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ASYMPTOTIC BEHAVIOUR OF SECOND-ORDER DIFFERENCE EQUATIONS

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Abstract

In this paper we prove several growth theorems for second-order difference equations.

1. Introduction

In this paper we study second-order nonlinear difference equations of the form

$$\Delta(c_{n-1}\Delta x_{n-1}) = d_n f(x_n) + g_n, \quad n \in \mathbf{N}, \tag{1.1}$$

where x_n is the desired solution, and c_n, d_n and g_n are given real sequences.

In Section 2 we give and cite several auxiliary results which we shall apply in the sections which follow.

In Section 3 we study the second-order linear difference equation

$$c_n x_{n+1} - b_n x_n + c_{n-1} x_{n-1} = 0, \quad n \in \mathbf{N}, \tag{1.2}$$

where x_n is the desired solution, and b_n and c_n are given real sequences. We investigate the asymptotic behaviour of the solutions of that equation under some conditions. We were motivated by [12] and [18].

This equation models, for example, the amplitude of oscillation of the weights on a discretely weighted vibrating string [2, p. 15–17].

A presentation of the results on similar problems for second-order differential equations can be found in [4].

In Section 4 we study the asymptotic behaviour of the second-order nonlinear difference equation (1.1).

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2. Auxiliary results

For an investigation into the asymptotic behaviour of the solution x_n, we need a few auxiliary lemmas. The first one is a discrete variant of the Bellman-Gronwall lemma. The continuous case of this lemma can be found in [3, 4] and [10]. Applications and further generalisations of this lemma can be found, for example, in [7, 11–13, 16–18, 20].

LEMMA 2.1 ([16, p. 112]). *If* $x_n, b_n, c_n \geq 0$, *and*

$$x_n \leq a_n + b_n \sum_{i=1}^{n-1} c_i x_i, \quad n \in \mathbf{N},$$

then

$$x_n \leq a_n + b_n \sum_{i=1}^{n-1} a_i c_i e^{\sum_{j=i+1}^{n-1} b_j c_j}, \quad n \in \mathbf{N}.$$

COROLLARY 2.2. *If* $x_n, c_n \geq 0$, *c is a positive constant, and*

$$x_n \leq c + \sum_{i=1}^{n-1} c_i x_i, \quad n \in \mathbf{N},$$

then

$$x_n \leq c \exp\left(\sum_{i=1}^{n-1} c_i\right), \quad n \in \mathbf{N}.$$

PROOF. By Lemma 2.1 we have

$$x_n \leq c + c \sum_{i=1}^{n-1} c_i e^{\sum_{j=i+1}^{n-1} c_j}, \quad n \in \mathbf{N}.$$

Applying the well-known inequality $x \leq e^x - 1, x \geq 0$, we obtain

$$\begin{aligned} x_n &\leq c\left(1 + \sum_{i=1}^{n-1}(e^{c_i} - 1)e^{\sum_{j=i+1}^{n-1} c_j}\right) \\ &= c\left(1 + \sum_{i=1}^{n-1}\left(e^{\sum_{j=i}^{n-1} c_j} - e^{\sum_{j=i+1}^{n-1} c_j}\right)\right) = c \exp\left(\sum_{i=1}^{n-1} c_i\right), \end{aligned}$$

as desired.

The following lemma was proved in [20].

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LEMMA 2.3. *If $x_n, c_n \geq 0$, c is a positive constant, $p \in [0, 1)$ and*

$$x_n \leq c + \sum_{i=1}^{n-1} c_i x_i^p, \quad n \in \mathbf{N},$$

then

$$x_n \leq \left(c^{1-p} + (1-p)\sum_{i=1}^{n-1} c_i\right)^{1/(1-p)}, \quad n \in \mathbf{N}.$$

The following lemma is a variant of the discrete version of Bihari's inequality [5]. This lemma generalises a discrete inequality in [11], see also [16, p. 114].

