J. Austral. Math. Soc. 20 (Series A) (1975), 18–24.

APPROXIMATION WITH NORMS DEFINED BY DERIVATIONS

J. M. BRIGGS

(Received 23 May 1972; revised 9 January 1973)

Communicated by J. B. Miller

A linear mapping D of the algebra of polynomial functions $P[0,1]$ into the algebra of all continuous complex-valued functions $C[0,1]$ is called a *derivation* provided $D(fg) = fD(g) + gD(f)$ for all polynomials f and g. The derivations of $P[0,1]$ into $C[0,1]$ are easily seen to be all mappings of the form D_w where w is a continuous function on $[0,1]$ and $D_w(f) = wf'$ (f' denotes the ordinary derivative of f). In fact, $w = D(x)$ where x is the coordinate function. Let D_w be such a derivation, and let $\| \cdot \|$ denote the supremum norm on $C[0,1]$. Then D_w gives rise to an algebra norm $\| \cdot \|_w$ on $P[0,1]$ defined by

$$\| f \|_w = \| f \| + \| D_w(f) \| = \| f \| + \| wf' \| \quad \text{for} \quad f \in P[0,1].$$

In this paper we study the algebra of all continuous functions on $[0,1]$ which are $\| \cdot \|_w$-approximable by polynomials; that is, those functions which are pointwise limits of $\| \cdot \|_w$-Cauchy sequences of polynomials. Let $C^1(w)$ denote the algebra of all such functions. For comparison purposes, we define two other algebras of functions. For $w \in C[0,1]$ let $\mathscr{Z}(w)$ denote the zero set of w. Let C_w^1 denote the subalgebra of $C[0,1]$ consisting of all f such that (i) $f'(y)$ exists for each $y \in [0,1] \backslash \mathscr{Z}(w)$, and (ii) the function wf' is continuous on $[0,1]$ where $(wf')(y) = 0$ if $y \in \mathscr{Z}(w)$, $(wf')(y) = w(y)f'(y)$ if $y \in [0,1] \backslash \mathscr{Z}(w)$. Finally, let AC_w be the subalgebra of C_w^1 consisting of absolutely continuous functions.

The following are the main results of this paper. Two algebras $C^1(w_1)$ and $C^1(w_2)$ are equal if and only if there exists a bounded function h on $[0,1]$ which is bounded away from zero such that $w_2 = hw_1$. The method of approximation described in this paper generalizes both uniform approximation of continuous functions and the familiar method of approximation of once continuously differentiable functions, since $C^1(w) = C[0,1]$ if and only if $w \equiv 0$, and $C^1(w) = C^1[0,1]$ if and only if w is never zero. As the following results indicate, the zero set of w plays an important role in what can be approximated. There exists a non-constant function f such that $wf' = 0$ if and only if $\mathscr{Z}(w)$ is uncountable; $C^1(w) = AC_w$ if and only if $1/w \in L^1[0,1]$. If the boundary of $\mathscr{Z}(w)$ is countable, then $C^1(w) = C^1_w$. Finally, as an example shows, if $\mathscr{Z}(w)$ is not suitably simple, then we should not expect that $C^1(w)$ will equal C^1_w.

The author expresses appreciation for the suggestions made by the referees in reading this paper; the present statement and proof of Theorem 6 are due to them.

