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SEMIGROUPS OF HIGH RANK

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1. Introduction

By the *rank* $r(S)$ of a finite semigroup S we shall mean the minimum cardinality of a set of generators of S. For a group G, as remarked in [**3**], one has $r(G) \leqq \log_2 |G|$, the bound being attained when G is an elementary abelian 2-group. By contrast, we shall see that there exist finite semigroups S for which $r(S) \geqq |S| - 1$. In the hope that it will not be considered too whimsical, we shall refer to a finite semigroup S of maximal rank (i.e. for which $r(S) = |S|$) as *royal*; a semigroup of next-to-maximal rank (i.e. for which $r(S) = |S| - 1$) will be called *noble*.

It is possible to extend these ideas to infinite semigroups by defining S to be *royal* if $\langle S \backslash \{s\} \rangle \subset S$ (properly) for all s in S. Equally, S is said to be *noble* if $\langle S \backslash \{s_1, s_2\} \rangle \subset S$ for all $s_1 \neq s_2$ in S, but there exists z in S such that $\langle S \backslash \{z\} \rangle = S$. The element z featuring in this definition will be called a *superfluous* element.

The structure of royal semigroups, which are necessarily bands, is given by Theorem 2.2. A noble semigroup S is defined as *singly* or *doubly* noble according as S has exactly one or exactly two superfluous elements. (As is shown in Theorem 3.7, there are no other possibilities.) The main part of this paper is devoted to an elucidation of the structure of singly noble semigroups. These need not be bands, but the description in Section 4 of singly noble bands turns out to be a major step in understanding the structure of singly noble semigroups in general. The main results are Theorems 4.11, 4.14, 4.15, 5.6 and 5.10.

Unexplained terms in semigroup theory will be found in [**2**].

2. Royal semigroups

Let S be a royal semigroup. If $a \neq a^2$ in S then $S \backslash \{a^2\}$ generates S, contrary to assumption. Hence S is a band and so by the Clifford–McLean Theorem [**1**, **4**, **2**] is a semilattice Y of rectangular bands E_α $(\alpha \in Y)$:

$$S = \mathcal{S}[Y; \{E_\alpha : \alpha \in Y\}].$$

We now have a lemma which applies also to noble bands.

Lemma 2.1. *Let $S = \mathcal{S}[Y; \{E_\alpha : \alpha \in Y\}]$ be a royal or noble band. Then each E_α is either a left zero or a right zero semigroup.*

Proof. Denoting the class of right zero semigroups by **RZ** and the class of left zero semigroups by **LZ**, let us suppose by way of contradiction that some E_α does not belong to $\mathbf{RZ} \cup \mathbf{LZ}$. Then E_α contains a copy $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of the 2×2 rectangular band, in which

$$e_{ij} e_{kl} = e_{il}.$$

Since $e_{12} = e_{11} e_{22}$ and $e_{21} = e_{22} e_{11}$ it is clear that $S \setminus \{e_{12}, e_{21}\}$ generates S, contrary to hypothesis.

The next observation is that if $S = \mathscr{S}[Y; \{E_\alpha : \alpha \in Y\}]$ is royal then Y is a chain. For if Y has a branch-point γ, i.e. a point for which there exist $\alpha, \beta > \gamma$ such that $\alpha\beta = \gamma$, then $E_\alpha E_\beta \subseteq E_\gamma$ and so there exists at least one element z of E_γ having an expression $z = xy$ with x in E_α, y in E_β and so certainly with $x \neq z$, $y \neq z$. Thus $S \setminus \{z\}$ generates S, contrary to assumption.

In stating the first main theorem it is convenient to use the symbol Ω to denote the *class of all non-zero cardinal numbers*. We shall also use the symbols R, L to stand for *right* and *left*, in a way that will be clear.

Theorem 2.2. *Let $(Y, \leqq)$ be a chain and let $M: Y \to \Omega$, $H: Y \to \{R, L\}$ be maps. For each α in Y let E_α be a set with cardinality $M(\alpha)$, endowed with the structure of a right zero or a left zero semigroup according as $H(\alpha) = R$ or $H(\alpha) = L$. Define a multiplication on $S = \bigcup \{E_\alpha : \alpha \in Y\}$ by the rule that*

$$xy = yx = y \ (x \in E_\alpha, y \in E_\beta, \alpha > \beta).$$

Then $S = \mathrm{Roy}(Y, M, H)$ is a royal semigroup.

Conversely, every royal semigroup is isomorphic to one constructed in this way.

