.

It is convenient for our discussion to take the outer boundary as the photosphere. In purely linear theory the boundary condition that must be applied there is simply that the envelope eigenfunction matches smoothly onto the corresponding causal eigenfunction in the atmosphere. But if nonlinearities are important in the atmosphere, this condition must be rejected.

If that were all there is to say, a solution beneath the photosphere could be found for any frequency $\omega$, and the dispersion relation measured by Deubner *et al.* (1979), for example, could be rationalized with any model of the Sun. Therefore, if the class of acceptable solar models is to be restricted, the solutions of the oscillation equations must be constrained further.

Beneath the photosphere ($r < R$) we represent the perturbation by

$$\mathbf{y}(\mathbf{r}, t) = (y_1, y_2) \equiv \mathrm{Re}\,\{[r^{-1}\xi(r), (g\rho r)^{-1}\delta p(r)]S_{lm}(\theta, \phi)e^{i\omega t}\} \tag{2.1}$$

with respect to spherical polar co-ordinates $(r, \theta, \phi)$, where $g(r)$ is the gravitational acceleration, $\rho(r)$ is the undisturbed density, $\xi$ and $\delta p$ are the oscillation eigenfunctions representing vertical displacement and Lagrangian pressure perturbation, $S_{lm}$ is a tesseral harmonic of degree $l$ and order $m$, and $t$ is time. Above the photosphere we represent the two linearly independent solutions by

$$\mathbf{y}^- = (y_1^-, y_2^-), \qquad \mathbf{y}^+ = (y_1^+, y_2^+)\,. \tag{2.2}$$

These are what Hill *et al.* (1978) call $\beta_-$ and $\beta_+$, and are respectively the causal and noncausal linear eigenfunctions. We suppose them to be normalized such that $y_1^-(R) = 1$ and $y_1^+(R) = 1$. In the spirit of Rosenwald and Hill (1980) we now introduce a parameter $\lambda$ such that the actual nonlinear atmospheric solution evaluated at the photosphere ($r = R$) is

$$\mathbf{y}_\lambda = y_1(R)[(1-\lambda)\mathbf{y}^- + \lambda\mathbf{y}^+]\,. \tag{2.3}$$

Continuity of displacement and Lagrangian pressure perturbation at the photosphere is then given by

$$\mathbf{y} = \mathbf{y}_\lambda \quad \text{at} \quad r = R\,. \tag{2.4}$$

Notice that $\lambda$ merely parameterizes the result of performing an appropriate nonlinear analysis of the atmosphere, the details of which we do not specify. The ratio $\lambda/(1-\lambda)$ of the contributions from $\mathbf{y}^+$ and $\mathbf{y}^-$ at the photosphere can thus be regarded as a formal measure of the degree of nonlinearity. Clearly a value of $\lambda$ exists for any frequency $\omega$. Linear theory results when $\lambda = 0$. Our task now is to determine what reasonable restrictions on $\lambda$ should be imposed.

It is certainly plausible that nonlinear processes in the atmosphere should depend on the frequency $\omega$, the horizontal wavenumber $k = [l(l+1)]^{1/2}R^{-1}$ and, of course, the amplitude $A$ of the oscillations. However, since the scale height of the atmosphere is much less than $k^{-1}$, any localized nonlinearity cannot depend on $k$; in the atmosphere all $p$ modes resemble radial oscillations. Moreover, the observations of Deubner *et al.* (1979) of five-minute oscillations of high degree, and the observations reported by Grec *et al.* (1980) of five-minute oscillations of low degree (Fossat and Grec, private communication) show no systematic frequency variation with changing amplitude*. Therefore, provided we confine our attention to those modes whose amplitudes are large enough to have been detected, we may safely infer that $\lambda$ is a function of $\omega$ alone. Notice that the assumption about the dependence of $\lambda$ on $A$ is not that $\lambda$ is independent of $A$ for all $A$, but simply that its value does not vary substantially in the range of interest. In the analysis that follows, that value is permitted to be quite different from the value at $A = 0$.