LEMMA 2.4. *Assume that x_n, a_n, b_n and c_n are positive sequences, and that a_n and b_n satisfy the conditions*

$$1 \leq \frac{a_{n+1}}{a_n} \leq M, \quad 1 \leq \frac{b_{n+1}}{b_n} \leq M, \quad n \in \mathbf{N}, \tag{2.1}$$

and

$$x_n \leq a_n + b_n \sum_{i=1}^{n-1} c_i g(x_i), \quad n \in \mathbf{N}, \tag{2.2}$$

where the real function $g(x)$ is continuous, nondecreasing and $g(x) \geq x$, for $x > 0$. Then

$$x_n \leq G^{-1}\left(G(a_1) + M \ln \frac{a_n b_n}{a_1 b_1} + \sum_{i=1}^{n-1} b_{i+1} c_i\right), \quad n \in \overline{1, n_0}, \tag{2.3}$$

where $G(u) = \int_\varepsilon^u (ds/g(s))$ and

$$n_0 = \sup\left\{ j \;\middle|\; G(a_1) + M \ln \frac{a_j b_j}{a_1 b_1} + \sum_{i=1}^{j-1} b_{i+1} c_i \in G(\mathbf{R}_+) \right\}.$$

PROOF. Let $R_n = b_n \sum_{i=1}^{n-1} c_i g(x_i)$, $s_n = \sum_{i=1}^{n} c_i g(x_i)$ and $v_n = R_n + a_n$. We can write (2.2) in the following form: $x_n \leq a_n + R_n$, $n \in \mathbf{N}$. From that we get

$$\begin{aligned} 0 \leq v_{n+1} - v_n &= b_{n+1}(s_{n-1} + c_n g(x_n)) - b_n s_{n-1} + a_{n+1} - a_n \\ &= (b_{n+1} - b_n)s_{n-1} + b_{n+1} c_n g(x_n) + a_{n+1} - a_n \\ &= \frac{b_{n+1} - b_n}{b_n} R_n + b_{n+1} c_n g(x_n) + a_{n+1} - a_n. \end{aligned} \tag{2.4}$$

By the mean value theorem we have

$$G(v_{n+1}) - G(v_n) = (v_{n+1} - v_n)/g(\zeta_n), \tag{2.5}$$

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for some $\zeta_n \in (v_n, v_{n+1})$. From (2.4) and (2.5) we obtain

$$G(v_{n+1}) - G(v_n) = \frac{1}{g(\zeta_n)}\left(\frac{b_{n+1}-b_n}{b_n}R_n + b_{n+1}c_n g(x_n) + a_{n+1} - a_n\right)$$
$$\leq \frac{b_{n+1}-b_n}{b_n} + b_{n+1}c_n + \frac{a_{n+1}-a_n}{g(a_n)}, \tag{2.6}$$

since $g(a_n) \leq g(v_n) \leq g(\zeta_n)$.

Summing (2.6) from 1 to $n-1$, we obtain

$$G(v_n) \leq G(a_1) + \sum_{i=1}^{n-1}\frac{b_{i+1}-b_i}{b_i} + \sum_{i=1}^{n-1} b_{i+1}c_i + \sum_{i=1}^{n-1}\frac{a_{i+1}-a_i}{g(a_i)}.$$

From the conditions of the theorem we get

$$G(v_n) \leq G(a_1) + M\sum_{i=1}^{n-1}\frac{b_{i+1}-b_i}{b_{i+1}} + \sum_{i=1}^{n-1} b_{i+1}c_i + M\sum_{i=1}^{n-1}\frac{a_{i+1}-a_i}{a_{i+1}}.$$

Since every positive nondecreasing sequence y_n satisfies the following inequality:

$$\sum_{i=1}^{n-1}\frac{y_{i+1}-y_i}{y_{i+1}} \leq \int_{y_1}^{y_n}\frac{dt}{t} = \ln\frac{y_n}{y_1},$$

we obtain

$$G(v_n) \leq G(a_1) + M\ln\frac{a_n b_n}{a_1 b_1} + \sum_{i=1}^{n-1} b_{i+1}c_i,$$

and so (2.3) follows.