Proof. For the direct part, it is a routine matter to show that the multiplication on $S = \bigcup \{E_\alpha : \alpha \in Y\}$ is associative and that S is royal. As regards the converse, we have already seen that a royal semigroup S must be a chain Y of semigroups E_α in $\mathbf{RZ} \cup \mathbf{LZ}$. We may therefore define (for each α in Y)

$$M(\alpha) = |E_\alpha|, \qquad H(\alpha) = \begin{cases} R & \text{if} \quad E_\alpha \in \mathbf{RZ} \\ L & \text{if} \quad E_\alpha \in \mathbf{LZ}, \ |E_\alpha| > 1. \end{cases}$$

For any x, y in S we must have either $xy = x$ or $xy = y$, since otherwise $S \setminus \{xy\}$ would generate S. So if we now take $\alpha > \beta$, $x \in E_\alpha$, $y \in E_\beta$ we know that

$$E_\alpha E_\beta \subseteq E_\beta, \qquad E_\beta E_\alpha \subseteq E_\beta$$

and so the only possibility is

$$xy = yx = y.$$

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Remark. It is easy to verify that every subsemigroup and every homomorphic image of a royal semigroup is royal. The class **Roy** of royal semigroups is not, however, a variety, since a direct product of royal semigroups need not be royal. The class **Roy** is contained in the variety $[x^2=x, axya=axaya]$ of *regular* bands. See Petrich [5] for information on varieties of bands.

3. Noble semigroups: preliminaries

Let S be a noble semigroup, generated by $S\backslash\{z\}$. As in the introduction, we refer to z as a *superfluous* element. It is clear that for all s in $S\backslash\{z\}$

$$s^2=s \quad \text{or} \quad s^2=z; \tag{3.1}$$

for any other value for s^2 would imply that $S\backslash\{s^2,z\}$ generates S. More generally, and for the same reason,

$$st=s \quad \text{or} \quad st=t \quad \text{or} \quad st=z \tag{3.2}$$

for all s,t in $S\backslash\{z\}$.

Also trivial is

Lemma 3.3. *Let S be a noble semigroup, generated by $S\backslash\{z\}$. Then there exist g,h in. $S\backslash\{z\}$ such that $gh=z$.*

Proof. Otherwise $S\backslash\{z\}$ is closed under multiplication and does not generate S.

The next result is not quite so obvious:

Lemma 3.4. *Let S be a noble semigroup, generated by $S\backslash\{z\}$. Then $z^2=z$.*

Proof. Suppose by way of contradiction that $z^2=g(\in S\backslash\{z\})$. Then

$$gz=zg(=z^3) \tag{3.5}$$

If $S=\{g,z\}$ is of order 2 then $g^2=z$ by Lemma 3.3 and either $gz=zg=g$ or $gz=zg=z$. In the former case

$$g^2z=z^2=g, \qquad g(gz)=g^2=z,$$

which contradicts associativity; in the latter case associativity again fails, since

$$z^2g=g^2=z, \qquad z(zg)=z^2=g.$$

Accordingly, suppose that $|S|\geqq 3$, and let $h\in S\backslash\{g,z\}$. By (3.1) we have either $h^2=z$ or $h^2=h$. In fact we must have $h^2=h$, for $h^2=z$ implies that $h^4=g$ and hence that $S\backslash\{g,z\}$ generates S. Thus all elements of $S\backslash\{g,z\}$ are idempotent.

Continuing with our supposition that $z^2=g$ we now consider two cases: (1) $g^2=g$; (2) $g^2=z$. In case (1) we consider the expression g_1g_2 for z. Such an expression must exist (with $g_1,g_2\in S\backslash\{z\}$) by Lemma 3.3. If both g_1,g_2 are distinct from g then

$$g=z^2=(g_1g_2)^2$$

implies that $S\backslash\{z,g\}$ generates S. Hence we have either

$$z=gg_2 \quad \text{or} \quad z=g_1g. \tag{3.6}$$

In the former case we get

$$gz=g(gg_2)=g^2g_2=gg_2=z$$

and hence

$$z^2=zgg_2=gzg_2=zg_2=(gg_2)g_2=gg_2^2=gg_2=z, \qquad \text{by (3.5)}$$

a contradiction to the assumption that $z^2=g$. A similar contradiction arises if we assume at (3.6) that $z=g_1g$. Thus case (1) cannot arise.

In case (2) it follows by (3.5) that

$$(gz)^2=g^2z^2=zg=gz$$

and so gz, being idempotent, must be distinct from both g and z. This implies that $S\backslash\{z,gz\}$ generates S and so again gives a contradiction. Thus case (2) cannot arise either. We have now shown as required that $z^2=z$.

Next, we have

Theorem 3.7. *Let S be a noble semigroup. Then either there is a unique superfluous element or there are precisely two such elements. In the latter case S must be a band.*

Proof. Suppose that $S\backslash\{z\}$ generates S. We consider two cases:

(1) there exists g in $S\backslash\{z\}$ such that $g^2=z$;
(2) $g^2=g$ for all g in $S\backslash\{z\}$.

