13

Anomalies

A characteristic feature of relativistic quantum field theories is that symmetries of the classical theory are not always present after quantization. We do not mean here the spontaneous breaking that is characterized by a non-invariant vacuum and by the presence of the Goldstone bosons. Rather we mean a situation where there is no conserved current for the symmetry despite the absence of any terms in the action that appear to break the symmetry. Such breaking of a symmetry is called anomalous.

If the classical action is invariant, then a naive application of Noether's theorem gives us a conserved current. That is, there is no anomalous symmetry breaking. What prevents the argument from being correct is the presence of UV divergences. The current is a composite operator, i.e., a product of elementary fields at the same point, and to define it, some kind of regularization and renormalization is needed. The renormalization may invalidate the equations used to prove Noether's theorem.

For simplicity, we will consider only global symmetries, as opposed to local, or gauge, symmetries. The simplest cases of global symmetries were considered in Chapter 9. These could be treated by using an ultra-violet regulator that preserved the symmetry. The proof of Noether's theorem can then be made in the cut-off theory. We showed that only symmetric counterterms are needed. Consequently the symmetry remains good after the cut-off is removed.

However, not all symmetries can be preserved after regularization. The case which we will treat in this chapter is that of chiral symmetries. These are transformations that act independently on the left- and right-handed components of Dirac fields. These are particularly interesting because sometimes the anomalous breaking of chiral symmetries cancels. Indeed there is a theorem, first proved by Adler & Bardeen (1969), that if anomalous breaking of a chiral symmetry is zero to one-loop order then it is zero to all orders.

Our treatment will use dimensional regularization. Chiral symmetry is valid in the physical space-time dimension $d = 4$, but not when $d \neq 4$. The anomaly in, say, an equation of current conservation will be an operator with

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in effect a coefficient proportional to $d-4$, where d is the space-time dimension. This would vanish at $d=4$, were it not for the existence of ultra-violet divergences which allow the anomaly operator to have a pole at $d=4$ so that a non-zero anomaly results at $d=4$. We will see explicitly how this works.

There are two key issues. The first is to derive simple forms for the anomaly. Of these the most dramatic is the Adler–Bardeen theorem that tells us that in some cases there is complete cancellation of the anomaly. The second issue is to derive the results in a form that is applicable to the physical theory, i.e., after renormalization and removal of the cut-off. Even though a particular derivation depends on the choice of a particular regularization scheme, the final results must be independent of this choice.

Note that to prove existence of an anomaly, it is not sufficient to say that the symmetry in question is broken in the regulated theory. The breaking may go away after removal of the cut-off. For example, if one uses a lattice cut-off, then Poincaré invariance is lost. However, one must prove that Poincaré invariance is restored in the renormalized continuum limit, if the theory is to agree with real world phenomena.

Aside from the case of chiral transformations, there are a number of other important situations where there are anomalies. One of the simplest is that of dilatations. These are scale transformations on space-time: $x^\mu \to \lambda x^\mu$. A classical Lagrangian is scale-invariant if it contains no dimensional parameters, like a mass scale. But to cut-off ultra-violet divergences we necessarily introduce a mass scale. The symmetry is necessarily broken in the regulated theory and the question arises of whether the symmetry remains broken after the theory is renormalized and the cut-off is removed. This answer is, in general, yes, if the theory has interactions. The reason is that there is, in fact, a mass scale hidden in the renormalized theory, as we saw when we discussed dimensional transmutation and the renormalization group, in Chapter 7. Detailed treatments can be found in the literature (see Collins (1976), Brown (1980) and references therein). The simple form for the Ward identity is known as the Callan–Symanzik equation (Callan (1970) and Symanzik (1970b)). The information contained in the Callan–Symanzik equation is in fact also contained in the renormalization-group equation that we studied in Chapter 7.

Other situations which we will not treat include the following: chiral gauge theories (see Costa *et al.* (1977), Bandelloni *et al.* (1980), Piguet & Rouet (1981)), conformal transformations (Sarkar (1974)) and supersymmetries (Piguet & Rouet (1981), Clark, Piguet & Sibold (1979, 1980), and Piguet & Sibold (1982a, b, c)).

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13.1 Chiral transformations

We will consider QCD with two flavors of quark: up and down. (We could have more flavors, but no essentially new ideas would be needed.) The Lagrangian is (2.11.7). If the quarks were all massless then the classical theory is invariant under the following transformations of the quark fields:

$$\psi \to \exp\{i[\tfrac{1}{2}(1-\gamma_5)(\omega_L^0 + \omega_L^a t^a) + \tfrac{1}{2}(1+\gamma_5)(\omega_R^0 + \omega_R^a t^a)]\}\psi. \quad (13.1.1)$$

Here the matrices t^a are the generators of the isospin group acting on the flavor indices. The transformations (13.1.1) form a group that we will call $U(1)_L \otimes U(1)_R \otimes SU(2)_L \otimes SU(2)_R$. The symmetry of the QCD Lagrangian under these transformations is broken by mass terms for the quarks. Since the masses of the u and d quarks are small, the chiral symmetries are only weakly broken.

These symmetries and their breaking were understood well before the advent of QCD – see, for example, Treiman, Gross & Jackiw (1972). Treatments of chiral symmetries in the light of QCD can be found in Marciano & Pagels (1978) and in Llewellyn-Smith (1980). Thus it is unnecessary to go into details here. What we will emphasize is how the potential for anomalies arises.

Since these transformations involve γ_5, they are in some sense coupled to the spin structure of the theory. Since spin is related to the symmetries of space-time, we can expect trouble when the theory is regulated, for imposition of an ultra-violet cut-off must alter the space-time structure.

Notice that there is a $U(1) \otimes SU(2)$ subgroup not involving γ_5; these transformations have $\omega_L = \omega_R$. For them the treatment of Chapter 9 is correct. The corresponding Noether currents are

$$\begin{aligned} j^\mu &= Z\bar{\psi}\gamma^\mu\psi, \\ j_a^\mu &= Z\bar{\psi}\gamma^\mu t^a\psi. \end{aligned} \quad (13.1.2)$$

The conserved charge derived from j^μ is the conserved quark number, while the transformations generated by j_a^μ are just ordinary isospin transformations.

For the other generators of chiral transformations, let us define axial currents

$$\begin{aligned} j_5^\mu &= Z\bar{\psi}\tfrac{1}{2}[\gamma^\mu, \gamma_5]\psi, \\ j_{a5}^\mu &= Z\bar{\psi}\tfrac{1}{2}[\gamma^\mu, \gamma_5]t^a\psi. \end{aligned} \quad (13.1.3)$$

In four dimensions $\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu$, so that we could have written $\gamma^\mu\gamma_5$ in place of the commutator in (13.1.3). However, because we will use dimensional regularization, we must use the form (13.1.3) in order to ensure that the currents are hermitian.

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To define the theory we must regulate its ultra-violet divergences, and for this we will use dimensional regularization. We will see that the regulated theory is not invariant under the transformations generated by the axial currents j_5^μ and j_{5a}^μ. This allows the possibility of an anomaly. When the cut-off is removed we will find that we can arrange for the non-singlet currents j_{a5}^μ to be without anomaly. Although we will not demonstrate it, it is true that the singlet current necessarily has an anomalous divergence.

Our remarks above were addressed to the case that all the quarks are massless. But in the real world, there are quark mass terms in the Lagrangian, so we now generalize our discussion. The vector singlet current j^μ is the current for quark number – i.e., 1/3 of baryon number – so this remains an exact symmetry. The $SU(2)$ symmetry given by the vector currents is broken by quark mass differences:

$$\partial_\mu j_a^\mu = -\mathrm{i}\bar{\psi}[t^a, M]\psi, \tag{13.1.4}$$

where M is the quark mass matrix. Since the masses of the u and d quarks are small we have an approximate isospin symmetry of strong interactions. By the theory given in Chapter 9, the currents j_a^μ and j^μ, as defined by (13.1.2), are finite, since the breaking is from mass terms (Symanzik (1970a)).