Preliminaries

In this paper we use the general theory of Banach algebras for two purposes: to obtain norm estimates and to use localization. Let A be a semi-simple, commutative Banach algebra over C with identity which we consider as an algebra of complex-valued continuous functions on its maximal ideal space $\mathscr{M}(A)$ via the Gelfand representation. A is called *regular* provided that for each closed set F in $\mathscr{M}(A)$ and point p not in F, there exists an element $f \in A$ such that $f(p) = 1$ and $f \mid F = 0$ (where $\mid$ denotes the restriction). If g is a continuous function on $\mathscr{M}(A)$ and $p \in \mathscr{M}(A)$, we say that g *belongs locally* to A at p provided there exists a neighborhood U of p and an element $f \in A$ such that $f \mid U = g \mid U$. It is well known that if A is regular, then a continuous function g belongs to A if and only if g belongs locally to A at each poibt of M(A) (see, for instance, page 224 of Naimark (1964)). Another fact which will be useful gives a comparison between the topologies of a Banach algebra and its subalgebras. Let A_1 and A_2 be commutative Banach algebras with norms $\| \cdot \|_1$ and $\| \cdot \|_2$ respectively such that A_2 is semi-simple and $A_1 \subseteq A_2$. Then there exists a constant M such atht $\| a \|_2 \leqq M \| a \|_1$ for all $a \in A_1$. (This is a consequence of Theorem 2.5.17 of Rickart(1960).)

Let A and B be commutative algebras over $\mathbf{C}$ with the identity of A contained in B and $B \subseteq A$. A linear mapping D of B into A is called a *derivation* if $D(fg) = fD(g) + gD(f)$ for all $f, g \in B$. Notice that since B contains the identity, the kernel of D must contain the constants. We say that D is *almost injective* if $\ker(D) = \mathbf{C}$.

Let $C[0,1]$ denote the algebra of all continuous, complex-valued functions on $[0,1]$ with the supremum norm $\| \cdot \|$, and $C^1[0,1]$ the algebra of complex-valued, continuously differentiable functions on $[0,1]$ with the norm $\| \cdot \|_1$ defined by $\| f \|_1 = \| f \| + \| f' \|$ where f' denotes the derivative of f. Let $AC[0,1]$ denote the algebra of complex-valued, absolutely continuous functions on $[0,1]$ with the norm $\| \cdot \|'$ defined by $\| f \|' = \| f \| + \int_0^1 |f'|$. These three algebras are regular, semi-simple, commutative Banach algebras with identity having $[0,1]$ as their maximal ideal spaces and each containing the dense subalgebra $P[0,1]$ of polynomials. (See pages 300–303 of Rickart (1960); also see Theorem 2 of Loy (1970).)

Derivations and Approximation

The derivation D_w of $P[0,1]$ into $C[0,1]$ has a natural extension $\tilde{D}_w$ to the algebra C_w^1: for each $f \in C_w^1$, let $\tilde{D}_w(f) = wf'$ as defined earlier. Then it is easy to see that $\tilde{D}_w$ is a derivation of C_w^1 into $C[0,1]$ which extends D_w. Define a norm $\|\cdot\|_w$ on C_w^1 by $\|f\|_w = \|f\| + \|\tilde{D}_w(f)\| = \|f\| + \|wf'\|$. Since $\tilde{D}_w$ is a derivation, $\|\cdot\|_w$ is submultiplicative. Hence, C_w^1 is a normed algebra; furthermore, it is easily verified that it is a Banach algebra and that D_w is a closed derivation of C_w^1 into $C[0,1]$. Let $C^1(w)$ be the closure in C_w^1 of $P[0,1]$; that is, $C^1(w)$ consists of those functions in $C[0,1]$ which can be approximated in this norm $\|\cdot\|_w$. The algebras C_w^1 and $C^1(w)$ give examples of algebras of derivable elements (see p. 310 of Loy (1970)). If we let $M = \max\{1, \|w\|\}$, then $\|p\| \leqq \|p\|_w \leqq M\|p\|_1$ for all $p \in P[0,1]$. The next theorem is a simple consequence of this inequality.

THEOREM 1. $C^1[0,1] \subseteq C^1(w) \subseteq C_w^1 \subseteq C[0,1]$.

Furthermore, each of these algebras is semi-simple and regular, and each has $[0,1]$ as its maximal ideal space.

Notice that $C^1(w) = C^1(|w|)$, $C_w^1 = C_{|w|}^1$, and $AC_w = AC_{|w|}$. Hence, when it is convenient for computing these algebras, we may assume that $w \geqq 0$. The remainder of this paper will be devoted to comparing and describing these algebras.