In case (1) the superfluous element z must be unique. For suppose that t $(\neq z)$ is also superfluous; then $t^2=t$ by Lemma 3.4 and so $t\neq g$; hence $g,z\in S\backslash\{t\}$ with $g^2=z$ and so g and t violate condition (3.1).

In case (2) we note that by Lemma 3.4 the semigroup S is a band. We now show that S cannot have more than two superfluous elements. By Lemma 3.3 we have

$$z=gh \tag{3.8}$$

where $g,h\in S\backslash\{z\}$. In fact g and h must be distinct since S is a band. It follows that

$$gz=z=zh. \tag{3.9}$$

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The only possible candidates for alternative generating sets are $S\backslash\{g\}$ and $S\backslash\{h\}$; for if $s\notin\{z,g,h\}$ then $S\backslash\{s\}$ must contain all three of z,g,h and so by (3.8) must violate (3.2). Let us now suppose that $S\backslash\{g\}$, $S\backslash\{h\}$ are both generating sets; we shall see that this leads to a contradiction. By Lemma 3.3 we have

$$g=xy,$$

where $x,y\in S\backslash\{g\}$. If x,y are both in $S\backslash\{z\}$ we can immediately deduce that $S\backslash\{z,g\}$ generates S, in contradiction to our supposition that S is noble. Hence either

$$g=zy \quad \text{or} \quad g=xz, \tag{3.10}$$

where $x,y\in S\backslash\{z,g\}$. Suppose first that $g=zy$. Then if $y\neq h$ the equation $g=zy$ (in which $g,z,y\in S\backslash\{h\}$) violates condition (3.2) for the generating set $S\backslash\{h\}$. Hence $y=h$. By the same token, if we assume that $g=xz$ in (3.10) we find that $x=h$. Hence we can refine (3.10) to

$$g=zh \quad \text{or} \quad g=hz.$$

In fact $g=zh$ is impossible because of (3.9), and so $g=hz$. But then (3.9) gives

$$z=gz=hz^2=hz=g,$$

which is again a contradiction. The only conclusion is that $S\backslash\{g\}$ and $S\backslash\{h\}$ cannot simultaneously be generating sets. This completes the proof of Theorem 3.7.

It is convenient now to define a noble semigroup S as *singly noble* if it has a unique superfluous element, and *doubly noble* if it has precisely two such elements. We shall not discuss doubly noble semigroups in this paper. Such semigroups do, however, exist, as the following example makes clear.

Example. (3.11) Let $S=\{a_1,a_2,z_1,z_2\}\subseteq\mathscr{T}(\{1,2,3,4\})$, where

$$a_1=\begin{pmatrix}1&2&3&4\\1&1&4&4\end{pmatrix},\quad a_2=\begin{pmatrix}1&2&3&4\\2&2&3&3\end{pmatrix},$$

$$z_1=\begin{pmatrix}1&2&3&4\\3&3&3&3\end{pmatrix},\quad z_2=\begin{pmatrix}1&2&3&4\\4&4&4&4\end{pmatrix}.$$

Then S has Cayley table

	a_1	a_2	z_1	z_2
a_1	a_1	a_2	z_1	z_2
a_2	a_1	a_2	z_1	z_2
z_1	z_2	z_1	z_1	z_2
z_2	z_2	z_1	z_1	z_2

and is clearly doubly noble, generated by either $S\backslash\{z_1\}$ or $S\backslash\{z_2\}$ but not by any subset of cardinality 2.

4. Singly noble bands

A singly noble semigroup need not be a band, but, as will be clear in Section 5, the major step in understanding the structure of singly noble semigroups in general is the study of singly noble bands. We begin with what at first seems to be a very particular example.

Let A be a right zero semigroup and let Z be a left zero semigroup, where $|A|\geqq 1$, $|Z|\geqq 2$. Let P be a subset of Z such that $|P|\geqq 2$ and let z_1 be a fixed element of P. Define a multiplication on the disjoint union $S=A\cup Z$ by the rules

$$\left.\begin{aligned} &za=z\ (z\in Z, a\in A)\\ &az=\begin{cases} z_1 & (a\in A, z\in P)\\ z & (a\in A, z\in Z\backslash P)\end{cases}\end{aligned}\right\} \tag{4.1}$$

It is possible to check directly that this is an associative multiplication. It is easy also to see that S is a singly noble band, with superfluous element z_1 and with two $\mathscr{J}$-classes, namely A and Z. The semigroup is in fact determined by the sets A, Z and P and by the fact that A and Z are respectively in **RZ** and **LZ**. To draw attention to these facts and also to the unique superfluous element z_1 we write

$$S=SNB_L^R(A,Z;P;z_1). \tag{4.2}$$

The dual semigroup, in which the upper $\mathscr{J}$-class is left zero and the lower $\mathscr{J}$-class is right zero, is denoted by

$$SNB_R^L(A,Z;P;z_1). \tag{4.3}$$

In a very similar way we construct a singly noble band in the case where A and Z are both in **RZ**, where P is a subset of Z such that $|P|\geqq 2$ and where $z_1\in P$. Here the specification is

$$\left.\begin{aligned} &az=z\quad (a\in A, z\in Z)\\ &za=\begin{cases} z_1 & (a\in A, z\in P)\\ z & (a\in A, z\in Z\backslash P).\end{cases}\end{aligned}\right\} \tag{4.4}$$