The axial symmetries are broken by the anomaly as well as by the quark mass terms:

$$\begin{aligned}\partial_\mu j_5^\mu &= -2\mathrm{i}\bar{\psi}\gamma_5 M\psi + \text{anomaly},\\ \partial_\mu j_{a5}^\mu &= -\mathrm{i}\bar{\psi}\gamma_5\{t^a, M\}\psi.\end{aligned} \tag{13.1.5}$$

The u- and d-quark masses are light enough to give us an approximate $SU(2)\otimes SU(2)$ symmetry. The axial part appears to be spontaneously broken as well as explicitly broken. The abnormally light mass of the pion is usually taken to mean that it would be a Goldstone boson if $m_u = m_d = 0$. See Marciano & Pagels (1978) and Gasser & Leutwyler (1982) for more details.

13.2 Definition of γ_5

In order to formulate consistently the dimensional regularization of theories with fermions we had to define an infinite set of matrices γ^μ, ($\mu = 0, 1, 2, \ldots$). As we saw in Chapter 4, they satisfied the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1, \quad \mathrm{tr}\, 1 = 4. \tag{13.2.1}$$

Our task is now to find a generalization of the matrix which at $d = 4$ is called γ_5. Its four dimensional definition is

$$\gamma_5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3; \tag{13.2.2}$$

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and it satisfies the anticommutation relation

$$\{\gamma^\mu, \gamma_5\} = 0 \quad (d = 4). \tag{13.2.3}$$

It would be natural to assume that this relation (13.2.3) can be maintained for arbitrary values of d. Unfortunately an inconsistency arises, as we will now show, when we wish to compute the trace of γ_5 with a product of the ordinary γ^μ's. The ultimate result will be that stated in Section 4.6, where we used the definition (13.2.2) of γ_5 for all values of d. Then γ_5 has mixed commutation and anticommutation relations, (4.6.3).

We now demonstrate the inconsistency in trace calculations, starting with $\operatorname{tr}\gamma_5$:

$$\begin{aligned} d\operatorname{tr}(\gamma_5) &= \operatorname{tr}(\gamma_5\gamma^\mu\gamma_\mu) \\ &= \operatorname{tr}(\gamma_\mu\gamma^5\gamma_\mu) \\ &= -\operatorname{tr}(\gamma_5\gamma_\mu\gamma^\mu) \\ &= -d\operatorname{tr}(\gamma_5). \end{aligned} \tag{13.2.4}$$

In the first and last lines we used

$$\gamma_\mu\gamma^\mu = \tfrac{1}{2}\{\gamma_\mu, \gamma^\mu\} = g^\mu_\mu 1 = d.$$

In the second line we used cyclicity of a trace, and in the third line we assumed (13.2.3). From (13.2.4) we see that $\operatorname{tr}\gamma_5 = 0$ except at $d = 0$. Now when we apply dimensional regularization we wish to obtain a result that is a meromorphic function of d. Hence we must have $\operatorname{tr}\gamma_5 = 0$ for all d.

Similarly

$$\begin{aligned} d\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu &= \operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma^\lambda \\ &= \operatorname{tr}\gamma^\lambda\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda \\ &= -\operatorname{tr}\gamma_5\gamma^\lambda\gamma_\mu\gamma_\nu\gamma_\lambda \\ &= -2g^\lambda_\mu\operatorname{tr}\gamma_5\gamma_\nu\gamma_\lambda + 2g^\lambda_\nu\operatorname{tr}\gamma_5\gamma_\mu\gamma_\lambda - d\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu \\ &= -2\operatorname{tr}\gamma_5\{\gamma_\mu, \gamma_\nu\} + (4-d)\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu \\ &= (4-d)\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu. \end{aligned} \tag{13.2.5}$$

Here we used the result $\operatorname{tr}\gamma_5 = 0$. Hence we find that $(d-2)\operatorname{tr}\gamma_5\gamma_\mu\gamma_\nu$ is zero.

Now, the same technique can be used to prove that

$$(4-d)\operatorname{tr}(\gamma_5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = 0. \tag{13.2.6}$$

At $d = 4$ this equation permits the usual non-zero trace of γ_5 with four other Dirac matrices. However, if the trace is to be meromorphic in d, equation (13.2.6) shows that it must be zero at $d = 4$, and we can therefore not obtain normal physics at $d = 4$. This is the inconsistency referred to at the start of this section.

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We are therefore forced to drop one of the hypotheses that led to (13.2.6). Candidate hypotheses for removal include:

(1) the anticommutation relation (13.2.3) ('t Hooft & Veltman (1972a)),
(2) the use of dimensional regularization for fermion loops (Bardeen (1972), and Chanowitz, Furman & Hinchliffe (1979)).

These last authors have shown how to calculate with a totally anticommuting γ_5. Bardeen chooses to use a regulator other than dimensional regularization for all fermion loops. On the other hand, Chanowitz *et al.* regulate fermion loops with an even number of γ_5's dimensionally. Their procedure is useful for low-order graphs, since the Ward identities are preserved for graphs without fermion loops. However, we then lose the use of dimensional regularization as a complete regulator; the theorems that we derived in Section 6.6 and in Chapter 12 no longer apply. The details of higher-order calculations by this method have not been spelled out.

Therefore let us follow 't Hooft & Veltman (1972a) and Breitenlohner & Maison (1977a) and change the anticommutation relation. In fact, we may use the definition (13.2.2) for all values of d, just as stated in Section 4.6. Our definition is, of course, not completely Lorentz covariant, since the first four dimensions are picked out as special. But this is not an overwhelming objection, for our actual physics is confined to these dimensions. An important advantage of the definition is that it gives a concrete construction of γ_5. We are therefore guaranteed consistency.

One notational inconvenience arises. We have a set of matrices γ^μ for $\mu = 0, 1, 2, \ldots$. Usually we only refer explicitly to the first four; the rest are referred to collectively. But there is γ^μ with $\mu = 5$. It is not the same as γ_5 defined by (13.2.2). However, we bow to standard usage and use γ_5 to denote the matrix defined in (13.2.2). Confusion should be rare.

The γ_5 so defined has mixed commutation and anticommutation relations that follow from (13.2.2) and the properties of γ^μ. These were stated in (4.6.3). The derivation of (13.2.4)–(13.2.6) now fails, because γ_5 does not anticommute with all of the γ^μ's. We can derive the correct relations easily. Since the trace of an odd number of γ^μ's is zero, we have

$$\operatorname{tr}(\gamma^\mu\gamma_5) = 0, \quad \operatorname{tr}(\gamma_5\gamma^\kappa\gamma^\lambda\gamma^\mu) = 0, \text{ etc.} \tag{13.2.7}$$

We may read off the trace of γ_5 with an even number of γ^μ's from (4.5.13) and its relatives with more than four γ-matrices. Thus we have

$$\left.\begin{aligned} \operatorname{tr}(\gamma_5) &= 0, \\ \operatorname{tr}(\gamma_5\gamma^\mu\gamma^\nu) &= 0. \end{aligned}\right\} \tag{13.2.8}$$

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Finally

$$\operatorname{tr}(\gamma_5\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu) = 4\mathrm{i}\varepsilon_{\kappa\lambda\mu\nu}, \tag{13.2.9}$$

where $\varepsilon_{\kappa\lambda\mu\nu}$ is given by (4.6.2).

