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AN OPTIMAL SEQUENTIAL PROCEDURE FOR A BUYING–SELLING PROBLEM WITH INDEPENDENT OBSERVATIONS

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Abstract

We consider a buying–selling problem when two stops of a sequence of independent random variables are required. An optimal stopping rule and the value of a game are obtained.

Keywords: Optimal stopping; multiple stopping rule; value of a game; buying–selling problem

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1. Statement of the problem

Let $y_1, y_2, \dots, y_N$ be a sequence of random variables, where $y_n = f_n + \varepsilon_n$, $f_n = f(n)$, $n = 1, 2, \dots, N$, is a known deterministic function (for example, a trend), and $\{\varepsilon_n,\ n = 1, \dots, N\}$ is a sequence of independent random variables with $\mathrm{E}\,\varepsilon_n = 0$. The random variable y_n can be interpreted as the value of an asset at time n. We observe these random variables sequentially and have to decide when we must stop. The first stop means the buying of an asset and the second stop signifies the selling of an asset. Our decision to stop depends on the observations already made, but does not depend on the future, which is not yet known. After two stops at times m_1 and m_2, $1 \le m_1 < m_2 \le N$, we get a gain $Z_{m_1,m_2} = y_{m_2} - y_{m_1}$. If we do not buy anything until time N then we get the gain $Z_{m_1,m_2} = 0$, so we may assume that $m_1 = N$ and $m_2 = N + 1$. This implies that

$$1 \le m_1 \le N_1, \qquad N_1 = N,$$
$$m_1 < m_2 \le N_2(m_1), \qquad N_2(m_1) = \begin{cases} N & \text{if } m_1 < N, \\ N+1 & \text{if } m_1 = N. \end{cases}$$

The problem consists of finding a procedure for maximizing the expected gain. Notice that problems with one stop (either buying or selling) were comprehensively considered in [4].

2. Some results of the theory of optimal multiple stopping rules

Let $y_1, y_2, \dots$ be a sequence of random variables with known joint distribution. We are allowed to observe the y_n sequentially, stopping anywhere we please. If we stop at time m_1 after

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observations $(y_1, \dots, y_{m_1})$, then we begin to observe another sequence $y_{m_1,m_1+1}, y_{m_1,m_1+2}, \dots$ (depending on $(y_1, \dots, y_{m_1})$), and we must solve the problem of an optimal stopping of the new sequence. If we make i stops at times $m_1, m_2, \dots, m_i,\ 1 \le i \le k-1$, then we observe a sequence of random variables $y_{m_1,\dots,m_i,m_i+1}, y_{m_1,\dots,m_i,m_i+2}, \dots$ whose distribution depends on $(y_1, \dots, y_{m_1}, y_{m_1,m_1+1}, \dots, y_{m_1,m_2}, \dots, y_{m_1,\dots,m_i})$. Our decision to stop at times $m_i,\ i = 1, 2, \dots, k$, depends solely on the values of the basic random sequence already observed and not on any future values. After $k,\ k \ge 2$, stops, we receive the gain

$$Z_{m_1,\dots,m_k} = g_{m_1,\dots,m_k}(y_1, \dots, y_{m_1,m_1+1}, \dots, y_{m_1,\dots,m_k}),$$

where $g_{m_1,\dots,m_k}$ is a known function. We are interested in finding stopping rules which maximize our expected gain.

More formally, assume that we are given

(a) a probability space $(\Omega, \mathcal{F}, \mathrm{P})$;

(b) a nondecreasing sequence of σ-subalgebras $\{\mathcal{F}_{m_1,\dots,m_{i-1},m_i},\ m_i > m_{i-1}\}$ of σ-algebra $\mathcal{F}$ such that

$$\mathcal{F}_{m_1,\dots,m_{i-1}} \subseteq \mathcal{F}_{m_1,\dots,m_i} \subseteq \mathcal{F}_{m_1,\dots,m_{i-1},m_i+1},$$

for all $i = 1, 2, \dots, k$, with $0 =: m_0 < m_1 < \dots < m_{i-1}$;

(c) a random process

$$\{Z_{m_1,\dots,m_{k-1},m_k}, \mathcal{F}_{m_1,\dots,m_{k-1},m_k},\ m_k > m_{k-1}\},$$

for any fixed integer $m_1, \dots, m_{k-1},\ 1 \le m_1 < m_2 < \dots < m_{k-1}$.