We write

$$S=SNB_R^R(A,Z;P;z_1). \tag{4.5}$$

The dual semigroup, in which both $\mathscr{J}$-classes are in **LZ**, is written

$$SNB_L^L(A,Z;P;z_1). \tag{4.6}$$

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We can now prove the following theorem:

Theorem 4.7. *Let S be a singly noble band with superfluous element z_1 and with two $\mathcal{J}$-classes. Then (for some A, Z, P, z_1) S is isomorphic to exactly one of the semigroups listed in formulae* (4.2), (4.3), (4.5) *and* (4.6).

Proof. By Lemma 2.1 the $\mathcal{J}$-classes of S are both in $\mathbf{RZ} \cup \mathbf{LZ}$. They form a two-element chain and it is clear that z_1 must be in the lower $\mathcal{J}$-class. Denote the $\mathcal{J}$-classes by A, Z, with

$$AZ \subseteq Z, \qquad ZA \subseteq Z.$$

If $|Z| = 1$ the semigroup is simply a right or left zero semigroup A with a zero element z_1 adjoined, and such a semigroup is royal rather than noble. Hence $|Z| \geqq 2$.

There are now four cases, according to whether A and Z are in $\mathbf{RZ}$ or in $\mathbf{LZ}$, but because of duality it will not be necessary to consider more than two. Suppose first that $A \in \mathbf{RZ}$, $Z \in \mathbf{LZ}$. Thus

$$aa' = a', \qquad zz' = z$$

for all a, a' in A and all z, z' in Z. The product $z_1 a$ must be in Z and cannot be $z \neq z_1$, since then $S \backslash \{z\}$ would generate S. Hence

$$z_1 a = z_1$$

for all z in A, and it now follows easily that for all z in Z

$$za = (zz_1)a = z(z_1 a) = zz_1 = z.$$

Now by the same argument as we used for $z_1 a$ we must have

$$az_1 = z_1 \tag{4.8}$$

for all a in A. Since $S \backslash \{z_1\}$ generates S there must exist a_0 in A and z_0 $(\neq z_1)$ in Z such that

$$a_0 z_0 = z_1. \tag{4.9}$$

We now show that for each z $(\neq z_1)$ in Z we have the implication

$$Az \cap \{z_1\} \neq \emptyset \Rightarrow Az = \{z_1\}. \tag{4.10}$$

For suppose that $a'z = z_1$ for some a' in A. Then for all a in A we have $az \in \{z, z_1\}$. If $az = z$ then

$$z = az = (a'a)z = a'(az) = a'z = z_1,$$

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which is a contradiction. Hence $az=z_1$ for all a in A. We have proved the implication (4.10).

It follows that Z divides into two complementary subsets P and $Z\backslash P$ given by

$$P=\{z\in Z: Az=\{z_1\}\},$$

$$Z\backslash P=\{z\in Z: z\neq z_1, Az=\{z\}\}.$$

By (4.8) and (4.9) z_1 and z_0 belong to P. Hence $|P|\geqq 2$.

We have in fact shown that

$$S\simeq SNB_L^R(A,Z;P;z_1).$$

Suppose now that the $\mathscr{J}$-classes A and Z are both in **RZ**, so that

$$aa'=a', \qquad zz'=z'$$

for all a,a' in A and all z,z' in Z. Then as before we must have $az_1=z_1a=z_1$ for all a in A and so

$$az=a(z_1z)=(az_1)z=z_1z=z$$

for all a in A and z in Z. Again there exist z_0 $(\neq z_1)$ in Z and a_0 in A such that

$$z_0a_0=z_1,$$

and once again we can show that for each $z\neq z_1$ either $zA=\{z_1\}$ or $zA=\{z\}$. Thus again we find

$$P=\{z\in Z: zA=\{z_1\}\},$$

$$Z\backslash P=\{z\in Z: z\neq z_1, zA=\{z\}\},$$

with $z_0,z_1\in P$. Thus

$$S\simeq SNB_R^R(A,Z;P;z_1).$$

Note. In (4.2), (4.3), (4.5) and (4.6) we are insisting that P properly contains $\{z_1\}$. If we allow $P=\{z_1\}$ then the semigroup in each case becomes royal. We shall sometimes want to allow this degenerate case.