The relations (4.6.3) mean that the chiral transformations (13.1.1) no longer generate a symmetry if d is not equal to 4. To discuss the resulting problems, the following notation is useful (Breitenlohner & Maison (1977a)):

$$\left.\begin{aligned} \hat{g}_{\mu\nu} &= \begin{cases} g_{\mu\nu}, & \text{if } \mu \text{ and } \nu \text{ are 4 or larger,} \\ 0, & \text{otherwise;} \end{cases} \\ \hat{V}^\mu &= \hat{g}^{\mu\nu} V_\nu. \end{aligned}\right. \tag{13.2.10}$$

Here V^μ is any vector. Then $\hat{g}_{\mu\nu}$ is a projector onto the unphysical dimensions. Thus, for example,

$$\left.\begin{aligned} \hat{g}^\mu_\mu &= g^{\mu\nu}\hat{g}_{\mu\nu} = \hat{g}^{\mu\nu}\hat{g}_{\mu\nu} = d - 4, \\ \{\not{V}, \gamma_5\} &= \sum_{\mu > 3} V_\mu\{\gamma^\mu, \gamma_5\} = 2\hat{\not{V}}\gamma_5 = 2\gamma_5\hat{\not{V}}. \end{aligned}\right\} \tag{13.2.11}$$

We may also define projections onto the four physical dimensions:

$$\left.\begin{aligned} \bar{g}^{\mu\nu} &= \begin{cases} g^{\mu\nu} & \text{if } \mu \text{ and } \nu \text{ are less than 4,} \\ 0 & \text{otherwise;} \end{cases} \\ \bar{V}^\mu &= \bar{g}^{\mu\nu} V_\nu. \end{aligned}\right\} \tag{13.2.12}$$

The following results are elementary, but will prove useful:

$$\left.\begin{aligned} \hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}_\mu &= (6-d)\hat{\gamma}^\nu, \\ \hat{\gamma}^\mu\bar{\gamma}^\nu\hat{\gamma}_\mu &= (4-d)\bar{\gamma}^\nu, \\ \bar{\gamma}^\mu\bar{\gamma}^\nu\bar{\gamma}_\mu &= -2\bar{\gamma}^\nu, \\ \bar{\gamma}^\mu\hat{\gamma}^\nu\bar{\gamma}_\mu &= -4\hat{\gamma}^\nu. \end{aligned}\right\} \tag{13.2.13}$$

Let us define $\omega^a = (\omega^a_{\mathrm{L}} + \omega^a_{\mathrm{R}})/2$ and $\omega^a_5 = (-\omega^a_{\mathrm{L}} + \omega^a_{\mathrm{R}})/2$. Then the variation of $\mathscr{L}$ under the chiral transformations (13.1.1) is

$$\delta\mathscr{L} = \mathrm{i}\bar{\psi}(2\omega^0_5\gamma_5 + \omega^a[M, t^a] + \omega^a_5\gamma_5\{M, t^a\})\psi - 2\bar{\psi}\gamma_5\hat{\not{D}}(\omega^0_5 + \omega^a_5 t^a)\psi. \tag{13.2.14}$$

Hence the divergences of the axial currents are

$$\begin{aligned} \partial_\mu j^\mu_{5a} &= -\mathrm{i}\bar{\psi}\gamma_5\{t^a, M\}\psi + \bar{\psi}\gamma_5\hat{\not{D}}t^a\psi \\ &= -\mathrm{i}\bar{\psi}\gamma_5\{t^a, M\}\psi + \bar{\psi}\gamma_5\{t^a, \hat{\not{D}}\}\psi/2, \end{aligned} \tag{13.2.15}$$

$$\partial_\mu j^\mu_5 = -2\mathrm{i}\bar{\psi}\gamma_5 M\psi + \bar{\psi}\gamma_5\hat{\not{D}}\psi. \tag{13.2.16}$$

The second term in each equation can potentially give an anomaly when we let $d \to 4$.

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13.3 Properties of axial currents

There are a number of somewhat different situations in which axial currents appear. The original papers on chiral anomalies primarily addressed anomalies in Ward identities of the chiral currents of strong interactions. More general cases have since been worked out, with corresponding generalizations of the Adler–Bardeen theorem. In this section we will list the various cases and state what is known.

13.3.1 *Non-anomalous currents*

The following properties apply, for example, to the non-singlet axial currents in QCD:

(1) If there is no anomaly to one-loop order (as for the current j^{μ}_{a5}), then there is no anomaly to all orders. The Ward identities of the current with elementary fields have no anomalies.
(2) Under the same condition as in (1) there is no anomaly in the two-current Ward identities.
(3) Under the same condition there is an anomaly in a three-current Ward identity (like the one for $\partial_{\kappa}\langle 0|\,Tj^{\kappa}_{a5}(x)j^{\mu}_{b}(y)j^{\nu}_{c}(z)|0\rangle$). However, the only non-zero term in the anomaly is the one-loop contribution.

The theorem that the complete anomaly in these cases is determined by the one-loop value is due to Adler & Bardeen (1969).

The lack of anomalies in the Ward identities of one current with elementary fields is essential if the currents are to generate the correct transformation law for the fields. These transformations imply commutation relations for the currents. Since these commutators are also given by the Ward identities with two currents, the two-current identities must be anomaly-free.

No such consistency requirement applies to the three-current Ward identity. The value of its anomaly is related to the decay rate for $\pi^0 \to 2\gamma$, and the lack of higher-order corrections enables a successful prediction to be made easily. (See Adler (1970); for reviews from the point of view of QCD, see Marciano & Pagels (1978) and Llewellyn-Smith (1980).)

13.3.2 *Anomalous currents*

The singlet current j_5 has an anomalous divergence. It has the form

$$\partial_{\mu}j^{\mu}_{5} = C(g)G^{a}_{\mu\nu}\tilde{G}^{a\mu\nu} + \text{mass terms}, \tag{13.3.1}$$

where $G^{a}_{\mu\nu}$ is the gluon field strength tensor and $\tilde{G}^{a}_{\mu\nu}$ is its dual:

$$\tilde{G}^{a\mu\nu} = \varepsilon^{\mu\nu\kappa\lambda}G^{a}_{\kappa\lambda}. \tag{13.3.2}$$

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The Adler–Bardeen theorem asserts that the coefficient of $C(g)$ is equal to its one-loop value

$$C(g) = N_{\mathrm{fl}} g^2/(32\pi^2).$$

However, the operators j_5 and $G\tilde{G}$ need renormalization, and there is not an obvious natural renormalization condition. So the value of C is susceptible to change by redefinition of the renormalization prescription. We will not treat this case here.

13.3.3 *Chiral gauge theories*

Theories like the Weinberg–Salam theory of weak interactions have a gauged chiral symmetry. It is essential that there be no anomaly, for otherwise the theory is not renormalizable and loses other important properties (Gross & Jackiw (1972) and Korthals Altes & Perrottet (1972)). A generalization of the Adler–Bardeen theorem is that there is no anomaly to any order of perturbation theory if there is none to one-loop order. Proofs have been given by Becchi, Rouet & Stora (1976), and by Costa *et al.* (1977).

13.3.4 *Supersymmetric theories*

Supersymmetric theories have potential anomalies similar to the chiral anomalies. A completely general treatment has not yet been given, but many particular cases have been treated – see Piguet & Rouet (1981), Piguet & Sibold (1982a, b, c), Clark, Piguet & Sibold (1979, 1980), and Jones & Leveille (1982).

13.4 Ward identity for bare axial current

Without use of the equations of motion the divergence of the non-singlet current j^μ_{a5} is

$$\begin{aligned} \partial_\mu j^\mu_{a5} &= \mathrm{i}Z_2\bar{\psi}\gamma_5 t^a(\mathrm{i}\not{D} - M_0)\psi + hc \\ &\quad + \mathrm{i}Z_2\bar{\psi}\{M_0, t^a\}\gamma_5\psi + \tfrac{1}{4}Z_2\bar{\psi}t^a\{\overleftrightarrow{\not{D}}, \gamma_5\}\psi \\ &= D^a_{\mathrm{em}} + D^a_M + D^a_{\mathrm{anom}}. \end{aligned} \tag{13.4.1}$$

The first term we will call the equation of motion term. When inserted in a Green's function with elementary fields it gives

$$\begin{aligned} &\langle 0|TD^a_{\mathrm{em}}(x)\prod A\prod_i \psi(y_i)\prod_j \bar{\psi}(z_j)|0\rangle \\ &= \sum_i \delta(x - y_i)\langle 0|T\prod A\prod\psi\prod\bar{\psi}|0\rangle_{\psi(y_i)\to -\gamma_5 t^a\psi} \\ &\quad + \sum_j \delta(x - z_j)\langle 0|T\prod A\prod\psi\prod\bar{\psi}|0\rangle_{\bar{\psi}(z_j)\to\bar{\psi}\gamma_5 t^a}. \end{aligned} \tag{13.4.2}$$

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The variations of the fields are just their chiral transformations multiplied by i. (We set $\omega_{\mathrm{R}}^a = -\omega_{\mathrm{L}}^a = \omega_5^a$ in (13.1.1) to obtain these transformations.) This equation (13.4.2) has the expected right-hand side for the Ward identity in the absence of anomalies.