In terms of the informal background of the first paragraph in this section, we can express the σ-algebra as follows:

$$\mathcal{F}_{m_1,\dots,m_i} = \sigma(y_1, \dots, y_{m_1}, y_{m_1,m_1+1}, \dots, y_{m_1,m_2}, \dots, y_{m_1,m_2,\dots,m_i}).$$

Following [3], we now give the required definitions and theorems.

Definition 1. A collection of integer-valued random variables $(\tau_1, \dots, \tau_i)$ is called an *i-multiple stopping rule*, $1 \le i \le k$, if the following conditions hold:

(a) $1 \le \tau_1 < \tau_2 \dots < \tau_i < \infty$ (P-a.s. (almost surely)),

(b) $\{\omega\colon \tau_1 = m_1, \dots, \tau_j = m_j\} \in \mathcal{F}_{m_1,\dots,m_j}$, for all $m_j > m_{j-1} > \dots > m_1 \ge 1$, $j = 1, 2, \dots, i$.

A k-multiple stopping rule with $k > 1$ is called a *multiple stopping rule*.

We use the following notation, where ξ represents an arbitrary random variable:

$$(\boldsymbol{m})_i = (m_1, m_2, \dots, m_i), \qquad (\boldsymbol{m})_1 = m_1,$$
$$\mathrm{E}_{(\boldsymbol{m})_i}\,\xi = \mathrm{E}(\xi \mid \mathcal{F}_{(\boldsymbol{m})_i}),$$
$$A_{(\boldsymbol{m})_i}\xi = \mathrm{E}_{(\boldsymbol{m})_i}\Big(\sup_{m_{i+1}} \mathrm{E}_{(\boldsymbol{m})_{i+1}}\Big(\cdots\Big(\sup_{m_{k-1}} \mathrm{E}_{(\boldsymbol{m})_{k-1}}\,\xi\Big)\cdots\Big)\Big).$$

The following condition is needed for the existence of all considered expectations.

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Condition 1.

$$\mathrm{E}\left(\sup_{m_1} A_{(\boldsymbol{m})_1}\left(\sup_{(\boldsymbol{m})_k} Z_{(\boldsymbol{m})_k}\right)\right) < +\infty.$$

We assume that Condition 1 is satisfied for the $Z_{(\boldsymbol{m})_k}$.

Let S_m be a class of multiple stopping rules $\boldsymbol{\tau} = (\tau_1, \dots, \tau_k)$ such that $\tau_1 \geq m$ (P-a.s.).

Definition 2. The function

$$v_m = \sup_{\boldsymbol{\tau} \in S_m} \mathrm{E}\, Z_{\boldsymbol{\tau}}$$

is called the *m-value of the game*. In particular, if $m = 1$ then $v = v_1$ is called the *value of the game*.

Definition 3. A multiple stopping rule $\boldsymbol{\tau}^* \in S_m$ is called an *optimal multiple stopping rule* in S_m if $\mathrm{E}\, Z_{\boldsymbol{\tau}^*}$ exists and $\mathrm{E}\, Z_{\boldsymbol{\tau}^*} = v_m$.

Condition 1 ensures the finiteness of v_m and the existence of $\mathrm{E}\, Z_{\boldsymbol{\tau}}$ for all $\boldsymbol{\tau} \in S_m$. The problem consists of finding an optimal multiple stopping rule and an m-value of the game v_m.