We have effectively described all singly noble bands with two $\mathscr{J}$-classes. Let us now consider an arbitrary singly noble band

$$S=\mathscr{S}[Y;\{E_\alpha:\alpha\in Y\}],$$

generated by $S\backslash\{z\}$. As pointed out in the paragraph preceding the statement of Theorem 2.2, at any branch-point γ of the semilattice Y there must be at least one element in E_j expressible as a product of two elements distinct from itself. In the case of a royal semigroup the effect of this observation was to conclude that Y must be a chain. Here we conclude that Y can have at most one branch-point, and that the branch-point (if any) must be at the $\mathscr{J}$-class containing z. Whether or not a branch-point occurs it will be useful to refer to the $\mathscr{J}$-class containing z as the *pivotal* $\mathscr{J}$-class. The element z itself will be called the *pivot*. If Y is a chain we shall say that S is a *catenary* singly noble band.

Let us first consider the catenary case and denote the pivotal $\mathscr{J}$-class by E_0; then 0 is not the maximum element of the chain Y. The structure of $E_\alpha \cup E_\beta (\alpha > \beta)$ must be that of a royal semigroup if $\beta \neq 0$. The structure of $E_\alpha \cup E_0$ $(\alpha > 0)$ is either that of a royal semigroup, or for some $P \subseteq E_0$ with $|P| \geqq 2$ is the appropriate $SNB(E_\alpha, E_0; P; z_1)$. We thus have a good deal of information about the global structure of S.

In fact we can be still more specific. Suppose that $\alpha > \beta > 0$ and suppose for definiteness that

$$E_\alpha, E_\beta \in \mathbf{RZ}, \qquad E_0 \in \mathbf{LZ}.$$

Suppose now that

$$E_\alpha \cup E_0 = SNB_L^R(E_\alpha, E_0; P; z_1),$$

$$E_\beta \cup E_0 = SNB_L^R(E_\beta, E_0; Q; z_1),$$

with $|P|, |Q| \geqq 2$. Since

$$E_\alpha \cup E_\beta = \mathrm{Roy}(\{\alpha, \beta\}, M, H)$$

(with $M(\alpha) = |E_\alpha|, M(\beta) = |E_\beta|$, $H(\alpha) = H(\beta) = R$) we have that

$$ab = ba = b$$

for all a in E_α, b in E_β. Hence, by (4.1), for b in E_β and z in P,

$$bz = (ba)z = b(az) = bz_1 = z_1.$$

It follows that $z \in Q$, and so $P \subseteq Q$. The same result applies to other mixtures of right and left zero semigroups.

We can now state the following theorem. As in Theorem 2.2, Ω denotes the class of all non-zero cardinal numbers.

Theorem 4.11. *Let $(Y, \leqq)$ be a chain and let 0 be a fixed non-maximal element of Y. Let $M: Y \to \Omega$ and $H: Y \to \{R, L\}$ be maps and suppose that $M(0) \geqq 2$. For each α in Y suppose that E_α is a set with $M(\alpha)$ elements having right or left zero semigroup structure according as $H(\alpha)$ is R or L. Let z_1 be a fixed element of E_0.*

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For each $\alpha>0$ *let* $P(\alpha)$ *be a subset of* E_0, *and suppose that*

(i) $z_1 \in P(\alpha)$ *for all* $\alpha>0$,
(ii) $0<\alpha\leqq\beta\Rightarrow P(\alpha)\supseteq P(\beta)$,
(iii) $|P(\alpha)|\geqq 2$ *for some* $\alpha>0$.

Define a multiplication on $E_\alpha\cup E_\beta$ $(\alpha>\beta\neq 0)$ *by the rule that*

$$E_\alpha\cup E_\beta=\mathrm{Roy}(\{\alpha,\beta\},M|\{\alpha,\beta\},H|\{\alpha,\beta\})$$

and on $E_\alpha\cup E_0$ $(\alpha>0)$ *by the rule that*

$$E_\alpha\cup E_0=SNB^{H(\alpha)}_{H(0)}(E_\alpha,E_0;P(\alpha);z_1).$$

Then the disjoint union $S=\bigcup\{E_\alpha:\alpha\in Y\}$ *is a catenary singly noble band.*

Conversely, every catenary singly noble band is isomorphic to one constructed in this way.

Proof. In effect we have already established the more difficult converse half of this result, provided we make as in the proof of Theorem 2.2 the convention that $H(\alpha)=R$ (say) whenever $|E_\alpha|=1$. The condition $P\subseteq Q$ obtained just before the statement of the theorem translates into property (ii) of the function P. If $|P(\alpha)|=1$ for any α (giving $|P(\beta)|=1$ for all $\beta\geqq\alpha$) then the corresponding

$$SNB^{H(\alpha)}_{H(0)}(E_\alpha,E_0;P(\alpha);z_1)$$

reduces simply to $\mathrm{Roy}(\{\alpha,0\},M|\{\alpha,0\},H|\{\alpha,0\})$, in accordance with the Note preceding this theorem. The condition (iii) merely ensures that the entire semigroup S is not royal.