The mass term on the right of (13.4.1) is expected; it is the non-anomalous breaking.

The anomaly term is D^a_{anom}. If we insert it in a Green's function with no divergences of any kind, then we may set $d = 4$. The result must be zero because D^a_{anom} vanishes if only the first four γ-matrices are used (for then $\{\not{D}, \gamma_5\} = 0$). But if the Green's function has an overall divergence, or if it has a divergent subgraph that contains the vertex for D^a_{anom}, then the result may be non-zero at $d = 4$, as we will see. The full Ward identity reads

$$\frac{\partial}{\partial x^\mu}\langle 0|Tj^\mu_{a5}(x)\prod A\prod\psi\prod\bar{\psi}|0\rangle = \text{right-hand side of (13.4.2)} + \text{mass term}$$
$$+\langle 0|TD^a_{\mathrm{anom}}(x)\prod A\prod\psi\prod\bar{\psi}|0\rangle. \qquad (13.4.3)$$

Recall that in the case of a symmetry such as the isospin $SU(2)$ of QCD that has no anomaly, we used its Ward identity to prove the current finite. The only possible counterterm for the current is proportional to itself, so finiteness of the divergence of the current, $\partial\cdot j$, implies finiteness of the current itself. The Ward identities imply that the divergence of the current is finite. However, for the axial currents the extra term in (13.4.3) prevents this argument from being made.

13.4.1 *Renormalization of operators in Ward identities*

Our aim will be to construct a finite current j^μ_{Ra5} that at $d = 4$ satisfies a non-anomalous Ward identity:

$$\frac{\partial}{\partial x^\mu}\langle 0|Tj^\mu_{Ra5}(x)\prod A\prod\psi\prod\bar{\psi}|0\rangle = \text{right-hand side of (13.4.2)}. \qquad (13.4.4)$$

The first step is to observe that the only counterterm to j^μ_{a5} is itself. No other operators have the correct dimension and quantum numbers. So we can define a minimally subtracted operator

$$\begin{aligned}[j^\mu_{a5}] &= Z_5 j^\mu_{a5}\\ &= \tfrac{1}{2}Z_5 Z_2\bar{\psi}[\gamma^\mu,\gamma_5]t^a\psi.\end{aligned} \qquad (13.4.5)$$

Throughout this section we will use square brackets to indicate minimal subtraction. So the renormalization factor Z_5 has the form

$$Z_5 = 1 + \text{poles at } d = 4.$$

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To obtain the operator j_{Ra5}, we will later show that we have to invoke a further finite renormalization.

We will show that the Ward identity (13.4.4) is not correct if j_{Ra5} is replaced by the minimally subtracted $[j_{a5}]$. Rather we must make a further finite renormalization to obtain a Ward identity without anomalies. Thus we have

$$\begin{aligned} j^{\mu}_{Ra5} &= z_5[j^{\mu}_{a5}] \\ &= \tfrac{1}{2} z_5 Z_5 Z_2 \bar{\psi}[\gamma^{\mu}, \gamma_5] t^a \psi, \end{aligned} \tag{13.4.6}$$

where $z_5(g)$ is a finite factor.

The anomaly operator D^a_{anom} is a dimension four scalar quantity, so there are several operators with which it can mix. It is proportional to the ε-tensor times a fourth rank tensor, which we will call $\Delta^{\kappa\lambda\mu\nu}$. Since $\varepsilon_{\kappa\lambda\mu\nu}$ appears nowhere in the Lagrangian, the tensor $\Delta^{\kappa\lambda\mu\nu}$ is invariant under all Lorentz transformations. Given these restrictions, a complete list of operators that mix with D^a_{anom} is

$$\partial_{\mu} j^{\mu}_{a5}, D^a_{\text{anom}}, \bar{\psi}\{M_0, t^a\}\gamma_5\psi.$$

No operator involving only ghost and gluon fields can be constructed such that it mixes with D^a_{anom}. The linearity in $\varepsilon_{\kappa\lambda\mu\nu}$ implies the presence of four factors of vector objects (derivatives or A-fields). Therefore the coefficient of the operator is independent of mass, by our usual results. Gauge invariance of D^a_{anom} allows a restricted set of gauge variant counterterms (see Chapter 12) none of which have low enough dimension to appear. Since j_{a5} is even under charge conjugation, so is D^a_{anom}. Therefore the only allowed counterterm proportional to quark mass has a flavor factor $\{M, t^a\}$ rather than the commutator $[M, t^a]$; this gives us the operator D^a_M that appears in (13.4.1).

We can therefore write the minimally subtracted operator corresponding to $[D^a_{\text{anom}}]$ as

$$[D^a_{\text{anom}}] = Z_a D^a_{\text{anom}} + Z_{a5}\partial\cdot j_{a5} + Z_{am} D^a_M. \tag{13.4.7}$$

The equation of motion operator is finite by itself – see (13.4.2) – so we have

$$[D^a_{\text{em}}] = D^a_{\text{em}}. \tag{13.4.8}$$

Note that the definition of D^a_{em} includes some counterterms, but these are manufactured from the wave-function, mass, and coupling renormalizations in the Lagrangian.

We also need the renormalization of the mass operator

$$[D^a_M] = Z_{5M} D^a_M. \tag{13.4.9}$$

It is somewhat unobvious that the renormalization is multiplicative. The easiest method is to examine the γ-matrix structure of self-energy graphs with

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an insertion of $\bar{\psi}t^a\gamma_5\psi$ or $\bar{\psi}\gamma_5\psi$. These operators can then be shown to be multiplicatively renormalizable with a common factor.

By use of (13.4.6)–(13.4.9) we can express the equation (13.4.1), for the divergence of the bare axial current, in terms of renormalized operators to find

$$\begin{aligned}[D^a_{\text{em}}] = {} & Z_5^{-1}(1 - Z_{a5}Z_a^{-1})\partial_\mu[j^\mu_{a5}] \\ & - [D^a_M]Z_{5M}^{-1}(1 + Z_{aM}Z_a^{-1}) - Z_a^{-1}[D^a_{\text{anom}}]. \qquad (13.4.10)\end{aligned}$$

The renormalized operators on the right are all linearly independent, so the only way a linear combination of them can equal the finite left-hand side is for the coefficients to be finite. Since we use minimal subtraction this implies:

$$\left.\begin{aligned} Z_a &= 1, \\ Z_{aM} &= Z_{5M} - 1, \\ Z_{a5} &= 1 - Z_5. \end{aligned}\right\} \qquad (13.4.11)$$

We therefore have the renormalized Ward identity:

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\langle 0|T[j^\mu_{a5}]\textstyle\prod A\prod\psi\prod\bar{\psi}|0\rangle = {} & \text{r.h.s. of (13.4.2)} \\ & + \langle 0|T([D^a_M] + [D^a_{\text{anom}}])\textstyle\prod A\prod\psi\prod\bar{\psi}|0\rangle, \\ & \qquad (13.4.12)\end{aligned}$$

which apparently still has an anomaly. Before showing how the anomaly in fact disappears, let us examine some low-order graphs.

13.5 One-loop calculations

The tree approximation for the two-point Green's function of $[D^a_{\text{anom}}]$ is given by Fig. 13.5.1. To save algebra we will set quark masses to zero. The graph's value is

$$\frac{\mathrm{i}\not{p}'}{p'^2}(-\mathrm{i})(\hat{\not{p}}' + \hat{\not{p}})\gamma_5 t^a \frac{\mathrm{i}\not{p}}{p^2}. \qquad (13.5.1)$$

If we let the external momenta be physical, i.e., in the first four dimensions, then this vanishes. The vertex for D^a_{anom} has the property we define as evanescence: it vanishes when the cut-off is removed and we go to the physical renormalized theory. We will formulate a precise definition of evanescence later, when we have understood the subtleties associated with inserting the vertex inside loops.