The sequences $\{V_{(\boldsymbol{m})_i}\}$ and $\{X_{(\boldsymbol{m})_i}\}$, $i = 1, 2, \dots, k$, are needed for constructing the multiple stopping rules $\boldsymbol{\tau}^*$. Let $T_{(\boldsymbol{m})_i}$ be a class of i-multiple stopping rules $(\boldsymbol{\tau})_i = (\tau_1, \dots, \tau_i)$, $i = 1, 2, \dots, k$, with $\tau_1 = m_1, \dots, \tau_{i-1} = m_{i-1}$, $\tau_i \geq m_i$ (P-a.s.). Let $T_{(\boldsymbol{m})_1} =: T_{m_1}$ denote the class of all stopping times τ_1 such that $\tau_1 \geq m_1$ (P-a.s.). We set $X_{(\boldsymbol{m})_k} = Z_{(\boldsymbol{m})_k}$ and define, by backward induction on i from $i = k$,

$$\begin{aligned} V_{(\boldsymbol{m})_i} &= \operatorname*{ess\,sup}_{(\boldsymbol{\tau})_i \in T_{(\boldsymbol{m})_i}} \mathrm{E}_{(\boldsymbol{m})_i} X_{(\boldsymbol{\tau})_i}, \\ X_{(\boldsymbol{m})_{i-1}} &= \mathrm{E}_{(\boldsymbol{m})_{i-1}} V_{(\boldsymbol{m})_{i-1}, m_{i-1}+1}, \qquad i = k, k-1, \dots, 1, \end{aligned} \tag{1}$$

where $X_0 := 0$.

Remark 1. We emphasize that most of the statements in this section are valid almost surely. We shall make no mention of this in what follows.

Let us now establish some properties of the sequences $\{V_{(\boldsymbol{m})_i}\}$ and $\{X_{(\boldsymbol{m})_i}\}$. It follows from results of the general theory of optimal stopping (see, for example, [1] and [2]) that $V_{(\boldsymbol{m})_i}$ satisfies the recursion equation

$$V_{(\boldsymbol{m})_i} = \max\{X_{(\boldsymbol{m})_i}, \mathrm{E}_{(\boldsymbol{m})_i} V_{(\boldsymbol{m})_{i-1}, m_i+1}\}.$$

The following theorem gives the existence conditions and the structure of an optimal multiple stopping rule in S_m.

Theorem 1. ([3].) *Let Condition 1 be satisfied. We put*

$$\tau_i^* = \inf\{m_i > m_{i-1} \colon V_{(\boldsymbol{m})_i} = X_{(\boldsymbol{m})_i}\},$$

for $i = 1, 2, \ldots, k$, on the set $D_{i-1} = \{\omega\colon \tau_1^* = m_1, \ldots, \tau_{i-1}^* = m_{i-1}\}$, where it is assumed that $\tau_i^*(\omega) = +\infty$ on $\{\omega\colon \tau_{i-1}^*(\omega) = +\infty\}$, $m_0 = m - 1$, and $D_0 = \Omega$. In that case, if the random vector $\boldsymbol{\tau}^* = (\tau_1^*, \ldots, \tau_k^*)$ is finite with probability one, then $\boldsymbol{\tau}^* \in S_m$ is an optimal multiple stopping rule.

The following theorem gives the characterization of the m-value v_m by means of the sequence $\{V_m\}$.

Theorem 2. ([3].) *If Condition 1 is satisfied, then $v_m = \mathrm{E}\, V_m$.*

We now consider a finite case. Let

$$\{Z_{(\boldsymbol{m})_k},\ 1 \le m_1 \le N_1, m_1 < m_2 < N_2(m_1), \ldots, m_{k-1} < m_k \le N_k(m_1, \ldots, m_{k-1})\}$$

be a family of random variables, where N_1 and $N_i(\cdot)$, $i = 2, \ldots, k$, are natural numbers. As in the general theory of optimal stopping [1], we define the sequence $V_{(\boldsymbol{m})_i}$ by backward induction from the recursion equations