LEMMA 2.5 ([14, p. 281]). *Let $v_n > 0$ and assume that the series $\sum_{i=1}^{+\infty} u_n$ and $\sum_{i=1}^{+\infty} v_n$ converge. Then*

$$\lim_{n\to+\infty}\frac{u_n}{v_n} = c \quad\Rightarrow\quad \lim_{n\to+\infty}\frac{\sum_{i=n}^{+\infty} u_i}{\sum_{i=n}^{+\infty} v_i} = c.$$

3. The linear equation case

We are now in a position to formulate and to prove the main results in the case of a linear equation. In what follows we exclude the trivial solution from consideration.

Observe that the difference equation $x_{n+1} - 2x_n + x_{n-1} = 0$ has a general solution in the form $x_n = an + b$ for some $a, b \in \mathbf{R}$. In the following theorem we give

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one sufficient condition, such that the difference equation (1.2) has solutions which approach those of $x_{n+1} - 2x_n + x_{n-1} = 0$. An equivalent result was proved in [6, p. 377]. We present here a different proof which follows the lines of the proof given in [4] in the continuous case. The proof essentially appears in [9, Theorem 7.17] but contains a gap. Hence we present here a correct proof.

THEOREM 3.1. *Consider* (1.2) *where* $\sum_{i=1}^{+\infty} i(|1 - c_i| + |2 - b_i|) < +\infty$. *Then the general solution is asymptotic to* $an + b$ *as* $n \to \infty$, *where* a *or* b *may be zero, but not both simultaneously.*

PROOF. Without loss of generality we may suppose $c_n > 0$, $n \in \mathbf{N} \cup \{0\}$. Let us write (1.2) in the following form:

$$\Delta(c_{n-1}\Delta x_{n-1}) = d_n x_n. \tag{3.1}$$

It is clear that $d_n = b_n - c_n - c_{n-1}$. Let $y_n = x_{n+1} - x_n$. Then from (3.1) we have

$$c_n y_n - c_{n-1} y_{n-1} = d_n x_n, \qquad n \in \mathbf{N}. \tag{3.2}$$

Summing (3.2) from 1 to $n - 1$, we obtain

$$x_n - x_{n-1} = \frac{1}{c_{n-1}} \left(c_0 y_0 + \sum_{i=1}^{n-1} d_i x_i \right). \tag{3.3}$$

Now, summing (3.3) from 1 to n, we get

$$x_n = x_0 + c_0 y_0 \sum_{i=1}^{n} \frac{1}{c_{i-1}} + \sum_{i=1}^{n} \frac{1}{c_{i-1}} \left(\sum_{j=1}^{i-1} d_j x_j \right).$$

By the condition of the theorem, $c_n \to 1$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{c_{i-1}} = 1$$

and the sequences $(|1/c_n|)$ and $\left(\left|(1/n)\sum_{i=1}^{n} 1/c_{i-1}\right|\right)$ are bounded, for example, by $M > 0$.

It follows that

$$\begin{aligned}
\frac{|x_n|}{n} &\le \frac{|x_0|}{n} + |c_0|\,|x_1 - x_0|M + \frac{M}{n}\sum_{i=1}^{n}\sum_{j=1}^{i-1}|d_j|\,|x_j| \\
&= \frac{|x_0|}{n} + |c_0|\,|x_1 - x_0|M + \frac{M}{n}\sum_{i=1}^{n}(n-i)|d_i|\,|x_i| \\
&\le |x_0| + |c_0|\,|x_1 - x_0|M + M\sum_{i=1}^{n-1} i|d_i|\frac{|x_i|}{i} \\
&\le (|x_0| + |c_0|\,|x_1 - x_0|M)\exp\left(M\sum_{i=1}^{n-1} i|d_i|\right) \\
&\le (|x_0| + |c_0|\,|x_1 - x_0|M)\exp\left(M|1 - c_0| + 3M\sum_{i=1}^{\infty} i(|1 - c_i| + |b_i - 2|)\right) \\
&= M_1 < \infty, \qquad (3.4)
\end{aligned}$$

where in the third inequality we applied Corollary 2.2.