We recall that the observed diagnostic $(k - \omega)$ diagram can be reproduced by the linear eigenfrequencies of a solar model with a substantial convection zone (Berthomieu *et al.*, 1980; Lubow *et al.*, 1980). This model we call model A. The question we must answer is whether for a model with a shallow convection zone (which we call model C), whose linear eigenfrequencies do not agree with observation, a function $\lambda(\omega)$ exists such that

$$\omega_n(k, 0; \mathrm{A}) = \omega_n(k, \lambda; \mathrm{C}) \tag{2.5}$$

for all $n$ and $k$, where $\omega_n(k, \lambda; \mathrm{M})$ denotes the $f$-mode eigenfrequency ($n = 0$) or $p$-mode eigenfrequency of order $n > 0$ of oscillations of solar model M, having horizontal wavenumber $k$ in the photosphere and computed with linearized theory beneath the photosphere subject to the boundary condition (2.4). If such a function were to exist, it would not be implausible that model C is a good model of the Sun; then the postulate of Rosenwald and Hill would deserve further consideration.

## 3. A Simple Illustrative Example

We first approximate the outer layers of the Sun by an isothermal atmosphere of perfect gas supported by a polytrope of index $\mu$. Because we are restricting attention to modes of high degree, we also make the plane-parallel approximation, and take the gravitational acceleration $g$ to be constant. With respect to Cartesian co-ordinates $\mathbf{x} \equiv (x, y, z)$, with $z$ increasing downwards, the equilibrium state is defined by:

$$p = p_0 e^{z/H}, \qquad \rho = \rho_0 e^{z/H} \tag{3.1}$$

when $z < 0$, and

$$p = p_0(1 + z/z_0)^{\mu+1}, \qquad \rho = (\mu + 1)(gz_0)^{-1}p_0(1 + z/z_0)^{\mu} \tag{3.2}$$

* We recognise that some of the apparent amplitude variation observed is a product of interference between modes with similar frequencies. That does not alter our conclusion.

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when $z > 0$, where $p$ and $\rho$ are pressure and density, $H = p_0/g\rho_0$ is the scale height of the isothermal atmosphere, and $p_0$, $\rho_0$ and $z_0$ are constants.

The constant $z_0$ is a measure of the depth of the transition between the isothermal and polytropic regions of the model; if the atmosphere were absent, $p$ and $\rho$ would vanish at $z = -z_0$. Notice that we have not imposed the condition $(\mu + 1)p_0 = g\rho_0 z_0$, which would imply continuity of density, and hence of temperature, at $z = 0$. We have in mind treating the superadiabatic boundary layer at the top of the solar convection zone simply as a temperature discontinuity. Our justification for so doing is the insensitivity of $p$-mode eigenfrequencies to details of the structure of that boundary layer (Berthomieu *et al.*, 1980); it is upon the temperature jump across that boundary layer that the eigenfrequencies depend. Thus we assume that the polytrope is adiabatically stratified, taking $\mu = (\gamma - 1)^{-1}$, where $\gamma$ is the adiabatic exponent $(\partial \ln p/\partial \ln p)_{\mathrm{ad}}$ of the gas. Notice that $z_0 = (\mu + 1)(T_0/T_a)H$, where $T_0$ is the limit of the temperature in the polytrope as $z \to 0$, and $T_a$ is the temperature of the isothermal atmosphere. Thus $z_0$ and $H$ have similar magnitudes.