LEMMA 2. *If $C^1(w_1) \subseteq C^1(w_2)$, then $\mathscr{Z}(w_1) \subseteq \mathscr{Z}(w_2)$.*

PROOF. If there were a point x_0 in $\mathscr{Z}(w_1)$ but not in $\mathscr{Z}(w_2)$, then, because x_0 is not in $\mathscr{Z}(w_2)$, every function in $C^1(w_2)$ would be continuously differentiable in some neighborhood of x_0. We show that this leads to a contradiction if $C^1(w_1) \subseteq C^1(w_2)$. More generally, suppose that $f'(x)$ exists for all f in $C^1(w)$ but that $w(x) = 0$. Since $f'(x) = \lim(f(x_n) - f(x))/(x_n - x)$ when $\lim x_n = x$, the uniform boundedness principle yields the existence of a constant M such that $|f'(x)| \leqq M(\|f\| + \|wf'\|)$ for all f in $C^1(w)$. Let U be a neighborhood of x for which $\sup_U |w| < 1/2M$. Then for all f in $C^1[0,1]$ which are constant outside U, we have $|f'(x)| \leqq M\|f\| + (1/2)\|f'\|$; but it is easy to see that there are such f with $f'(x) = \|f'\|$ and $\|f\|$ arbitrarily small, thus reaching a contradiction.

THEOREM 3. $C^1(w_1) \subseteq C^1(w_2)$ *if and only if there exists a bounded function h on $[0,1]$ such that $h|\mathscr{Z}(w_1) = 1$ and $w_2 = hw_1$.*

PROOF. Suppose that $w_2 = hw_1$ where h is bounded by $M \geqq 1$. If $p \in P[0,1]$, then $\|p\|_{w_2} \leqq M\|p\|_{w_1}$. Hence $C^1(w_1) \subseteq C^1(w_2)$.

Now suppose that $C^1(w_1) \subseteq C^1(w_2)$. By semisimplicity there exists a constant $M > 1$ such that $\|f\|_{w_2} \leqq M\|f\|_{w_1}$ for all f in $C^1[0,1]$. We claim that w_2/w_1 is bounded by M outside $\mathscr{Z}(w_1)$. If not, there exists an interval I, disjoint from

$\mathscr{Z}(w_1)$, such that $|w_2/w_1| \geqq N$ on I, where $N > M$. Then, for any f in $C^1[0,1]$ which is constant outside I, we have

$$\|f\|_{w_2} = \|f\| + \|w_1(w_2/w_1)f\| \geqq \|f\| + N\|w_1 f'\|$$

and thus

$$M(\|f\| + \|w_1 f'\|) \geqq \|f\| + N\|w_1 f'\|$$

or

$$\|w_1 f'\| \leqq (M-1)/(N-M)\|f\| \quad \text{for all such } f.$$

It then must be true that

$$\|f'\| \leqq (M-1)/((N-M)\min_I |w_1|)\|f\| = K\|f\|$$

which is clearly impossible. Thus w_2/w_1 is bounded and the conclusion of the theorem follows since $\mathscr{Z}(w_1) \subseteq \mathscr{Z}(w_2)$.

Corollary 4. *$C^1(w_1) = C^1(w_2)$ if and only if there exists a function h on $[0,1]$ which is both bounded above and bounded away from zero such that $w_2 = hw_1$.*

Corollary 5. *$C^1(w) = C[0,1]$ if and only if $w \equiv 0$; $C^1(w) = C^1[0,1]$ if and only if w is never zero.*

Before comparing the algebras $C^1(w)$, C^1_w, and AC_w, we characterize when the derivation $\tilde{D}_w$ is almost injective.