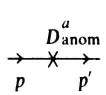

Fig. 13.5.1. Tree approximation for two-point Green's function of $[D^a_{\text{anom}}]$.

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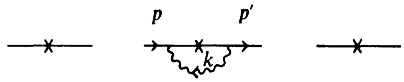

(a) (b) (c)

Fig. 13.5.2. Graphs up to order g^2 for two-point Green's function of j^μ_{a5}.

Next let us examine the graphs for the two-point function of the current j^μ_{a5}, as given in Fig. 13.5.2. Graph (*a*) has the value

$$\frac{i\not p'}{p'^2}\bar{\gamma}^\mu\gamma_5 t^a \frac{i\not p}{p^2}. \tag{13.5.2}$$

Taking the divergence is the same as multiplying by $i(p'-p)^\mu$. The result is

$$\begin{aligned}
&\frac{i\not p'}{p'^2}(\bar{\not p}' - \bar{\not p})i\gamma_5 t^a \frac{i\not p}{p^2}\\
&= \frac{i\not p'}{p'^2}(\bar{\not p}'\gamma_5 + \gamma_5\bar{\not p})it^a\frac{i\not p}{p^2}\\
&= \frac{i\not p'}{p'^2}(\not p'\gamma_5 + \gamma_5\not p - \hat{\not p}'\gamma_5 - \gamma_5\hat{\not p})it^a\frac{i\not p}{p^2}\\
&= -\gamma_5 t^a i\not p/p^2 - i(\not p'/p'^2)\gamma_5 t^a + (13.5.1).
\end{aligned} \tag{13.5.3}$$

When we set $\hat{\not p} = \hat{\not p}' = 0$, to get the four-dimensional result, we obtain the lowest-order case of the chiral Ward identity.

The next order graph is Fig. 13.5.2(*b*). Its value, with the external propagators amputated, is

$$\Gamma_{2b} = \frac{ig^2 C_F}{16\pi^4}(2\pi\mu)^{4-d}\int d^d k \frac{\gamma_\nu(\not p' + \not k)\bar{\gamma}^\mu\gamma_5 t^a(\not p + \not k)\gamma^\nu}{k^2(p'+k)^2(p+k)^2}. \tag{13.5.4}$$

This is evidently divergent. It is easy to calculate the pole at $d=4$:

$$\begin{aligned}
\text{pole}(\Gamma_{2b}) &= \frac{g^2 t^a C_F}{32\pi^2}\,\text{pole}\left\{\frac{1}{4-d}\gamma_\nu\gamma_\kappa\bar{\gamma}^\mu\gamma_5\gamma^\kappa\gamma^\nu\right\}\\
&= \frac{g^2 t^a C_F}{32\pi^2}\,\text{pole}\left\{\frac{1}{4-d}\bar{\gamma}^\mu\gamma_5(d-6)^2\right\}\\
&= \frac{g^2}{8\pi^2}\frac{C_F t^a}{4-d}\bar{\gamma}^\mu\gamma_5.
\end{aligned} \tag{13.5.5}$$

Here we have twice used the result that

$$\begin{aligned}
\gamma_\kappa\bar{\gamma}^\mu\gamma_5\gamma^\kappa &= \bar{\gamma}_\kappa\bar{\gamma}^\mu\gamma_5\bar{\gamma}^\kappa + \hat{\gamma}_\kappa\bar{\gamma}^\mu\gamma_5\hat{\gamma}^\kappa\\
&= 2\bar{\gamma}^\mu\gamma_5 - \hat{\gamma}_\kappa\hat{\gamma}^\kappa\bar{\gamma}^\mu\gamma_5\\
&= (6-d)\bar{\gamma}^\mu\gamma_5.
\end{aligned} \tag{13.5.6}$$

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There is a counterterm graph implicit in the definition $j^\mu_{a5} = Z_2\bar{\psi}\bar{\gamma}^\mu\gamma_5 t^a\psi$, with the quark wave-function renormalization given by

$$Z_2 = 1 - \frac{g^2 C_{\mathrm{F}}}{8\pi^2(4-d)} + O(g^4). \tag{13.5.7}$$

The resulting counterterm graph Fig. 13.5.2(c) therefore cancels the UV divergence of graph (b), leaving a finite result. No additional renormalization is needed:

$$Z_5 = 1 + O(g^4). \tag{13.5.8}$$

Let us next take the divergence of (13.5.5) plus its counterterm, by multiplying by $\mathrm{i}(p'-p)_\mu$. It is left as an exercise for the reader to verify that the Ward identity (13.4.3) holds at this order. What we will do is examine the graphs of order g^2 for the Green's function of D^a_{anom}. These are listed in Fig. 13.5.3. Note that the definition includes a covariant derivative:

$$\begin{aligned}\tfrac{1}{2}Z_2\bar{\psi}\{\overleftrightarrow{\not{D}},\gamma_5\}t^a\psi &= \tfrac{1}{2}\bar{\psi}Z_2\{\overleftrightarrow{\not{\partial}},\gamma_5\}t^a\psi - \mathrm{i}g_0 Z_2\bar{\psi}\{\not{A}_0,\gamma_5\}t^a\psi \\ &= Z_2\bar{\psi}\hat{\overleftrightarrow{\not{\partial}}}\gamma_5\psi - 2\mathrm{i}g_0 Z_2\bar{\psi}\hat{\not{A}}_0\gamma_5 t^a\psi.\end{aligned} \tag{13.5.9}$$

The $\bar{\psi}\hat{\not{A}}\gamma_5\psi$ term gives rise to the graphs (b) and (c).

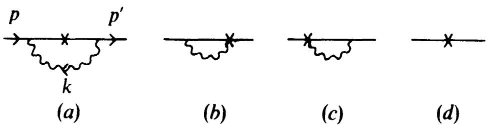

Fig. 13.5.3. Graphs of order g^2 for two-point Green's function of $[D^a_{\mathrm{anom}}]$.

Graph (a) equals

$$\begin{aligned}\Gamma_{3a} &= \frac{C_{\mathrm{F}} g^2 t^a}{16\pi^4}(2\pi\mu)^{4-d}\int \mathrm{d}^d k\,\frac{\gamma_\nu(\not{p}'+\not{k})(\hat{\not{p}}'+\hat{\not{p}}+2\hat{\not{k}})\gamma_5(\not{p}+\not{k})\gamma^\nu}{k^2(p'+k)^2(p+k)^2} \\ &= \frac{\mathrm{i}g^2 C_{\mathrm{F}}}{16\pi^2}t^a(2\pi\mu)^{4-d}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\times \\ &\quad\times\{\Gamma(3-d/2)D^{d/2-3}\gamma_\nu[\not{p}'(1-y)-\not{p}x][\hat{\not{p}}'(1-2y)+\hat{\not{p}}(1-2x)] \\ &\quad\times\gamma_5[\not{p}(1-x)-\not{p}'y]\gamma^\nu \\ &\quad-\tfrac{1}{2}\Gamma(2-d/2)D^{d/2-2}\gamma_\nu[2(\not{p}'(1-y)-\not{p}x)\hat{\gamma}_\kappa\gamma_5\gamma^\kappa \\ &\qquad+2\gamma_\kappa\hat{\gamma}^\kappa\gamma_5(\not{p}(1-x)-\not{p}'y) \\ &\qquad+\gamma_\kappa(\hat{\not{p}}'(1-2y)+\hat{\not{p}}(1-2x))\gamma_5\gamma^\kappa]\gamma^\nu\},\end{aligned} \tag{13.5.10}$$

where

$$D = -p^2 x(1-x) - p'^2 y(1-y) + 2p\cdot p' xy. \tag{13.5.11}$$

If it were not that there is an ultra-violet divergence given by the

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$\Gamma(2-d/2)$, we could let p and p' be four-dimensional, and then set $d=4$ to obtain zero. (Note that $\hat{\gamma}_\kappa\gamma^\kappa = \hat{\gamma}_\kappa\hat{\gamma}^\kappa = d-4$.) The pole prevents this argument from being made. First of all, notice that the pole is

$$\begin{aligned}\mathrm{pole}(\Gamma_{3a}) &= -\frac{ig^2 C_\mathrm{F}}{96\pi^2} t^a \,\mathrm{pole}\left\{\frac{1}{4-d}\gamma_\nu\gamma_\kappa(\hat{\not{p}}' + \hat{\not{p}})\gamma_5\gamma^\kappa\gamma^\nu\right\} \\ &= \frac{-3ig^2 C_\mathrm{F}}{8\pi^2} t^a \frac{(\hat{\not{p}}' + \hat{\not{p}})}{4-d}. \end{aligned} \tag{13.5.12}$$