The polytropic layer is supposed to represent that part of the convection zone within which the $p$ modes are trapped. In reality $\mu$ varies, due to ionization, and the value we adopt is intended to be representative of the trapping region. Changing the structure of the superadiabatic boundary layer in a realistic solar model, which can be achieved by changing the mixing length, for example, changes both the temperature jump across that boundary layer and the temperature stratification beneath. This changes the depth of the convection zone. It also moves the ionization zones, and so changes $\mu$. Hence, in general, two distinct polytropic models should differ in both $z_0$ and $\mu$. Nevertheless, to keep the analysis simple, we consider explicitly only the effect of changing $z_0$. We do, however, allow for the possibility that $\gamma$ is different in the atmosphere and the convection zone. Thus we use $\gamma$ to specify the constant adiabatic exponent in the isothermal atmosphere, and use only $\mu$ in the adiabatically stratified polytrope beneath.

We next calculate the adiabatic normal modes of oscillation of the model. We adopt the divergence of the velocity, $\chi$, as dependent variable, and seek nontrivial separable solutions of the form

$$\chi(\mathbf{x}, t) = \mathrm{Re}[X(z)e^{ikx + i\omega t}] . \tag{3.3}$$

In so doing we are ignoring the $f$ modes. The vertical component of the associated displacement eigenfunction can be written

$$\xi(\mathbf{x}, t) = (gk^3)^{-1/2}\,\mathrm{Re}[\Xi(z)e^{ikx + i\omega t}] . \tag{3.4}$$

We observe that the equation of continuity and the linearized adiabatic equation of state together imply $i\omega\delta p = -\gamma p\chi$. Thus continuity of $\delta p$ and $\xi$ at $z = 0$ is obtained by requiring that $\Xi$ and $X$ be continuous. Derivations of the equations of motion and discussions of some aspects of their solutions for isohermal and polytropic atmospheres are given by Lamb (1932). The problem treated here is just a straightforward generalization of Lamb's analysis.

In the isothermal atmosphere, two independent solutions are

$$X_+ = e^{\kappa_+ z/H}, \qquad X_- = e^{\kappa_- z/H}, \tag{3.5}$$

where

$$\kappa_\pm = -\tfrac{1}{2}\left[\!\left[ 1 \pm \{1 - 4\gamma^{-1}[\sigma^2 - (\gamma - 1)\sigma^{-2}]Hk + 4H^2k^2\}^{1/2} \right]\!\right] \tag{3.6}$$

and $\sigma^2 = (gk)^{-1}\omega^2$. The solutions $X_+$ and $X_-$ correspond respectively to the $\beta_+$ and $\beta_-$ solutions of Hill *et al.* (1978). The associated displacement amplitudes are

$$\Xi_\pm = i\gamma\sigma(\sigma^2 - 1)^{-1}(\kappa_\pm + 1 + \sigma^{-2}Hk)e^{\kappa_\pm z/H}. \tag{3.7}$$

We confine attention to modes with $\omega < \omega_c$, where $\omega_c^2 = \gamma g/4H$ is the square of Lamb's critical cutoff frequency for radial modes in the atmosphere. Thus $\kappa_+$ and $\kappa_-$ are real.

In the polytropic envelope the disturbance is given by

$$X = e^{-\zeta}U(-\alpha, \mu + 2, 2\zeta), \tag{3.8}$$

$$\Xi = \frac{i\gamma\sigma}{(\mu + 1)(\sigma^2 - 1)}\zeta e^{-\zeta}\left[\frac{\mathrm{d}U}{\mathrm{d}\zeta} + \left(\frac{\mu + 1}{\zeta} - \sigma^{-2} - 1\right)U\right], \tag{3.9}$$

where

$$2\alpha = \mu^{-1}\sigma^2 - (\mu + 2) \tag{3.10}$$

and $\zeta = k(z_0 + z)$. Here $U$ is the confluent hypergeometric function that vanishes as $\zeta \to \infty$ (e.g. Abramowitz and Stegun, 1964).