From (3.4) we obtain

$$\sum_{i=1}^{n-1}|d_i|\,|x_i| \le M_1\sum_{i=1}^{n-1} i|d_i| \le M_1\sum_{i=1}^{+\infty} i|d_i| < \infty. \qquad (3.5)$$

By (3.3), we can conclude that there exists $\lim_{n\to\infty}(x_n - x_{n-1}) = a$. If this limit is not zero, we have $x_n \sim an$ as $n \to \infty$. In particular, $x_n \ne 0$ for sufficiently large n. To ensure that $\lim_{n\to\infty}(x_n - x_{n-1})$ is not zero, we may choose x_1 and x_0 such that

$$|c_0|\,|x_1 - x_0| - M_1\sum_{i=1}^{+\infty} i|d_i| > 0.$$

Further, we shall use the fact that

$$z_n = x_n\left(C + \sum_{i=1}^{n-1}\prod_{j=1}^{i-1}\frac{c_j x_j}{c_{j+1}x_{j+2}}\right) = x_n\left(C + x_1x_2c_1\sum_{i=1}^{n-1}\frac{1}{c_i x_i x_{i+1}}\right)$$

is another solution, linearly independent of x_n; see, for example, [15, p. 160]. Therefore

$$z_n = x_n\left(\sum_{i=n}^{+\infty}\frac{1}{c_i x_i x_{i+1}}\right)$$

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is another solution, linearly independent of x_n. It is well-defined since $x_n \neq 0$ for sufficiently large n and $x_n \sim an$ as $n \to \infty$. By Lemma 2.5, we obtain

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{x_n}{n} \lim_{n\to\infty} \frac{\left(\sum_{i=n}^{+\infty} 1/(c_i x_i x_{i+1})\right)}{1/n} = a \lim_{n\to\infty} \frac{1/(c_n x_n x_{n+1})}{1/(n(n+1))} = \frac{1}{a}.$$

Thus the solution az_n is asymptotic to 1 as $n \to \infty$, and therefore every solution of our difference equation is asymptotic to $an + b$ as $n \to \infty$.

If y_n is an arbitrary solution of (1.2) and if $\lim_{n\to+\infty} y_n$ is finite, then $y_n = cz_n$, $n \in \mathbf{N}$ for some $c \in \mathbf{R}$. Thus if $\lim_{n\to+\infty} y_n = 0$ we obtain $c = 0$, that is, y_n is a trivial solution. In the other cases $\lim_{n\to+\infty} y_n = \infty$ and so $a \neq 0$.

Before formulating the following result we would like to point out that recently W. Trench investigated principal and nonprincipal solutions of the nonoscillatory equation (3.1) in [19].

Let us investigate what happens in the case of $\sum_{i=1}^{+\infty} i|d_i| = +\infty$. The simplest case is when $d_n = 1/n^\alpha$, $\alpha \in (0, 2]$ and $c_n = 1$ for all $n \in \mathbf{N}$.

THEOREM 3.2. *Consider the equation*

$$x_{n+1} - 2x_n + x_{n-1} = d_n x_n, \tag{3.6}$$

where $d_i = c/i^\alpha$, $i \in \mathbf{N}$, $c \in \mathbf{R}$, $\alpha \in (0, 2]$. *Then for every solution of* (3.6) *the asymptotic formula*

(1) $x_n = \mathcal{O}\left(n^{|c|+1}\right)$, *for* $\alpha = 2$,
(2) $x_n = \mathcal{O}\left(ne^{|c|n^{2-\alpha}/(2-\alpha)}\right)$, *for* $\alpha \in (0, 2)$

holds.