As regards the direct half of the theorem, the only issue is whether the multiplication on S is associative. Let $a\in E_\alpha$, $b\in E_\beta$, $c\in E_\gamma$. If $\alpha=\beta=\gamma$ then $(ab)c=a(bc)$ by associativity within a right or left zero semigroup. If $|\{\alpha,\beta,\gamma\}|=2$ then $(ab)c=a(bc)$ by associativity within a royal or singly noble semigroup with two $\mathscr{J}$-classes. If α,β,γ are all distinct then all cases are automatic by the properties of royal semigroups except

$$\alpha>\beta>0,\ \beta>\alpha>0,\ \alpha>0>\beta,\ \beta>0>\alpha,$$
$$\alpha>\gamma>0,\ \gamma>\alpha>0,\ \alpha>0>\gamma,\ \gamma>0>\alpha,$$
$$\beta>\gamma>0,\ \gamma>\beta>0,\ \beta>0>\gamma,\ \gamma>0>\beta.$$

This is a tedious verification and it will suffice to give an example. Suppose that $\alpha>\beta>0$ and that $H(\alpha)=H(\beta)=R$, $H(0)=L$. Then E_α, E_β are right zero semigroups, while E_0 is left zero. If $a\in E_\alpha$, $b\in E_\beta$, $z\in E_0$ then $P(\alpha)\subseteq P(\beta)$ and we have

$$(ab)z=bz=\begin{cases} z_1 & \text{if} \quad z\in P(\beta)\\ z & \text{if} \quad z\notin P(\beta).\end{cases}$$

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On the other hand

$$a(bz)=\begin{cases} az_1 & \text{if} \quad z\in P(\beta) \\ az & \text{if} \quad z\notin P(\beta) \end{cases}$$

$$=\begin{cases} z_1 & \text{if} \quad z\in P(\beta) \\ z & \text{if} \quad z\notin P(\beta), \end{cases}$$

since $z\notin P(\beta)$ certainly implies $z\notin P(\alpha)$. Thus $(ab)z=a(bz)$ as required. Other cases are either similar or easier.

The singly noble band described in the theorem will be denoted by

$$S=SNB(Y;M,H;P). \tag{4.12}$$

Remark. If we drop condition (iii) in the statement of the theorem we may have $P(\alpha)=\{z_1\}$ for all $\alpha>0$. In this case S reduces to Roy(Y,M,H).

In order to study singly noble bands that are not catenary we require some more terminology. A singly noble band will be called *feminine* if the pivotal $\mathcal{J}$-class consists of a single element, and *masculine* otherwise. (The feminine bands have narrower waists.) Theorem 4.11 indicates that all catenary singly noble bands are masculine, but we shall see that this need not be so for the non-catenary case.

Let $S=\mathcal{S}[Y;\{E_\alpha:\alpha\in Y\}]$ be a singly noble band in which Y is not a chain. Denote the unique branch-point of Y by 0 and the pivot by $z_1\,(\in E_0)$. We shall in fact limit ourselves to the case where S is *arboreal*, i.e. where the underlying semilattice Y is a *tree*. The more general case would not be impossible to describe, but the details might be tedious and in order to keep this paper to a reasonable length we shall not tackle it here.

In effect the restriction to the arboreal case means that $\{\alpha\in Y:\alpha\geqq 0\}$ is the union of chains Z_i $(i\in I)$, with

$$Z_i\cap Z_j=\{0\} \qquad (i\neq j).$$

Since Y is not a chain we must have $|I|\geqq 2$.

Suppose now that S is both masculine and arboreal. We define $M\colon Y\to\Omega$ by $M(\alpha)=|E_\alpha|$ and note that $M(0)\geqq 2$. Also, we define $H\colon Y\to\{R,L\}$ by the rule that

$$H(\alpha)=\begin{cases} R & \text{if} \quad E_\alpha\in \mathbf{RZ} \\ L & \text{if} \quad E_\alpha\in \mathbf{LZ} \text{ and } |E_\alpha|>1. \end{cases}$$

For each i in I the structure of the catenary singly noble (or royal) band $S_i=\bigcup\{E_\alpha:\alpha\in Z_i\}$ determines a map $P_i\colon Z_i\to\mathcal{P}(E_0)$ satisfying conditions (i) and (ii) of Theorem 4.11: that is, $z_1\in P_i(\alpha)$ for all α in Z_i; and for all α,β in Z_i

$$0<\alpha\leqq\beta\Rightarrow P_i(\alpha)\supseteq P_i(\beta).$$

Then

$$S_i \simeq SNB(Z_i; M|Z_i, H|Z_i; P_i),$$

the case where S_i is royal being covered by the eventuality that $P_i(\alpha)=\{z_1\}$ for all α in Z_i.