The eigenvalues $\sigma_n$ of $\sigma$ are determined by demanding that at $\zeta = \zeta_0 \equiv kz_0$ the solutions (3.8) and (3.9) be continuous with an appropriate combination of (3.5) and (3.7). We take that combination as in (2.3). We recall that $kH \ll 1$ for modes with $\omega < \omega_c$ and that $z_0$ is comparable with $H$. Hence $\zeta_0 \ll 1$. We can therefore expand $\kappa_\pm$ in powers of $kH$, and $U$ in powers of $\zeta_0$, retaining only the leading significant terms. The result is

$$\sigma_n^2 \simeq s_n^2 - \frac{2\Gamma(\mu + n + 1)}{\mu\Gamma(\mu + 1)\Gamma(\mu + 2)\Gamma(n)} \frac{1 - K}{K}(2kz_0)^{\mu + 1}, \tag{3.11}$$

where

$$s_n^2 = 1 + 2n/\mu, \tag{3.12}$$

$$K(n, \lambda) = 1 - \lambda - [(1 - 2\lambda)\gamma^{-1}(s_n^2 - s_n^{-2}) + 2(\lambda - 1)s_n^{-2}]Hk, \tag{3.13}$$

$n$ is a positive integer and $\Gamma$ is the gamma function. The second term on the right-hand side of Equation (3.11) is the change in $\sigma_n^2$ produced by replacing a complete polytrope, extending to $z = -z_0$, with the composite model considered here. This change is small compared with $\sigma_n^2$.

We specify our reference model A by setting $z_0 = z_A$. Its dimensionless eigenfrequencies $\sigma_{An}$ are given by Equations (3.11) and (3.12) with $\lambda = 0$, and we pretend that these agree with observation. We now consider model C, with $z_0 = z_C$. Its frequencies

$\sigma_{Cn}$ are determined by choosing $\lambda(\omega)$ such that $\sigma_{CN} = \sigma_{AN}$ for some $N$, which yields, to leading order in $Hk$,

$$\lambda = -\tfrac{1}{4}[1 + (2\gamma - 1)s_N^{-4}]\left[1 - \left(\frac{z_A}{z_C}\right)^2\right]\left(\frac{\omega}{\omega_c}\right)^2 . \tag{3.14}$$

To assess how close the frequencies $\sigma_{Cn}$ of the presumed nonlinear modes are to $\sigma_{An}$, we compare their differences with the corresponding differences $\sigma_{Cn}^{(L)} - \sigma_{An}$, where $\sigma_{Cn}^{(L)}$ are the linear eigenfrequencies of model C, computed with $\lambda = 0$. The result is

$$\frac{\sigma_{Cn} - \sigma_{An}}{\sigma_{Cn}^{(L)} - \sigma_{An}} \simeq \frac{(2\gamma - 1)(s_n^{-4} - s_N^{-4})}{1 + (2\gamma - 1)s_n^{-4}} . \tag{3.15}$$

Except when $n = N$, this is not zero. Thus the frequencies $\sigma_{Cn}$ cannot be made to coincide with the linear eigenfrequencies of model A. Moreover, it is evident that the magnitude of the deviation is comparable with the difference between the linear eigenfrequencies of models A and C. This is true also if the polytropic index $\mu$ is permitted to be different in the two models. Moreover, it is not possible to vary both $z_0$ and $\mu$ in such a way as to make the frequencies of the two models agree for all values of $n$.

## 4. Analysis of a More Realistic Pair of Solar Models

Two solar envelope models were computed using the fast numerical programme described by Belvedere *et al.* (1980). The models were integrated inwards from the 'surface', where $r = R_s$ and $T = 4900$ K, to $r = R_0 \equiv R/2$, adjusting the conditions at $r = R_s$ to ensure that $r = R$ and $T = 5770$ K at an optical depth of $\frac{2}{3}$. A mesh spacing of 0.05 electron pressure scale heights was used. Model A was chosen to have abundances of helium and heavy elements: $Y, Z = 0.25, 0.02$, and a constant mixing length to pressure scale height ratio $\alpha = 1.65$ was adopted. Its convection zone was 220 Mm deep. Model C was computed with the same composition, but with $\alpha = 0.65$, and had a convection zone 40 Mm deep. The models resemble models A and C of Christensen-Dalsgaard *et al.* (1979), the main difference being in the chemical composition of model C beneath the convection zone.