PROOF. Let $y_n = x_{n+1} - x_n$. As in Theorem 3.1 we have

$$x_n - x_{n-1} = y_{n-1} = y_0 + c\sum_{i=1}^{n-1} \frac{1}{i^\alpha} x_i. \tag{3.7}$$

Now, summing (3.7) from 1 to n, and by a simple calculation we obtain

$$x_n = x_0 + n(x_1 - x_0) + c\sum_{i=1}^{n-1} (n-i)\frac{1}{i^\alpha} x_i. \tag{3.8}$$

It follows that

$$|x_n| \leq |x_0| + n|x_1 - x_0| + n|c|\sum_{i=1}^{n-1} \frac{1}{i^\alpha} |x_i|$$

and further

$$\frac{|x_n|}{n} \le \frac{|x_0|}{n} + |x_1 - x_0| + |c| \sum_{i=1}^{n-1} \frac{1}{i^\alpha} |x_i|$$
$$\le |x_0| + |x_1 - x_0| + |c| \sum_{i=1}^{n-1} \frac{1}{i^{\alpha-1}} \frac{|x_i|}{i}.$$

Applying the discrete Bellman-Gronwall lemma, we obtain

$$\frac{|x_n|}{n} \le (|x_0| + |x_1 - x_0|) \exp\left(|c| \sum_{i=1}^{n-1} \frac{1}{i^{\alpha-1}} \right).$$

We have

$$\sum_{i=1}^{n-1} \frac{1}{i^{\alpha-1}} \le \begin{cases} 1 + \int_1^{n-1} \frac{dt}{t^{\alpha-1}}, & \text{for } \alpha \in (1, 2], \\ \int_1^n \frac{dt}{t^{\alpha-1}}, & \text{for } \alpha \in (0, 1]. \end{cases}$$

Thus we have

$$\sum_{i=1}^{n-1} \frac{1}{i} \le 1 + \ln(n-1),$$

$$\sum_{i=1}^{n-1} \frac{1}{i^{\alpha-1}} \le \begin{cases} 1 + \frac{(n-1)^{2-\alpha} - 1}{2-\alpha}, & \text{for } \alpha \in (1, 2), \\ \int_1^n t^{1-\alpha} dt = \frac{n^{2-\alpha} - 1}{2-\alpha}, & \text{for } \alpha \in (0, 1]. \end{cases}$$

From all of the above, the result follows.

EXAMPLE 1. Consider the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \frac{2}{n^2} x_n, \quad n \ge 1.$$

This equation is derived from (3.6) by putting $c = 2$. Its solution is $x_n = n^2$.

EXAMPLE 2. Consider the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \frac{6}{n^2} x_n, \quad n \ge 1.$$

This equation is derived from (3.6) by putting $c = 6$. Its solution is $x_n = n^3$.

QUESTION 1. These two examples motivate us to conjecture that in Theorem 3.2

$$x_n = \mathcal{O}\left(n^{(1+\sqrt{4|c|+1})/2}\right)$$

holds. Is it really so and for what $c \in \mathbf{R}$ does this formula hold?

These examples show that for a fixed α the growth of the solution of (3.6) really depends on the parameter c.

THEOREM 3.3. *There exists a sequence d_n such that $\lim_{n\to+\infty} d_n = 0$, $\sum_{i=1}^{+\infty} i|d_i| = +\infty$ and for some solutions of* (3.6), *$n^k \prec |x_n|$ holds for every $k \in \mathbf{N}$.*

PROOF. Consider the equation

$$x_{n+1} - 2x_n + x_{n-1} = \frac{1}{n^\alpha}x_n, \quad \alpha \in (0, 2).$$

We know that

$$x_n = x_0 + n(x_1 - x_0) + \sum_{i=1}^{n-1}(n-i)\frac{1}{i^\alpha}x_i. \tag{3.9}$$