In fact the functions P_i $(i \in I)$ are not independent. To see this, let us look at α in Z_i, β in Z_j (with $i \neq j$) and let us suppose for definiteness that

$$H(\alpha)=R, \qquad H(\beta)=H(0)=L.$$

Then $E_\alpha \cup E_0$, $E_\beta \cup E_0$ are respectively

$$SNB_L^R(E_\alpha, E_0; P_i(\alpha); z_1), \qquad SNB_L^L(E_\beta, E_0; P_j(\beta); z_1).$$

For $a \in E_\alpha$, $b \in E_\beta$ we must have

$$ab=ba=z_1.$$

since $E_\alpha E_\beta, E_\beta E_\alpha \subseteq E_0$ and since z_1 is the unique superfluous element of S. Hence for all z in E_0

$$(ab)z=z_1 z=z_1,$$

while

$$a(bz)=\begin{cases} az_1 & \text{if} \quad z \in P_j(\beta) \\ az & \text{if} \quad z \notin P_j(\beta) \end{cases}$$

$$=\begin{cases} z_1 & \text{if} \quad z \in P_i(\alpha) \cup P_j(\beta) \\ z & \text{otherwise.} \end{cases}$$

Thus associativity fails unless

$$P_i(\alpha) \cup P_j(\beta) = E_0 \tag{4.13}$$

for all $i \neq j$ and all α in Z_i, β in Z_j.

We have therefore proved the converse half of the following theorem:

Theorem 4.14. *Let Y be a tree with a single branch-point* 0 *and suppose that $\{\alpha \in Y: \alpha \geqq 0\}$ is a union of chains Z_i $(i \in I)$, with*

$$|I| \geqq 2, \quad |Z_i| \geqq 2 \ (i \in I), \quad Z_i \cap Z_j = \{0\} \qquad (i \neq j).$$

Let

$$S_i = SNB(Z_i; M_i, H_i; P_i) \qquad (i \in I)$$

be a catenary singly noble or royal band with $\mathscr{J}$-classes E_α $(\alpha \in Z_i)$, pivotal $\mathscr{J}$-class E_0 and pivot z_1, and suppose that

$$S_i \cap S_j = E_0 \qquad (i \neq j).$$

Suppose also that for all $i \neq j$ and all α in Z_i, β in Z_j,

$$P_i(\alpha) \cup P_j(\beta) = E_0.$$

Let $\bar{Y} = \{\alpha \in Y : \alpha < 0\}$ and let

$$T = \mathrm{Roy}(\bar{Y}, M, H)$$

be a royal band, where $T \cap S_i = \emptyset$ for all i in I.

Define a multiplication on

$$S = T \cup \bigcup \{S_i : i \in I\}$$

by the rule that

$$(S_i \backslash E_0)(S_j \backslash E_0) = \{z_1\} \qquad (i \neq j),$$

$$xy = yx = y \quad \text{whenever} \quad x \in S_i,\ y \in T.$$

Then S is an arboreal masculine singly noble band.

Conversely, every arboreal masculine singly noble band is isomorphic to one of this kind.

Proof. Here the verification of associativity (which is all that remains to be proved) is much easier than in Theorem 4.11 and is omitted altogether.

For feminine singly noble bands we obtain the following theorem:

Theorem 4.15. *Let R_i $(i \in I)$ be non-trivial royal bands intersecting in a common minimum $\mathscr{J}$-class $\{z\}$, where $|I| \geqq 2$. Let R_0 be a royal band whose maximum $\mathscr{J}$-class is $\{z\}$. Let*

$$S = R_0 \cup \bigcup \{R_i : i \in I\}$$

and define a multiplication on S by

$$R_i R_j = \{z\} \qquad (i, j \in I,\ i \neq j),$$

$$xy_0 = y_0 x = y_0 \qquad (x \in R_i,\ i \in I,\ y_0 \in R_0).$$

Then S is an arboreal feminine singly noble band.

Conversely, every arboreal feminine singly noble band is isomorphic to one constructed in this way.

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Proof. The verification that S is associative is trivial. Also S is singly noble, generated by $S\backslash\{z\}$, and is feminine since the pivotal $\mathcal{J}$-class consists of a single element z. It is arboreal, since the underlying semilattice has one branch-point at 0 from which at least two chains diverge upwards.

To establish the converse half, suppose that $S=\mathcal{S}[Y;\{E_\alpha:\alpha\in Y\}]$ is an arboreal feminine singly noble band. We know from Theorem 4.11 that a catenary singly noble band must be masculine; hence Y has a branch-point, and this must be at the pivotal $\mathcal{J}$-class $E_0=\{z\}$. In our previous notation, write $\{\alpha\in Y:\alpha\geqq 0\}$ as $\bigcup\{Z_i:i\in I\}$, where the Z_i are chains and where $Z_i\cap Z_j=\{0\}$ if $i\neq j$. Then for each i in I the catenary subsemigroup

$$R_i=\bigcup\{E_\alpha:\alpha\in Z_i\}$$

cannot be singly noble (since it is feminine) and so must be royal. This applies also to

$$R_0=\bigcup\{E_\alpha:\alpha\leqq 0\}.$$

The result is now clear.