The linearized adiabatic oscillation equations [Unno *et al.*, 1979, Equations (17.9), (17.10); Cox, 1980, Equations (17.50), (17.51)] were integrated outwards from $r = R_0$ using a fourth-order Runge–Kutta algorithm. At $r = R_s$ the solutions were matched onto the adiabatic solutions for a plane parallel isothermal atmosphere at $T = 4900$ K. For each integration $\omega$ was specified, and in cases where the boundary condition (2.4) had to be satisfied, the eigensolution was found by Newton–Raphson iteration on $\omega$.

First, the eigenfrequencies $\omega_{An}$ of model A were computed with $\lambda = 0$. Then, for some value $N$ of $n$, the frequencies $\omega_{Cn}$ of model C were forced to coincide with them. This, of course, required no iteration: the frequency $\omega_{CN}$ was chosen to be $\omega_{AN}$, and subsequently $\lambda(\omega)$ was computed by insisting that the photospheric condition (2.4) be

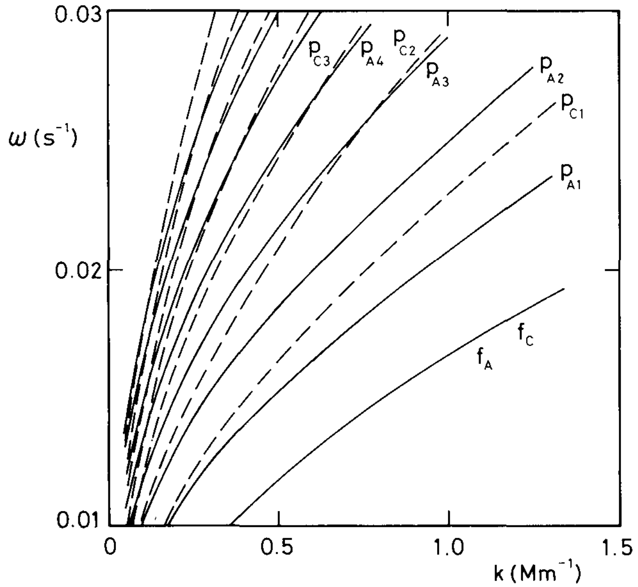


Fig. 1. Theoretical diagnostic diagrams for models A and C of the solar envelope; the frequencies $\omega$ are the eigenfrequencies of the linear normal mode analysis ($\lambda = 0$), and are expressed as continuous functions of the wave number $k$. Model A is our standard model; for the purposes of the argument presented in this paper its linear eigenfrequencies, indicated by the continuous curves, may be regarded as being in agreement with observation. Model C has a shallow convection zone; its eigenfrequencies are represented by the broken curves. In the frequency range plotted the $f$-mode frequencies of the two models differ by no more than a few parts per thousand.

satisfied. Finally, the remainder of the frequencies of model C were computed, using the same function $\lambda(\omega)$ in condition (2.4).

The linear eigenfrequencies of models A and C are shown in Figure 1. The frequencies of model A do not agree exactly with observation, due to the simplifying approximations used in the construction of the equilibrium model. Nevertheless, we shall pretend that they do. The linear $p$-mode eigenfrequencies of model C differ from observation substantially. The $f$-mode frequencies do agree, because these are essentially independent of stratification when $kR \gg 1$.

In Figure 2 we show for several orders $N$ our measures $(1 - \lambda)/\lambda$ of the nonlinearity required to make the frequencies of model C coincide with the linear eigenfrequencies of model A. Infinities of $(1 - \lambda)/\lambda$ signify a pure $\beta_-$ solution, and zeros signify a pure $\beta_+$ solution.