Let $x_0 = 0$ and $x_1 = 1$. It is easy to see that in that case $x_n \ge 0$ for every $n \in \mathbf{N}$. Thus we have $x_n \ge n$, for $n \in \mathbf{N}$. Applying this in (3.9) we obtain

$$x_n \ge n + \sum_{i=1}^{n-1}(n-i)\frac{1}{i^\alpha}i = n + n\sum_{i=1}^{n-1}\frac{1}{i^{\alpha-1}} - \sum_{i=1}^{n-1}\frac{1}{i^{\alpha-2}}.$$

Since

$$\sum_{i=1}^{n-1}\frac{1}{i^\beta} \sim \int_1^{n-1}\frac{dt}{t^\beta} \sim \frac{n^{1-\beta}}{1-\beta}, \quad \text{for } \beta \ne 1$$

we have that there is a $c_1 > 0$ such that

$$x_n \ge n + c_1 n^{3-\alpha} \ge c_1 n^{3-\alpha}, \tag{3.10}$$

for all $n \in \mathbf{N}$. Hence $n^\beta \prec |x_n|$, for $\beta < 3 - \alpha$. Applying (3.10) in (3.9) we obtain

$$x_n \ge n + c_1\sum_{i=1}^{n-1}(n-i)\frac{1}{i^\alpha}i^{3-\alpha} = n + nc_1\sum_{i=1}^{n-1}\frac{1}{i^{2\alpha-3}} - c_1\sum_{i=1}^{n-1}\frac{1}{i^{2\alpha-4}},$$

from which it follows that there is a $c_2 > 0$ such that $x_n \ge n + c_2 n^{5-2\alpha}$, for all $n \in \mathbf{N}$.

Repeating the previous procedure and by induction we obtain that for every $k \in \mathbf{N}$, there is a constant $c_k > 0$ such that

$$x_n \ge n + c_k n^{2k+1-k\alpha} \ge c_k n^{(2-\alpha)k+1},$$

for every $n \in \mathbf{N}$. From this and since $\alpha \in (0, 2)$ the result follows.

4. The nonlinear equation case

In this section we shall study the asymptotic behaviour of the second-order nonlinear difference equation (1.1).

THEOREM 4.1. *Consider* (1.1) *where*

(a) $c_n \geq \delta > 0$, $n \geq n_0$;
(b) g_n *is an arbitrary real sequence*;
(c) d_n *is a real sequence such that* $\sum_{i=1}^{+\infty} i|d_i| < +\infty$;
(d) f *is a real function such that* $|f(x)| \leq L|x|^\alpha$, $x \in \mathbf{R}$, *for some* $L > 0$ *and some* $\alpha \in [0, 1]$.

Then the following asymptotic formula holds:

$$x_n = \mathcal{O}\left(n + n\sum_{i=1}^{n-1}(n-i)|g_i|\right) \quad as\ n \to +\infty.$$

PROOF. Let $y_n = c_n(x_{n+1} - x_n)$. Then from (1.1) we have

$$y_n - y_{n-1} = d_n f(x_n) + g_n, \quad n \in \mathbf{N}. \tag{4.1}$$

As in Theorem 3.1 we can obtain

$$x_n = x_0 + c_0 y_0 \sum_{i=1}^{n} \frac{1}{c_{i-1}} + \sum_{i=1}^{n} \frac{1}{c_{i-1}} \left(\sum_{j=1}^{i-1}(d_j f(x_j) + g_j)\right).$$

By conditions (a), (d) and some simple calculations, we obtain

$$\begin{aligned}|x_n| &\leq |x_0| + n|c_0|\,|x_1 - x_0|M + M\sum_{i=1}^{n-1}(n-i)|g_i| + nM\sum_{i=1}^{n-1}|d_i|\,|f(x_i)| \\ &\leq |x_0| + n|c_0|\,|x_1 - x_0|M + M\sum_{i=1}^{n-1}(n-i)|g_i| + nML\sum_{i=1}^{n-1}|d_i|\,|x_i|^\alpha,\end{aligned}$$

where M is an upper bound for the sequence $(|1/c_n|)$.