The manipulations on the Dirac matrices are easy to do incorrectly, so let us be careful. We need the following result

$$\begin{aligned}\gamma_\kappa\hat{\gamma}^\mu\gamma_5\gamma^\kappa &= \bar{\gamma}_\kappa\hat{\gamma}^\mu\gamma_5\bar{\gamma}^\kappa + \hat{\gamma}_\kappa\hat{\gamma}^\mu\gamma_5\hat{\gamma}^\kappa \\ &= \bar{\gamma}_\kappa\bar{\gamma}^\kappa\hat{\gamma}^\mu\gamma_5 - \hat{\gamma}_\kappa\hat{\gamma}^\kappa\hat{\gamma}^\mu\gamma_5 + 2\hat{\gamma}^\mu\gamma_5 \\ &= (10-d)\hat{\gamma}^\mu\gamma_5. \end{aligned} \tag{13.5.13}$$

In the first line we split γ^κ into a four-dimensional piece $\bar{\gamma}^\kappa$ and a $(d-4)$-dimensional piece $\hat{\gamma}^\kappa$. Then in the second line we used the commutators or anticommutators of $\bar{\gamma}^\kappa$ and $\hat{\gamma}^\kappa$ with $\hat{\gamma}^\mu$ and γ_5.

The graphs of Fig. 13.5.3(b) and (c) may be evaluated similarly. The sum of the pole terms for all three graphs is

$$\mathrm{pole}(\Gamma_{3a} + \Gamma_{3b} + \Gamma_{3c}) = \frac{-ig^2 C_\mathrm{F}}{8\pi^2} t^a \frac{(\hat{\not{p}}' + \hat{\not{p}})\gamma_5}{4-d}, \tag{13.5.14}$$

which is cancelled by the counterterm Fig. 13.5.3(d). This is in agreement with our general result (13.4.11).

We are now ready to compute the value at $d=4$ of the sum of the graphs of Fig. 13.5.3. Considerable simplification occurs. Since $\hat{p}$ and $\hat{p}'$ are now zero, the term in (13.5.10) that multiplies $\Gamma(3-d/2)$ vanishes. Similarly the last term multiplying $\Gamma(2-d/2)$ gives zero. The remaining two terms have a factor $\hat{\gamma}^\kappa\hat{\gamma}_\kappa = d-4$, which cancels the pole to leave a finite result:

$$\Gamma_{3a}\big|_{d=4,\hat{p}=\hat{p}'=0} = \frac{ig^2 C_\mathrm{F}}{8\pi^2} t^a(\bar{\not{p}}' - \bar{\not{p}})\gamma_5. \tag{13.5.15}$$

Similarly Γ_{3b} and Γ_{3c} give

$$(\Gamma_{3b} + \Gamma_{3c})\big|_{d=4,\hat{p}=\hat{p}'=0} = \frac{ig^2 C_\mathrm{F}}{8\pi^2} t^a(\bar{\not{p}}' - \bar{\not{p}})\gamma_5. \tag{13.5.16}$$

Effectively Fig. 13.5.3 sums to $g^2 C_\mathrm{F}/(4\pi^2)$ times the vertex for $\partial\cdot j_{a5}$. It is easy to understand why the result should be of this form. Without the loop integration, the vertex for D^a_anom vanishes when $\hat{k} = \hat{p} = \hat{p}' = 0$. When we include the integration over the components of $\hat{k}$, we can get a non-zero value for the graphs even if $\hat{p} = \hat{p}' = 0$. However, the

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evanescence property of the basic vertex implies that it has effectively a factor $d-4$. We only get a finite result by multiplying by an ultra-violet divergence – so the effect at $d=4$ is of a local operator.

A general theory of evanescent operators can be worked out. The results simply generalize what we have learnt from examples:

(1) We define an evanescent vertex as one that is finite and that vanishes in a tree graph when we set $d=4$ and when all momenta and polarizations are four-dimensional.
(2) A Green's function or a graph or an operator is evanescent if it is finite at $d=4$ and if it vanishes when its external momenta and polarizations are four-dimensional.
(3) Consider a graph containing an evanescent vertex. If the graph is completely finite then it is evanescent. ('Completely finite' means that the graph and all its subgraphs have negative degree of divergence.)
(4) A renormalized operator $[E]$ whose basic vertex is evanescent has the following expansion:

$$[E] = \sum_V C_{EV}[V] + \text{evanescent operators.} \tag{13.5.17}$$

The sum is over operators V whose basic vertices are non-evanescent. The only operators that are needed are the ones that according to the usual power-counting and symmetry requirements will mix with E under renormalization. The general proof is left as an exercise to the reader.

13.6 Non-singlet axial current has no anomaly

13.6.1 *Reduction of anomaly*

The only operator that can appear on the right-hand side of (13.5.17) for the case $E=D^a_{\text{anom}}$ is $\partial\cdot[j_{a5}]$. The restrictions are that it be pseudoscalar, isovector, gauge invariant and of dimension at most four. (If we had non-zero quark masses, then the operator $[D^a_M]$ could also appear.) So we have

$$[D^a_{\text{anom}}] = C(g)\partial\cdot[j^\mu_{a5}] + \text{evanescent,} \tag{13.6.1}$$

which, when substituted into the renormalized identity (13.4.12), gives

$$\frac{\partial}{\partial x^\mu}\langle 0|T(1-C)[j^\mu_{a5}]\prod A\prod\psi\prod\bar{\psi}|0\rangle$$
$$= \text{r.h.s. of (13.4.2)} + \text{evanescent.} \tag{13.6.2}$$

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So we should define the renormalized current

$$j^{\mu}_{Ra5} = (1 - C)[j^{\mu}_{a5}]$$
$$= (1 - C)Z_5^{-1} j^{\mu}_{a5}, \tag{13.6.3}$$

which is (13.4.6) with $z_5 = 1 - C$. For the physical four-dimensional theory this implies that j^{μ}_{Ra5} has Ward identities with no anomaly, viz. (13.4.4), as we wished to prove.

From our calculations of Fig. 13.5.3 we see that

$$C = g^2 C_{\mathrm{F}}/(4\pi^2) + O(g^4). \tag{13.6.4}$$

Our proof has been long and involves a general theory of evanescent operators summarized in Section 13.5. The basic idea, however, is simple. The only way an anomaly can appear in the physical theory is when a divergence cancels an effective factor of $d - 4$ for the evanescence of an anomaly. The anomaly in the four-dimensional theory is a local operator, and the only possible operators are those which power-counting would allow as counterterms to $\partial \cdot j$.

In the case of our isovector current, the only such operator is $\partial \cdot j$ itself. So a finite renormalization (13.6.3) serves to eliminate the anomaly at $d = 4$.

13.6.2 *Renormalized current has no anomalous dimension*

Let us apply the renormalization-group operator $\mu \mathrm{d}/\mathrm{d}\mu$ to the Ward identity (13.4.4). For the right-hand side we get

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \text{right-hand side} = \text{right-hand side } \{\Sigma \text{ anomalous dimensions of fields}\}.$$

To get the same result for the left-hand side, the current j^{μ}_{Ra5} must have zero anomalous dimension (when $d = 4$):

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} j_{Ra5} = 0. \tag{13.6.5}$$

This is a sensible result: the current j^{μ}_{Ra5} is a physical object, and it should not depend on how we parametrize the theory by a renormalized coupling.