$$V_{(\boldsymbol{m})_{i-1}, N_i(m_1,\ldots,m_{i-1})} = X_{(\boldsymbol{m})_{i-1}, N_i(m_1,\ldots,m_{i-1})}, \tag{2}$$

$$V_{(\boldsymbol{m})_i} = \max\{X_{(\boldsymbol{m})_i}, \mathrm{E}_{(\boldsymbol{m})_i}\, V_{(\boldsymbol{m})_{i-1}, m_i+1}\}, \tag{3}$$

for $1 \le m_1 \le N_1, \ldots, m_{i-1} < m_i \le N_i(m_1, \ldots, m_{i-1})$. As before, $X_{(\boldsymbol{m})_k} = Z_{(\boldsymbol{m})_k}$.

Using Theorem 1, we define the optimal multiple stopping rule $\boldsymbol{\tau}^*$. From Theorem 2, (2), and (3), we obtain the value v_m.

3. Main result

Let us consider our initial problem. The following theorem gives an optimal double stopping rule $\boldsymbol{\tau}^*$ and the value v.

Theorem 3. *Let $\{y_n\}$ and Z_{m_1,m_2} be defined as in Section 1. Let $v^{L,l}$ be the value of a game with l, $l \le 2$, stops and L, $L \le N$, steps. Then the value v is equal to $v^{N,2}$, where*

$$\begin{aligned} v^{n,2} &= \mathrm{E}(\max\{v^{n-1,1} - y_{N-n+1}, v^{n-1,2}\}), && 2 \le n \le N,\ v^{1,2} = 0,\\ v^{n,1} &= \mathrm{E}(\max\{y_{N-n+1}, v^{n-1,1}\}), && 1 \le n \le N-1,\ v^{0,1} = -\infty. \end{aligned}$$

We put

$$\begin{aligned} \tau_1^* &= \min\{\min\{m_1\colon 1 \le m_1 \le N-1, y_{m_1} \le v^{N-m_1,1} - v^{N-m_1,2}\}, N\},\\ \tau_2^* &= \min\{\min\{m_2\colon \tau_1^* < m_2 \le N, y_{m_2} \ge v^{N-m_2,1}\}, \{N+1\colon \tau_1^* = N\}\}; \end{aligned}$$

then $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$ is the optimal double stopping rule.

Proof. If $m_1 = N$ and $m_2 = N+1$, then $Z_{m_1,m_2} = 0$. It follows easily that $v^{1,2} = 0$.

Let us consider the case $1 \le m_1 \le N-1$, $m_1 < m_2 \le N$. From (3) and the independence of $y_1, \ldots, y_N$, we obtain

$$\begin{aligned} V_{m_1,m_2} &= \max\{X_{m_1,m_2}, \mathrm{E}_{m_1,m_2}\, V_{m_1,m_2+1}\}\\ &= \max\{y_{m_2} - y_{m_1}, \mathrm{E}_{m_1,m_2}\, V_{m_1,m_2+1}\}\\ &= \max\{y_{m_2}, \mathrm{E}_{m_1,m_2}\, V_{m_1,m_2+1} + y_{m_1}\} - y_{m_1}\\ &= \max\{y_{m_2}, v^{N-m_2,1}\} - y_{m_1}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} v^{N-m_2+1,1} &= \mathrm{E}_{m_1,m_2-1}(\max\{y_{m_2}, v^{N-m_2,1}\}) \\ &= \mathrm{E}(\max\{y_{m_2}, v^{N-m_2,1}\}), \qquad \text{for } m_1 < m_2 < N, \\ v^{N-m_2+1,1} &= v^{1,1} = \mathrm{E}\, y_N = f_N, \qquad\qquad \text{for } m_2 = N. \end{aligned}$$

Indeed, it follows from (2) that $V_{m_1,N} = X_{m_1,N}$. Hence, $v^{0,1} = -\infty$.