Let $A_n = \sum_{i=1}^{n-1}(n-i)|g_i|$. By the well-known inequality $|x|^\alpha \leq 1 + |x|$, $x \in \mathbf{R}$, $\alpha \in [0, 1]$, we have

$$|x_n| + 1 \leq c(1 + n + A_n) + cn\sum_{i=1}^{n-1}|d_i|(|x_i| + 1),$$

for some $c \geq 1$.

By Lemma 2.1 and condition (c), we obtain

$$
\begin{aligned}
|x_n| + 1 &\le c(1+n+A_n) + c^2 n \sum_{i=1}^{n-1} (1+i+A_i)|d_i| e^{c\sum_{j=i+1}^{n-1} j|d_j|} \\
&\le c(1+n+A_n) + c_1 n \sum_{i=1}^{n-1} (1+i+A_i)|d_i|, \qquad (4.2)
\end{aligned}
$$

where $c_1 = c^2 e^{c\sum_{i=1}^{+\infty} i|d_i|}$.

From (4.2) we obtain

$$\frac{|x_n|+1}{n+nA_n} \le c(1/n+1) + c_1 \sum_{i=1}^{n-1} (1+i)|d_i|, \qquad (4.3)$$

since A_i is nondecreasing. From (4.3), the result follows.

REMARK 1. If $\alpha > 1$, then the theorem does not hold.

EXAMPLE 3. Consider the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \frac{2}{n^{2\alpha}} x_n^\alpha, \quad n \ge 1, \ \alpha > 1.$$

This equation satisfies all conditions of Theorem 4.1 except $\alpha \in [0, 1]$. For this equation $x_n = n^2$ is a solution, but x_n is not $\mathcal{O}(n)$. Also $\lim_{n\to\infty}(x_{n+1} - x_n)$ is not finite.

EXAMPLE 4. Consider the difference equation

$$x_{n+1} - 2x_n + x_{n-1} = \frac{6}{n^{3\alpha-1}} x_n^\alpha, \quad n \ge 1, \ \alpha > 1.$$

For this equation $x_n = n^3$ is a solution, but x_n is not $\mathcal{O}(n)$.

THEOREM 4.2. *Consider* (1.1) *where*

(a) $c_n \ge \delta > 0$, $n \ge n_0$;
(b) g_n *is a real sequence such that* $\sum_{i=1}^{+\infty} |g_i| < +\infty$;
(c) d_n *is a real sequence such that* $\sum_{i=1}^{+\infty} i^\alpha |d_i| < +\infty$, *for some* $\alpha \in [0, 1)$;
(d) f *is a real function such that* $|f(x)| \le L|x|^\alpha$, $x \in \mathbf{R}$, *for some* $L > 0$.

Then for every solution x_n *of* (1.1), $x_n = \mathcal{O}(n)$ *as* $n \to +\infty$ *and the following limit is finite*:

$$\lim_{n\to+\infty} c_{n-1}(x_n - x_{n-1}).$$

PROOF. As in the proof of Theorem 4.1 we have

$$|x_n| \le |x_0| + n|c_0|\,|x_1 - x_0|M + M\sum_{i=1}^{n-1}(n-i)|g_i| + nML\sum_{i=1}^{n-1}|d_i|\,|x_i|^\alpha$$
$$\le c\left(1 + n + n\sum_{i=1}^{n-1}|g_i|\right) + cn\sum_{i=1}^{n-1}|d_i|\,|x_i|^\alpha,$$

for some $c > 0$. Since $\sum_{i=1}^{+\infty}|g_i| < +\infty$, we have

$$\frac{|x_n|}{n} \le c_1 + c\sum_{i=1}^{n-1}|d_i|\,|x_i|^\alpha \le c_1 + c\sum_{i=1}^{n-1} i^\alpha|d_i|\left(\frac{|x_i|}{i}\right)^\alpha.$$