Finally we show in Figure 3 the complete diagnostic diagrams for model C constructed with the parameterized nonlinear boundary conditions (2.4) defined with the functions $\lambda(\omega)$ shown in Figure 2. Once again, the results are compared with the linear eigenfrequencies of model A, which we presume to be in agreement with observation. The deviations between the frequencies of the models are, in all cases, comparable with the deviations between the corresponding linear eigenfrequencies, as we found also for the simple model in Section 3.

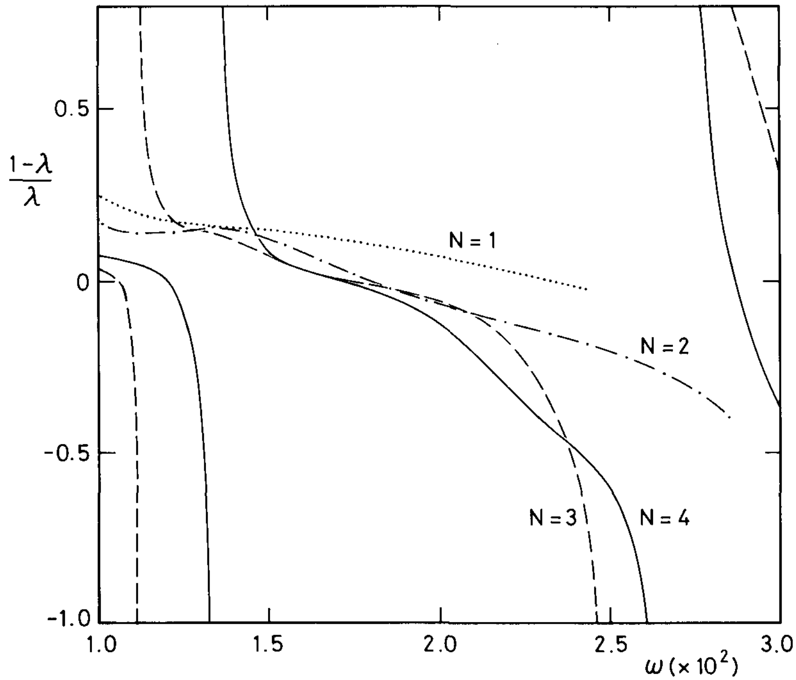


Fig. 2. Examples of the measures $(1 - \lambda)/\lambda$ of the nonlinearity that we have presumed to be present in the atmosphere of model C. They were constructed in such a way as to make the frequencies of oscillation of model C agree with the linear eigenfrequencies of model A for all $p$ modes of order $N$ as indicated. Infinities of $(1 - \lambda)/\lambda$ occur where the eigenfrequencies of models A and C intersect in Figure 1.

## 5. Summary and Conclusion

We have argued that the effects of atmospheric nonlinearities on the highest-amplitude five-minute oscillations depend only on frequency. Granted that this is so, we have shown that the oscillation frequencies of a particular solar model with a shallow convective zone, no matter what form the nonlinearities take, cannot be made to coincide with the linear eigenfrequencies of another model constructed to be in fair agreement with observation.

We have not demonstrated explicitly that this result is true for all solar models that differ substantially from our standard. Nevertheless, an analysis of a highly simplified model of the outer layers of the sun, consisting of an isothermal atmosphere on a polytropic envelope, suggests that this is indeed a general result. To find a nonlinear model whose frequencies agree with observation for $n_0$ values of the order $n$ of the modes is likely to require the adjustment of at least $n_0$ parameters of the model to which the frequencies are sensitive, and possibly more. Sensitivity analyses of the linear eigenfrequencies by Berthomieu *et al.* (1980) and Lubow *et al.* (1980) suggest that such parameters do not exist. That is why we have not pursued the matter further.