Theorem 6. *$\tilde{D}_w$ is almost injective if and only if $\mathscr{Z}(w)$ is countable.*

Proof. Suppose $\mathscr{Z}(w)$ is uncountable. Then it contains a perfect set K with empty interior (see p. 228 of Sierpinski (1952)). Following Cantor, we can construct a nonconstant continuous function f which is constant on each interval of the complement of K, and for this f, we have $wf' = 0$. Suppose now that $\mathscr{Z}(w)$ is countable. If $wf' = 0$, then f is certainly constant on each interval of the complement of $\mathscr{Z}(w)$. Let U be the set of all points x such that f is constant in some neighborhood of x. Then the complement of U is a closed set without isolated points, since f is continuous, and contained in $\mathscr{Z}(w)$. Hence this set is empty or uncountable; but since $\mathscr{Z}(w)$ is countable, it is empty. Thus f is constant.

Our last task will be to describe what these algebras are in many cases.

Lemma 7. *$AC_w \subseteq C^1(w)$.*

Proof. Let $g \in AC_w$ and $\varepsilon > 0$. Since $g \in AC[0,1]$, there exists $f \in C^1[0,1]$ such that $\|f-g\| < \varepsilon/4$ and $\int_0^1 |f'-g'| < \varepsilon/4$. Let δ be such that if $S \subseteq [0,1]$ and meas$(S) < \delta$, then $\int_S |f'| + |g'| < \varepsilon/8$. Since w and wg' are zero on $\mathscr{Z}(w)$, there exists an open neighborhood U of $\mathscr{Z}(w)$ such that (i) meas $U \backslash \mathscr{Z}(w) < \delta$, (ii) $\sup_U |w| < \varepsilon/(4(1+f))$, and (iii) $\sup_U |wg'| < \varepsilon/8$. Let V be an open neighborhood of $\mathscr{Z}(w)$ such that $\bar{V} \subset U$. Choose a continuous function F such that $F = 1$ on $\bar{V}$, $F = 0$ outside U, and $0 \leqq F \leqq 1$ everywhere. Let $h = Ff' + (1-F)g'$. Then h is continuous, $h = f'$ on $\bar{V}$, $h = g'$ outside U, and $|h| \leqq |f'| + |g'|$ everywhere off $\mathscr{Z}(w)$. Let

$$G(x) = g(0) + \int_0^x h(t)dt.$$

Then G is in $C^1[0,1]$, and it is routine to verify that $\|g - G\|_w < \varepsilon$. The proof is complete since $C^1[0,1] \subseteq C^1(w)$.

THEOREM 8. *$C^1(w) = AC_w$ if and only if $1/w \in L^1[0,1]$.*

PROOF. Assume that $1/w \in L^1[0,1]$, and let $M = \max\{1, \int_0^1 |1/w|\}$. Then $\|f'\| \leqq M\|f\|_w$ for each $f \in C^1[0,1]$ where $\|\cdot\|'$ is the standard norm on $AC[0,1)$. Hence $C^1(w) \subseteq AC[0,1]$, and this in turn implies that $C^1(w) = AC_w$ by Lemma 7.

Now assume that $C^1(w) = AC_w$. Hence, $C^1(w) \subseteq AC[0,1]$ and there exists a constant $M > 1$ such that $\|f\|' \leqq M\|f\|_w$ for all $f \in C^1(w)$. Therefore,

$$(*) \qquad \|f\| + \int_0^1 |f'| \leqq M(\|f\| + \|wf'\|), \qquad f \in C^1[0,1].$$

We may assume that $w \geqq 0$. For each positive integer n, let $v_n \in C[0,1]$ be defined by $v_n = \min\{1/w, n\}$ and let $s_n = \int_0^1 v_n$. Then $s_n \to +\infty$ if and only if $1/w$ is not in $L^1[0,1]$. We claim that there exist $\{u_n\} \subset C[0,1]$ such that $|u_n| \leqq v_n$,

$$\int_0^1 |u_n| \geqq s_n - 1,$$

and

$$\left| \int_0^x u_n(t)dt \right| \leqq 1$$

for $x \in [0,1]$ and all n. To prove this, fix n, and let k be some integer larger than s_n. Subdivide $[0,1]$ into k successive disjoint intervals $I_1, \cdots, I_k$ such that