For $\alpha \in [0, 1)$, by Lemma 2.3 we get

$$\frac{|x_n|}{n} \le \left(c_1^{1-\alpha} + (1-\alpha)c\sum_{i=1}^{n-1} i^\alpha|d_i|\right)^{1/(1-\alpha)}$$
$$\le \left(c_1^{1-\alpha} + (1-\alpha)c\sum_{i=1}^{+\infty} i^\alpha|d_i|\right)^{1/(1-\alpha)} < +\infty,$$

thus the first part of the theorem follows.

From the above we know that there exists $M > 0$ such that $|x_n| \le Mn$, for every $n \in \mathbf{N}$. Summing (4.1) from $n+1$ to $n+p$, we obtain

$$y_{n+p} - y_n = \sum_{i=n+1}^{n+p} g_i + \sum_{i=n+1}^{n+p} d_i f(x_i).$$

Hence

$$|y_{n+p} - y_n| \le \sum_{i=n+1}^{n+p}|g_i| + \sum_{i=n+1}^{n+p}|d_i|\,|x_i|^\alpha$$
$$\le \sum_{i=n+1}^{n+p}|g_i| + M^\alpha\sum_{i=n+1}^{n+p} i^\alpha|d_i|.$$

By the conditions of the theorem and Cauchy's criteria we obtain the result.

REMARK 2. Theorem 4.2 is a generalisation of the main result in [8]. The result also holds in the case $\alpha = 1$ (see, for example, [1, Problem 6.24.40]). Using Corollary 2.2 instead of Lemma 2.3 in the above proof we can prove the theorem in this case.

REMARK 3. Example 3 shows that we cannot allow that $\sum_{i=1}^{+\infty} i^\alpha|d_i| = +\infty$, for some $\alpha \in [0, 1)$. Indeed, in that case $d_n = 2/n^{2\alpha}$ and $\sum_{i=1}^{+\infty} i^\alpha|d_i| = \sum_{i=1}^{+\infty} 2/i^\alpha = +\infty$, if $\alpha \in [0, 1)$. On the other hand $x_n = n^2$ is a solution such that $x_n \ne \mathcal{O}(n)$.

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THEOREM 4.3. *Consider* (1.1), *where* $g_n = 0$ *and* $c_n = 1$ *for all* $n \in \mathbf{N}$, f *is a real even nondecreasing function for* $x > 0$, $f(x) \geq |x|$ *for* $x \in \mathbf{R}$ *and* $\int^{+\infty} ds/f(s) = +\infty$. *Then for every solution* x_n *of* (1) *we have*

$$x_n = \mathcal{O}\left(G^{-1}\left(G(2c) + 4\ln n + \sum_{i=1}^{n-1}(i+1)|d_i|\right)\right) \quad \text{as } n \to +\infty,$$

where $c = \max\{1, |x_0|, |x_0 - x_1|\}$ *and* $G(u) = \int_\varepsilon^u ds/f(s)$, $\varepsilon \in (0, 1)$.

PROOF. As in Theorem 4.1 we have

$$|x_n| \leq |x_0| + n|x_1 - x_0|\,|c_0| + n\sum_{i=1}^{n-1}|d_i||f(x_i)|$$
$$\leq c(1+n) + cn\sum_{i=1}^{n-1}|d_i|f(|x_i|).$$

By Lemma 2.4, we get

$$|x_n| \leq G^{-1}\left(G(2c) + 2\ln\frac{n(n+1)}{2} + \sum_{i=1}^{n-1}(i+1)|d_i|\right), \quad \text{for } n \in \mathbf{N},$$

since $\int_{-\infty} ds/f(s) = \int^{+\infty} ds/f(s) = +\infty$, from which the result follows.

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