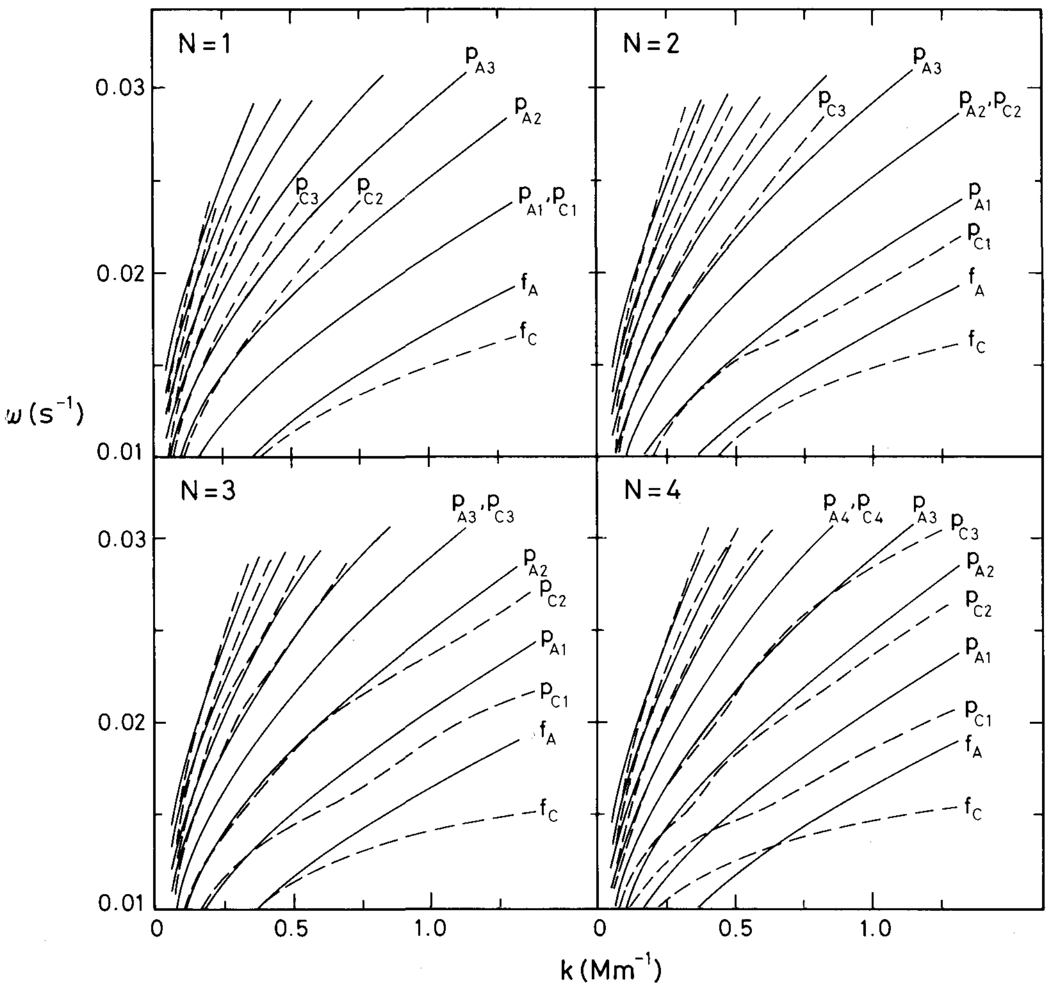


Fig. 3. Diagnostic diagrams for model C (broken curves) computed with the boundary condition (2.4) and the functions $\lambda(\omega)$ represented in Figure 2. The linear eigenfrequencies ($\lambda = 0$) of model A (continuous curves) are included for comparison.

We contend that the evidence we have provided renders it extremely unlikely that nonlinearities in the solar atmosphere can influence the five-minute oscillation eigenfrequencies to a noticeable extent, however much they may distort the eigenfunctions. We expect that this is so also for modes with longer periods, because for these the coupling between the atmosphere and the interior is even weaker that it is for the five-minute modes. Only the high-frequency modes that suffer trapping in the chromosphere are liable to be affected. Therefore we conclude that inferences concerning the internal structure of the Sun based on linear eigenfrequencies are probably correct.

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