Useful consequences follow, for the minimally subtracted current does have an anomalous dimension:

$$\mu \mathrm{d}[j_{a5}]/\mathrm{d}\mu = -\gamma_5 [j_{a5}], \tag{13.6.6}$$

where

$$\gamma_5(g) = \mu \mathrm{d} \ln(Z_5)/\mathrm{d}\mu$$
$$= [(2 - d/2)g + \beta(g)]\partial \ln(Z_5)/\partial g. \tag{13.6.7}$$

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Now, the coefficient C is a function of g; it is dimensionless and even in the presence of masses cannot depend on them, just like the renormalization factor Z_5. We also define it to have no cut-off dependence, since it is a factor between renormalized operators at $d = 4$.

We therefore have

$$\begin{aligned} 0 &= \mu \mathrm{d} j_{Ra5}/\mathrm{d}\mu \\ &= -\beta(\partial C/\partial g)[j_{a5}] - (1 - C)\gamma_5[j_{a5}], \end{aligned}$$

so that

$$\beta\partial(1 - C)/\partial g = \gamma_5(1 - C). \tag{13.6.8}$$

It follows that

$$1 - C = \exp\left\{\int_0^g \mathrm{d}g'\gamma_5(g')/\beta(g')\right\}, \tag{13.6.9}$$

where we used as a boundary condition the fact that C has a perturbation expansion starting at order g^2. In order that the integral in (13.6.9) be convergent, the order g^2 term in γ_5 must vanish.

So from the definition (13.6.7) it must be that Z_5 has no order g^2 term; this we know by explicit calculation. Moreover, we know from (13.6.4) the one-loop value of C, so that

$$\gamma_5 = -A_1 g^4 C_{\mathrm{F}}/(2\pi^2) + O(g^6), \tag{13.6.10}$$

where the one-loop term in β is $-A_1 g^3$. Hence (13.6.10) gives us a prediction of the leading divergence in Z_5:

$$Z_5 = 1 - \frac{A_1 C_{\mathrm{F}} g^4}{4\pi^2(4 - d)} + O(g^6). \tag{13.6.11}$$

The reader is invited to check this by Feynman graph calculations.

We may use the techniques of Chapter 7 to sum the divergences. We find the full renormalization factor of j^{ν}_{Ra5} to be

$$\begin{aligned} \frac{j^{\mu}_{Ra5}}{j^{\mu}_{a5}} &= (1 - C)Z_5^{-1} \\ &= \exp\left\{\int_0^g \mathrm{d}g'\gamma_5(g')\left[\frac{1}{\beta(g')} - \frac{1}{(d/2 - 2)g' + \beta(g')}\right]\right\} \\ &= \exp\left\{\int_0^g \mathrm{d}g' \frac{\gamma_5(g')(d/2 - 2)g'}{\beta(g')[(d/2 - 2)g' + \beta(g')]}\right\} \\ &= 1 + (d - 4)O[\ln(g^2/(4 - d))] \\ &\to 1 \quad \text{as } d \to 4. \end{aligned} \tag{13.6.12}$$

Evidently the Noether current is finite in the complete theory.

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13.7 Three-current Ward identity; the triangle anomaly

13.7.1 *General form of anomaly*

We consider the Green's function

$$D_{abc}^{\lambda\mu\nu}(p,p') \equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\, \mathrm{e}^{\mathrm{i}p\cdot x+\mathrm{i}p'\cdot y}\langle 0|\,Tj_{Ra5}^{\lambda}(0)j_b^{\mu}(x)j_c^{\nu}(y)|0\rangle. \quad (13.7.1)$$

Only connected graphs contribute; Lorentz invariance forces the vacuum expectation value of a current to be zero. The currents are all renormalized currents, so all subdivergences are cancelled by counterterms, and the only possible infinity in (13.7.1) is an overall divergence. In fact there is no overall divergence, as we will now show.

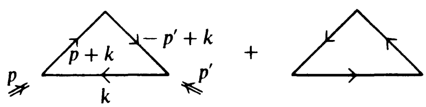

Fig. 13.7.1. Lowest-order graph for (13.7.1).

Individual graphs for (13.7.1) have a linear divergence, as can be seen from, say, Fig. 13.7.1. Any divergence must be linear in external momenta and proportional to the ε-tensor. The only possibility is

$$\varepsilon^{\kappa\lambda\mu\nu}[a(d)p_\kappa + b(d)p'_\kappa]. \quad (13.7.2)$$

There is also the constraint of conservation of the vector current. This is expressed by constructing a Ward identity in the dimensionally regularized theory.

Consider

$$p_\mu D^{\lambda\mu\nu} = -\mathrm{i}\int \mathrm{d}^dx\,\mathrm{d}^dy\,\mathrm{e}^{\mathrm{i}p\cdot x+\mathrm{i}p'\cdot y}\frac{\partial}{\partial x^\mu}\langle 0|\,Tj_{Ra5}^{\lambda}(0)j_b^{\mu}(x)j_c^{\nu}(y)|0\rangle. \quad (13.7.3)$$

By use of the result

$$\mathrm{i}\partial_\mu Z_2\bar\psi\gamma^\mu t^a\psi = Z_2\bar\psi t^a(\mathrm{i}\not{D} - M_0)\psi - Z\bar\psi(-\mathrm{i}\overleftarrow{\not{D}} - M_0)t^a\psi$$

and the equations of motion, we find

$$p_\mu D^{\lambda\mu\nu} = \int \mathrm{d}^dx\,\mathrm{i}\mathrm{e}^{\mathrm{i}(p+p')\cdot x}\langle 0|\,Tj_{Ra5}^{\lambda}(0)Z_2\bar\psi[t^c,t^b]\gamma^\nu\psi(x)|0\rangle$$
$$+\int \mathrm{d}^dy\,\mathrm{i}\mathrm{e}^{\mathrm{i}p'\cdot y}\langle 0|\,TZ_2\bar\psi[t^a,t^b]\bar\gamma^\lambda\gamma_5\psi\, j_c^\nu(y)|0\rangle. \quad (13.7.4)$$

In these equations we assumed that the currents j_b^μ and j_c^ν are conserved. Each of the terms in (13.7.4) is a Green's function of a vector and a pseudovector

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current. Parity invariance forces them to be zero, so

$$p_\mu D^{\lambda\mu\nu} = 0. \tag{13.7.5a}$$

Similarly

$$p'_\nu D^{\lambda\mu\nu} = 0. \tag{13.7.5b}$$

The counterterm (13.7.2) must therefore give zero when multiplied by p_μ or p'_ν. This forces the whole counterterm to be zero; the Green's function (13.7.1) is finite as it stands.

In the regulated theory the axial current is not conserved, so we cannot prove the Ward identity $(p + p')_\lambda D^{\lambda\mu\nu} = 0$ by the same manipulations. Indeed we have

$$(p + p')_\lambda D^{\lambda\mu\nu} = \text{commutator terms} + \int \mathrm{d}^4x \mathrm{d}^4y \mathrm{e}^{\mathrm{i}(p\cdot x + p'\cdot y)} \times \\ \times \langle 0 | T E_a(0) j_b^\mu(x) j_c^\nu(y) | 0 \rangle. \tag{13.7.6}$$

Here E is the evanescent operator in (13.6.2):

$$\begin{aligned} E &= [D^a_{\text{anom}}] - C\partial\cdot[j_{a5}] \\ &= [D^a_{\text{anom}}] - C(1 - C)^{-1}\partial\cdot j_{Ra5}. \end{aligned} \tag{13.7.7}$$

The commutator terms in (13.7.6) vanish, as in the Ward identity for the vector currents. Hence, finiteness of the left-hand side implies that the Green's function of E with j_b^μ and j_c^ν is finite. Even though graphs for it are quadratically divergent, the divergences cancel.