From (1) we obtain

$$\begin{aligned} X_{m_1} &= \mathrm{E}_{m_1} V_{m_1,m_1+1} \\ &= \mathrm{E}_{m_1}(\max\{y_{m_1+1}, v^{N-m_1-1,1}\} - y_{m_1}) \\ &= \mathrm{E}(\max\{y_{m_1+1}, v^{N-m_1-1,1}\}) - y_{m_1} \\ &= v^{N-m_1,1} - y_{m_1}. \end{aligned}$$

In the same way,

$$\begin{aligned} V_{m_1} &= \max\{X_{m_1}, \mathrm{E}_{m_1} V_{m_1+1}\} \\ &= \max\{v^{N-m_1,1} - y_{m_1}, \mathrm{E}_{m_1} V_{m_1+1}\} \\ &= \max\{v^{N-m_1,1} - y_{m_1}, v^{N-m_1,2}\}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} v^{N-m_1+1,2} &= \mathrm{E}_{m_1-1}(\max\{v^{N-m_1,1} - y_{m_1}, v^{N-m_1,2}\}) \\ &= \mathrm{E}(\max\{v^{N-m_1,1} - y_{m_1}, v^{N-m_1,2}\}), \qquad \text{for } m_1 < N-1, \\ v^{N-m_1+1,2} &= v^{2,2} = \mathrm{E}(\max\{f_N - y_{N-1}, 0\}), \qquad\qquad \text{for } m_1 = N-1. \end{aligned}$$

Taking into account Theorem 2, we obtain the value $v = v^{N,2}$.

Now, using Theorem 1, (4), and (5), we obtain the optimal double stopping rule $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$, where

$$\begin{aligned} \tau_1^* &= \min\{\min\{m_1 \colon 1 \le m_1 \le N-1,\, y_{m_1} \le v^{N-m_1,1} - v^{N-m_1,2}\}, N\}, \\ \tau_2^* &= \min\{\min\{m_2 \colon \tau_1^* < m_2 \le N,\, y_{m_2} \ge v^{N-m_2,1}\}, \{N+1 \colon \tau_1^* = N\}\}. \end{aligned}$$

This completes the proof.

Remark 2. This theorem can easily be generalized when more than two stops are required. In fact, having applied the theorem once to generate two stops, we can treat the time at the second stop as zero and reapply the theorem as many times as necessary.

4. Examples

We discuss here three examples in which we specify the distribution of the ‘noise’ component ε (i.e. any ε_n). We assume that the distribution of ε is either uniform, Laplace (double exponential), or normal; in each case we present a solution of the optimal double stopping problem.

Example 1. (*Uniform distribution.*) Let $\varepsilon_1, \ldots, \varepsilon_N$ be a sequence of independent random variables having uniform distribution $\mathrm{U}(-a, a)$, where $a > 0$ is a fixed number. From Theorem 3 we obtain

$$\begin{aligned} v^{n,1} &= \mathrm{E}(\max\{y_{N-n+1}, v^{n-1,1}\}) \\ &= \mathrm{E}(\max\{f_{N-n+1} + \varepsilon_{N-n+1}, v^{n-1,1}\}) \\ &= \mathrm{E}(\max\{\varepsilon_{N-n+1}, v^{n-1,1} - f_{N-n+1}\}) + f_{N-n+1} \\ &= \int_{-a}^{a} \max\{x, v^{n-1,1} - f_{N-n+1}\}(2a)^{-1}\,\mathrm{d}x + f_{N-n+1} \\ &= \int_{-a}^{c} c(2a)^{-1}\,\mathrm{d}x + \int_{c}^{a} x(2a)^{-1}\,\mathrm{d}x + f_{N-n+1} \\ &= (v^{n-1,1} - f_{N-n+1} + a)^2(4a)^{-1} + f_{N-n+1}, \end{aligned}$$

where $c = v^{n-1,1} - f_{N-n+1}$, $1 \le n \le N-1$, and $v^{0,1} = f_N - a$.