$$\int_{I_j} v_n = s_n/k.$$

Let w_n be defined by $w_n(x) = (-1)^{j+1}v_n(x)$ if $x \in I_j$, and choose a continuous function u_n with the same sign as w_n such that $|u_n| \leqq |w_n| = v_n$, and

$$\int_{I_j} |u_n| = \int_{I_m} |u_n| \geqq (s_n - 1)/k,$$

for $j, m = 1, \cdots, k$. It is clear that $\{u_n\}$ satisfy the claim. Let $f_n \in C^1[0,1]$ be defined by $f_n(y) = \int_0^y u_n(t)dt$. Then $f_n' = u_n$; hence, $\|wf_n'\| = \|wu_n\| \leqq \|wv_n\| \leqq 1$.

https://doi.org/10.1017/S1446788700023879 Published online by Cambridge University Press

Furthermore, by the claim, $\|f_n\| \leqq 1$ and

$$\int_0^1 |f_n'| \geqq s_n - 1.$$

Substituting in (*) gives that $s_n \leqq 2M + 1$. Hence $\{s_n\}$ is bounded, and thus $1/w \in L^1[0,1]$.

We now give an example of a large class of functions w such that $C^1(w)$ properly contains AC_w.

EXAMPLE. If $w \in C[0,1]$ and $w'(x_0) = 0$ at some point $x_0 \in \mathscr{Z}(w)$, then $C^1(w)$ properly contains AC_w. This is easily seen since, by using the definition of $w'(x_0) = 0$, one can show that $1/w$ is not in $L^1[0,1]$.

We shall give a theorem guaranteeing that under certain very general conditions on $\mathscr{Z}(w)$, $C^1(w) = C_w^1$. First, we discuss this problem. In what now follows, we use the result of Theorem 1 that $C^1(w)$ is a regular algebra. Suppose that $f \in C_w^1$. Then $f \in C^1(w)$ if and only if f locally belongs to $C^1(w)$ at each point of $[0,1]$. Let $\mathscr{N}(f)$ be the set of all points of $[0,1]$ at which f locally belongs to $C^1(w)$, and $\mathscr{S}(f) = [0,1] \setminus \mathscr{N}(f)$. Then $f \in C^1(w)$ if and only if $\mathscr{S}(f)$ is empty. But f is continuously differentiable in some neighborhood of each point of $[0,1] \setminus \mathscr{Z}(w)$. Hence $[0,1] \setminus \mathscr{Z}(w) \subseteq \mathscr{N}(f)$. Furthermore, it is easily seen that any continuous function on $[0,1]$ locally belongs to $C^1(w)$ at each point of the interior of $\mathscr{Z}(w)$. Thus the interior of $\mathscr{Z}(w)$ is contained in $\mathscr{N}(f)$. Hence, $\mathscr{S}(f)$ is a closed subset of the boundary of $\mathscr{Z}(w)$.

LEMMA 9. *If $x_0 \in [0,1]$ and if $f \in C_w^1$ belongs locally to $C^1(w)$ at each point of $[0,1] \setminus \{x_0\}$, then $f \in C^1(w)$.*

PROOF. (We give the proof in case $x_0 = 0$; from this one can see how to proceed in the other cases.) Let $g \in C_w^1$ belong locally to $C^1(w)$ at each $y \in (0,1]$, and let $\varepsilon > 0$. Choose $f \in C^1[0,1]$ and $a, c \in (0,1)$ such that (i) $\|f - g\| < \varepsilon/12$, (ii) $a < c$, (iii) $\sup_{y \in [0,c]} |(wg')(y)| < \varepsilon/12$, (iv) $\sup_{y \in [0,c]} |w(y)| < \varepsilon/12((1 + \|f'\|))$, and (v)

$$\int_0^c |f'| < \varepsilon/12.$$

But $g|[a,1]$ is an element of the Banach algebra of restrictions of functions in $C^1(w)$ to $[a,1]$ (where the norm of such a function is the infimum of the norms of functions in $C^1(w)$ agreeing with it on $[a,1]$). Hence, since the C^1-functions are also dense in this restriction algebra, there exists $h \in C^1[a,1]$ such that (vi)

$$\sup_{y \in [a,1]} |h(y) - g(y)| + \sup_{y \in [a,1]} |(wh')(y) - (wg')(y)| < \varepsilon/12.$$