Now, E is an evanescent operator. This means that its Green's functions with elementary fields vanish in the four-dimensional theory. The general theory of evanescent operators, which we summarized at the end of Section 13.5, then tells us that the only way that the right-hand side of (13.7.6) will fail to vanish is for E to be part of a graph or subgraph with overall degree of divergence at least zero. Now, the definition of E has ensured that these subgraphs are all evanescent. Hence we are left with the complete graphs. So we have (at $d = 4$)

$$\begin{aligned} (p + p')_\lambda D^{\lambda\mu\nu} &= A(g)\varepsilon_{\mu\nu\alpha\beta} p^\alpha p'^\beta \varepsilon_{abc} \\ &= \tfrac{1}{2} A \varepsilon_{\mu\nu\alpha\beta}(p + p')^\alpha (p - p')^\beta \varepsilon_{abc}. \end{aligned} \tag{13.7.8}$$

The tensor structure is the only one possible. The coefficient A is dimensionless at $d = 4$, so it can only be a function of g. Note that the right-hand side of (13.7.8) obeys vector current conservation, so that

$$\begin{aligned} p_\mu (p + p')_\lambda D^{\lambda\mu\nu} &= 0, \\ p'_\nu (p + p')_\lambda D^{\lambda\mu\nu} &= 0, \end{aligned}$$

as should be.

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13.7.2 One-loop value

The lowest-order value of A is easily computed from the graphs of Fig. 13.7.1.

$$A\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}p'^{\beta} = \tfrac{3}{4}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{tr}\,[(\hat{\not p} - \hat{\not p}' + 2\hat{\not k})\gamma_5(\not p + \not k)\gamma^{\mu}\not k\gamma^{\nu}(\not k - \not p')]}{k^2(p+k)^2(p'-k)^2}$$
$$+ \text{charge conjugate}. \tag{13.7.9}$$

The factor 3 is the number of quark colors. To evaluate this, notice that

$$\left.\begin{aligned}\mathrm{tr}\,(\gamma_5\gamma^{\kappa}\gamma^{\lambda}) &= 0,\\ \mathrm{tr}\,(\gamma_5\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}) &= 4\mathrm{i}\varepsilon^{\kappa\lambda\mu\nu}.\end{aligned}\right\} \tag{13.7.10}$$

Since $\varepsilon_{\kappa\lambda\mu\nu}$ is restricted to the first four dimensions, it follows that the trace of γ_5 with four Dirac matrices is zero if one of the matrices is a $\hat{\gamma}$:

$$\mathrm{tr}\,(\gamma_5\hat{\gamma}^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}) = 0. \tag{13.7.11}$$

Let us commute the $-\not p' + \not k$ in (13.7.9) to the left, and use (13.7.11) whenever possible. We will also set $\hat{p} = \hat{p}' = 0$. The result is

$$\begin{aligned}
&\mathrm{tr}\,[(\hat{\not p} - \hat{\not p}' + 2\hat{\not k})\gamma_5(\not p + \not k)\gamma^{\mu}\not k\gamma^{\nu}(-\not p' + \not k)]\\
&\quad = \mathrm{tr}\,[2\hat{\not k}\gamma_5(\not p + \not k)\gamma^{\mu}\not k(\not p' - \not k)\gamma^{\nu}] + 0\\
&\quad = \mathrm{tr}\,[2\hat{\not k}\gamma_5(\not p + \not k)\gamma^{\mu}\not k\not p'\gamma^{\nu}] + 0\\
&\quad = -\,\mathrm{tr}\,[2\hat{\not k}\gamma_5(\not p + \not k)\not k\gamma^{\mu}\not p'\gamma^{\nu}] + 0\\
&\quad = -\,\mathrm{tr}\,[2\hat{\not k}\gamma_5\not p\not k\gamma^{\mu}\not p'\gamma^{\nu}] + 0\\
&\quad = -\,\mathrm{tr}\,[2\hat{\not k}\not k\gamma_5\not p\gamma^{\mu}\not p'\gamma^{\nu}] + 0.
\end{aligned} \tag{13.7.12}$$

The terms indicated by '0' vanish by use of (13.7.11). The charge conjugate term gives an equal contribution.

We now have

$$A\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}p'^{\beta} = 3\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{tr}\,[\hat{\not k}\not k\gamma_5\not p\gamma^{\mu}\not p'\gamma^{\nu}]}{k^2(p'-k)^2(p+k)^2}. \tag{13.7.13}$$

This is now only a logarithmically divergent integral. After use of Feynman parameters the standard result (4.4.14) gives

$$\begin{aligned}
A\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}p'^{\beta} &= \frac{3\mathrm{i}}{16\pi^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\tfrac{1}{2}\Gamma(2 - d/2)\,\times\\
&\qquad\times\frac{\mathrm{tr}\,[\hat{\gamma}_{\kappa}\gamma^{\kappa}\gamma_5\not p\gamma^{\mu}\not p'\gamma^{\nu}]}{[-p^2x(1-x) - p'^2y(1-y) + 2p\cdot p'xy]} + O(d-4)\\
&= -\frac{3}{8\pi^2}\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}p'^{\beta}.
\end{aligned} \tag{13.7.14}$$

The evanescence of the vertex has effectively given a factor of $d - 4$ which

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cancels the UV pole to leave a finite result

$$A = -3/(8\pi^2) + O(g^2). \tag{13.7.15}$$

This result for the anomaly in the axial Ward identity for (13.7.1) was first found by Adler (1969), and Bell & Jackiw (1969).

13.7.3 *Higher orders*

There are, in fact, (Adler & Bardeen (1969)) no higher-order corrections to the anomaly for (13.7.1). We will follow the proof due to Zee (1972). The basic idea is simple. Each of the currents in (13.7.1) is RG invariant, and there is no overall counterterm. Therefore this Green's function is invariant when we make an RG transformation. The anomaly must therefore be invariant also. But the anomaly coefficient $A(g)$ depends on the coupling g and on no other parameter of the theory. We can change g arbitrarily by changing the renormalization mass μ. Hence A is independent of g.

This proof may easily be written out. Renormalization-group invariance of $D^{\lambda\mu\nu}_{abc}$ is the equation

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}D^{\lambda\mu\nu}_{abc} = 0.$$

Hence

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}A(g)\varepsilon_{abc}\varepsilon^{\mu\nu\alpha\beta}p_\alpha p'_\beta = (p+p')_\lambda\mu\frac{\mathrm{d}}{\mathrm{d}\mu}D^{\lambda\mu\nu}_{abc} = 0.$$

Since $\mu\mathrm{d}/\mathrm{d}\mu = \mu\partial/\partial\mu + \beta\partial/\partial g$, this gives

$$\beta\partial A/\partial g = 0.$$

Then A is independent of g, so it equals its lowest-order value:

$$A = -3/(8\pi^2) \text{ exactly.} \tag{13.7.16}$$

This is a very striking result. The proof we have given is very simple, but the reader should not suppose it is not a deep result. The whole power of the renormalization apparatus is needed for its derivation. We first had to show that there is no anomaly in the divergence of the axial current. Then we had to show that there was no counterterm needed to make $D^{\lambda\mu\nu}_{abc}$ finite. These results involved considerable cancelation of UV infinities. Since the anomaly in $\partial\cdot j_{a5}$ disappears when the UV cut-off is removed, it can affect a Ward identity only by being enhanced by a UV infinity which has not made its appearance earlier.

Thus the anomaly is associated with a UV pole implicit in the Feynman graphs. It is precisely for this reason that it must have the dependence on the

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parameters of the theory and on the external momenta that is characteristic of a renormalization counterterm. In particular, it is polynomial in the momenta and masses of the degree determined by UV power-counting. Once this is clear, the most general possible form of the anomaly is (13.7.8). The final step to show that A is independent of g is trivial.

An important phenomenological consequence of the anomaly is a calculation of the decay rate for $\pi^0 \rightarrow 2\gamma$ (see Marciano & Pagels (1978) and Llewellyn-Smith (1980)). The amplitude is proportional to the number of quark colors, so the decay rate is proportional to the square of this number. The measured rate in fact agrees with the standard theory that there are three colors.

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