Similarly,

$$\begin{aligned} v^{n,2} &= \mathrm{E}(\max\{v^{n-1,1} - y_{N-n+1}, v^{n-1,2}\}) \\ &= \mathrm{E}(\max\{v^{n-1,1} - f_{N-n+1} - \varepsilon_{N-n+1}, v^{n-1,2}\}) \\ &= \mathrm{E}(\max\{-\varepsilon_{N-n+1}, v^{n-1,2} - v^{n-1,1} + f_{N-n+1}\}) + v^{n-1,1} - f_{N-n+1} \\ &= \int_{-a}^{a} \max\{-x, d\}(2a)^{-1}\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\ &= -\int_{-a}^{-d} x(2a)^{-1}\,\mathrm{d}x + \int_{-d}^{a} d(2a)^{-1}\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\ &= (v^{n-1,2} - v^{n-1,1} + f_{N-n+1} + a)^2(4a)^{-1} + v^{n-1,1} - f_{N-n+1}, \end{aligned}$$

where $d = v^{n-1,2} - v^{n-1,1} + f_{N-n+1}$, $2 \le n \le N$, and $v^{1,2} = 0$.

In Table 1 we present some numerical results in the case $f_n = 0.1n + 10$, $N = 6$, and $a = 1$. Thus, we get the expected gain $v = v^{6,2} = 1.2070$ and $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$, where

$$\begin{aligned} \tau_1^* &= \min\{\min\{m_1 \colon 1 \le m_1 \le 5,\, y_{m_1} \le v^{6-m_1,1} - v^{6-m_1,2}\}, 6\}, \\ \tau_2^* &= \min\{\min\{m_2 \colon \tau_1^* < m_2 \le 6,\, y_{m_2} \ge v^{6-m_2,1}\}, \{7 \colon \tau_1^* = 6\} \equiv 7\}. \end{aligned}$$

For instance, if we observe the sequence

$$\{y_n,\ n = 1, \ldots, 6\} \colon 10.0384, 9.3296, 10.5656, 10.6310, 10.3470, 11.2767,$$

then $m_1 = 2$ (because $y_2 = 9.3296 \le v^{4,1} - v^{4,2} = 10.9334 - 0.8484 = 10.0850$) and $m_2 = 6$ (because $y_6 = 11.2767 \ge v^{0,1} = 9.6000$). This yields the gain $11.2767 - 9.3296 = 1.9471$.

TABLE 1: The values for the uniform distribution $\mathrm{U}(-1, 1)$.

n	0	1	2	3	4	5	6
$v^{n,1}$	9.6000	10.6000	10.8025	10.8918	10.9334	10.9512	—
$v^{n,2}$	—	0.0000	0.3025	0.6050	0.8484	1.0442	1.2070

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Example 2. (*Laplace distribution.*) Suppose that independent random variables $\varepsilon_1, \dots, \varepsilon_N$ are identically distributed by a Laplace (double exponential) distribution $\mathrm{L}(0, b)$ with probability density function

$$g(x) = \frac{1}{2b} \exp\left\{-\frac{|x|}{b}\right\}, \qquad x \in (-\infty, \infty),\ b > 0.$$

As above, using Theorem 3 we obtain

$$\begin{aligned}
v^{n,1} &= \mathrm{E}(\max\{y_{N-n+1}, v^{n-1,1}\}) \\
&= \mathrm{E}(\max\{f_{N-n+1} + \varepsilon_{N-n+1}, v^{n-1,1}\}) \\
&= \mathrm{E}(\max\{\varepsilon_{N-n+1}, v^{n-1,1} - f_{N-n+1}\}) + f_{N-n+1} \\
&= \int_{-\infty}^{\infty} \max\{x, v^{n-1,1} - f_{N-n+1}\} g(x)\,\mathrm{d}x + f_{N-n+1} \\
&= \int_{-\infty}^{c} c g(x)\,\mathrm{d}x + \int_{c}^{\infty} x g(x)\,\mathrm{d}x + f_{N-n+1} \\
&= v^{n-1,1} + \frac{b}{2} \exp\left\{\frac{f_{N-n+1} - v^{n-1,1}}{b}\right\},
\end{aligned}$$

where $c = v^{n-1,1} - f_{N-n+1}$, $1 \le n \le N - 1$, and $v^{0,1} = -\infty$.