Let b, $a < b < c$, be chosen so that (vii) $\int_b^c |h'| < \varepsilon/12$. Finally, choose $\phi \in C[0,c]$ such that (viii) $\phi(c) = h'(c)$, (ix) $|\phi(y)| \leqq |f'(y)|$ for $0 \leqq y \leqq b$, and

https://doi.org/10.1017/S1446788700023879 Published online by Cambridge University Press

(x) $|\phi(y)| \leqq |h'(y)|$ for $b \leqq y \leqq c$. Let $\psi(y) = \phi(y)$ for $0 \leqq y \leqq c$, and $\psi(y) = h'(y)$ for $c \leqq y \leqq 1$, and define

$$g^*(y) = f(0) + \int_0^y \psi(t)dt.$$

Then $g^* \in C^1[0,1]$, and it is routine to check that $\|g - g^*\|_w < \varepsilon$. The proof is complete.

LEMMA 10. *If* $f \in C_w^1$, *then* $\mathscr{S}(f)$ *is a perfect set.*

PROOF. Suppose that x_0 is an isolated point of $\mathscr{S}(f)$. Then there is a neighborhood of x_0 in which f locally belongs to $C^1(w)$ at each point except x_0. But we may assume that f locally belongs to $C^1(w)$ at each point of $[0,1]\backslash\{x_0\}$. (If not, one can find an element in C_w^1 agreeing with f in a neighborhood of x_0 for which it is true.) Then f must belong to $C^1(w)$ at x_0 by Lemma 9, and this contradiction proves the lemma.

The following theorem is clear from Lemma 10, the discussion preceding Lemma 9, and Theorem 120 of Sierpinski (1952).

THEOREM 11. *If the boundary of* $\mathscr{Z}(w)$ *is countable, then* $C^1(w) = C_w^1$.

The result of Theorem 11 and its proof are analogous to a theorem of Ditkin (see p. 226 of Naimark (1964)), although $C^1(w)$ is not an ideal in C_w^1. In addition, because there are nontrivial point derivations on C_w^1 at points where w is not zero, we see that Ditkin's condition will not hold in C_w^1 unless w is identically zero.

COROLLARY 12. *If* $\tilde{D}_w$ *is almost injective, then* $C^1(w) = C_w^1$.

PROOF. This follows from Theorem 11 and Theorem 6.

Theorem 11 says that $C^1(w) = C_w^1$ except possibly when $\mathscr{Z}(w)$ is a "complicated" set. We now give such an example where $C_w^1 \neq C^1(w)$.

EXAMPLE. Let $\mathscr{T}$ be the Cantor ternary set on $[0,1]$, and let w be defined by $w(y) = (\text{distance}(y, \mathscr{T}))^{1/3}$. Then $1/w \in L^1[0,1]$; hence, by Theorem 8, $C^1(w) = AC_w$. But since C_w^1 contains the Cantor ternary function, we see that $C^1(w) \neq C_w^1$.

References

R. J. Loy (1970), 'Maximal ideal spaces of Banach algebras of derivable elements', *J. Austral. Math. Soc.* **11**, 310–312.

M. A. Naimark (1964), *Normed rings* (Noordhoff, Groningen, The Netherlands, 1964).

C. E. Rickart (1960), *General theory of Banach algebras* (Van Nostrand, New York, 1960).

W. Sierpinski (1952), *General topology* (University of Toronto Press, Toronto, 1952).

Department of Mathematics
University of Nevada, Las Vegas
Las Vegas, Nevada 89154, U.S.A.