Likewise,

$$\begin{aligned}
v^{n,2} &= \mathrm{E}(\max\{v^{n-1,1} - y_{N-n+1}, v^{n-1,2}\}) \\
&= \mathrm{E}(\max\{v^{n-1,1} - f_{N-n+1} - \varepsilon_{N-n+1}, v^{n-1,2}\}) \\
&= \mathrm{E}(\max\{-\varepsilon_{N-n+1}, v^{n-1,2} - v^{n-1,1} + f_{N-n+1}\}) + v^{n-1,1} - f_{N-n+1} \\
&= \int_{-\infty}^{\infty} \max\{-x, d\} g(x)\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\
&= -\int_{-\infty}^{-d} x g(x)\,\mathrm{d}x + \int_{-d}^{\infty} d g(x)\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\
&= v^{n-1,2} + \frac{b}{2} \exp\left\{\frac{v^{n-1,1} - v^{n-1,2} - f_{N-n+1}}{b}\right\},
\end{aligned}$$

where $d = v^{n-1,2} - v^{n-1,1} + f_{N-n+1}$, $2 \le n \le N$, and $v^{1,2} = 0$.

In Table 2, we present some numerical results in the case $f_n = 100 - n$, $N = 5$, and $b = 1$. It follows that we have the expected gain $v = v^{5,2} = 0.7117$ and $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$, where

$$\begin{aligned}
\tau_1^* &= \min\{\min\{m_1 \colon 1 \le m_1 \le 4,\ y_{m_1} \le v^{5-m_1,1} - v^{5-m_1,2}\}, 5\}, \\
\tau_2^* &= \min\{\min\{m_2 \colon \tau_1^* < m_2 \le 5,\ y_{m_2} \ge v^{5-m_2,1}\}, \{6 \colon \tau_1^* = 5\} \equiv 6\}.
\end{aligned}$$

TABLE 2: The values for the Laplace distribution $\mathrm{L}(0, 1)$.

n	0	1	2	3	4	5
$v^{n,1}$	$-\infty$	95.0000	96.3591	97.3082	98.3069	—
$v^{n,2}$	—	0.0000	0.1839	0.4031	0.5704	0.7117

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In particular, if we observe the sequence

$$\{y_n, n = 1, \dots, 5\}\colon 99.2057, 97.1005, 96.8307, 95.6100, 95.7487,$$

then $m_1 = 5$ and $m_2 = 6$. We buy nothing, so the gain is $v = 0$.

Example 3. (*Normal distribution.*) Let $\varepsilon_1, \dots, \varepsilon_N$ be a sequence of independent random variables having normal distribution $\mathrm{N}(0, \sigma^2)$ with probability density function

$$h(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \qquad x \in (-\infty, \infty),\ \sigma > 0.$$

As before, from Theorem 3 we obtain

$$\begin{aligned}
v^{n,1} &= \mathrm{E}(\max\{y_{N-n+1}, v^{n-1,1}\}) \\
&= \mathrm{E}(\max\{f_{N-n+1} + \varepsilon_{N-n+1}, v^{n-1,1}\}) \\
&= \mathrm{E}(\max\{\varepsilon_{N-n+1}, v^{n-1,1} - f_{N-n+1}\}) + f_{N-n+1} \\
&= \int_{-\infty}^{\infty} \max\{x, v^{n-1,1} - f_{N-n+1}\} h(x)\,\mathrm{d}x + f_{N-n+1} \\
&= \int_{-\infty}^{c} c h(x)\,\mathrm{d}x + \int_{c}^{\infty} x h(x)\,\mathrm{d}x + f_{N-n+1} \\
&= c\Phi\left(\frac{c}{\sigma}\right) + \sigma\varphi\left(\frac{c}{\sigma}\right) + f_{N-n+1} \\
&= \sigma\psi\left(\frac{v^{n-1,1} - f_{N-n+1}}{\sigma}\right) + f_{N-n+1},
\end{aligned}$$

where $\psi(x) = \varphi(x) + x\Phi(x)$, $\varphi(x)$ is the density function of the standard normal distribution, $\Phi(x)$ is the distribution function of the standard normal distribution, $c = v^{n-1,1} - f_{N-n+1}$, $1 \le n \le N-1$, and $v^{0,1} = -\infty$.

In the same way,

$$\begin{aligned}
v^{n,2} &= \mathrm{E}(\max\{v^{n-1,1} - y_{N-n+1}, v^{n-1,2}\}) \\
&= \mathrm{E}(\max\{v^{n-1,1} - f_{N-n+1} - \varepsilon_{N-n+1}, v^{n-1,2}\}) \\
&= \mathrm{E}(\max\{-\varepsilon_{N-n+1}, v^{n-1,2} - v^{n-1,1} + f_{N-n+1}\}) + v^{n-1,1} - f_{N-n+1} \\
&= \int_{-\infty}^{\infty} \max\{-x, d\} h(x)\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\
&= -\int_{-\infty}^{-d} x h(x)\,\mathrm{d}x + \int_{-d}^{\infty} d h(x)\,\mathrm{d}x + v^{n-1,1} - f_{N-n+1} \\
&= \sigma\varphi\left(\frac{d}{\sigma}\right) + d\Phi\left(\frac{d}{\sigma}\right) + v^{n-1,1} - f_{N-n+1} \\
&= \sigma\psi\left(\frac{v^{n-1,2} - v^{n-1,1} + f_{N-n+1}}{\sigma}\right) + v^{n-1,1} - f_{N-n+1},
\end{aligned}$$

where $d = v^{n-1,2} - v^{n-1,1} + f_{N-n+1}$, $2 \le n \le N$, and $v^{1,2} = 0$.

TABLE 3: The values for the normal distribution N(0, 1).

n	0	1	2	3	4	5	6	7
$v^{n,1}$	$-\infty$	3.0615	5.6583	7.4079	8.1709	8.2954	8.2967	—
$v^{n,2}$	—	0.0000	0.0015	0.0183	0.1841	0.9497	2.6573	5.2368

In Table 3, we present some numerical results in the case $f_n = 8\sin(\pi n/8)$, $N = 7$, and $\sigma = 1$. It follows that the expected gain is $v = v^{7,2} = 5.2368$ and $\boldsymbol{\tau}^* = (\tau_1^*, \tau_2^*)$, where

$$\tau_1^* = \min\{\min\{m_1 : 1 \le m_1 \le 6,\ y_{m_1} \le v^{7-m_1,1} - v^{7-m_1,2}\}, 7\},$$
$$\tau_2^* = \min\{\min\{m_2 : \tau_1^* < m_2 \le 7,\ y_{m_2} \ge v^{7-m_2,1}\}, \{8 : \tau_1^* = 7\} \equiv 8\}.$$

For instance, if we observe the sequence

$$\{y_n,\ n = 1, \ldots, 7\}: 2.4894, 5.9438, 7.0311, 8.9202, 7.8443, 5.4808, 3.5506,$$

then $m_1 = 1$ (because $y_1 = 2.4894 \le v^{6,1} - v^{6,2} = 8.2967 - 2.6573 = 5.6394$) and $m_2 = 4$ (because $y_4 = 8.9202 \ge v^{3,1} = 7.4079$). This yields the gain $8.9202 - 2.4894 = 6.4308$.

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