5. Singly noble semigroups

In addition to singly noble bands we have two very obvious sources of singly noble semigroups. First, it is clear that $\mathbf{Z}_2$, the two element group, is singly noble, and that it is the only group with this property. Secondly, any null semigroup N, with $N^2=\{z\}$, is singly noble. It turns out that the structure of singly noble semigroups can be described in terms of singly noble bands, copies of $\mathbf{Z}_2$ and null semigroups.

Let S be a singly noble semigroup, generated by $S\backslash\{z\}$. We saw in Section 3 that $z^2=z$ and that for each g in $S\backslash\{z\}$ either $g^2=g$ or $g^2=z$. In fact we have

Lemma 5.1. *Let S be a singly noble semigroup, generated by $S\backslash\{z\}$, and let $B=\{s\in S:s^2=s\}$. Then B is a subsemigroup of S and is either a royal band or a singly noble band.*

Proof. Let $b,c\in B$. Then

$$bc\in\{b,c,z\}\subseteq B$$

and so B is a subsemigroup. The result is now clear.

Lemma 5.2. *Let S be a singly noble semigroup, generated by $S\backslash\{z\}$, and suppose that $|C|\geqq 2$, where $C=\{s\in S:s^2=z\}$. Then C is a singly noble subsemigroup of S.*

Proof. As above, if $s,t\in C$ then

$$st\in\{s,t,z\}\subseteq C$$

and so C is a subsemigroup. Being contained in a singly noble semigroup, it must be either royal or singly noble, but since it is not a band it cannot in fact be royal.

Let us now look a little more closely at C. For each c in C we have $c^2=z$ and so

$$cz=zc\ (=c^3).$$

Hence $C=G\cup N$, where

$$G=\{c\in C: cz=zc=c\},$$

$$N=\{c\in C: cz=zc=z\},$$

and $G\cap N=\{z\}$. Then G is closed under multiplication, since $c,d\in G$ implies

$$(cd)z=c(dz)=cd,\ z(cd)=(zc)d=cd,$$

and in fact G is a group with identity z, with $c=c^{-1}$ for all c in G. It must be singly noble or royal, being a subsemigroup of a singly noble semigroup, and hence $|G|\leqq 2$. That is, either $G=\{z\}$, or $G=\{z,c\}$, with $cz=zc=c$, $c^2=z^2=z$.

Next, we examine N. If $c,d\in N\backslash\{z\}$ then $cd\in\{c,d,z\}$. If $cd=c$ then

$$c=cd=(cd)d=cd^3=cz=z,$$

a contradiction, and we similarly get a contradiction from the assumption that $cd=d$. Hence $cd=z$ and so N is a null semigroup with z as zero element.

We have thus proved

Theorem 5.3. *Let S be a singly noble semigroup, generated by $S\backslash\{z\}$. Then there exist a royal or singly noble band B, a group G of order not greater than 2 with identity z, and a null semigroup N with zero element z such that*

$$B\cap G=B\cap N=G\cap N=\{z\}$$

and $S=B\cup G\cup N$. If B is royal then at least one of G,N is non-trivial.

Probing more deeply into the fine structure of S, let us now suppose that B is *masculine*, i.e. that the $\mathcal{J}$-class of z in B contains at least one other element z_2. Let us suppose (without essential loss of generality) that the $\mathcal{J}$-class $\{z,z_2,\ldots\}$ is in **RZ**, so that

$$zz_2=z_2,\quad z_2z=z.$$

Suppose now that $G=\{g,z\}$, of order 2. Then

$$z_2g=z_2(zg)=(z_2z)g=zg=g,$$

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and $gz_2 \in \{g, z_2, z\}$. We show that each of the three possibilities leads to a contradiction. First, $gz_2 = g$ gives

$$z = g^2 = g^2 z_2 = zz_2 = z_2,$$

which is not the case. Secondly, $gz_2 = z_2$ gives

$$g = gz = g(z_2 z) = (gz_2)z = z_2 z = z;$$

and thirdly, $gz_2 = z$ gives

$$g = zg = (gz_2)g = g(z_2 g) = g^2 = z.$$

Our conclusion is as follows.

Theorem 5.4. *If in Theorem 5.3 the band B is masculine then the group G is trivial.*

It may help in explaining our remaining results if we introduce some additional terminology. We shall say that a singly noble semigroup S, generated by $S \setminus \{z\}$, is *dexter* if the pivotal $\mathcal{J}$-class J_z is a right zero semigroup containing at least two elements, *sinister* if J_z is a left zero semigroup containing at least two elements, *feminine* if $J_z = \{z\}$, and *balanced* if $J_z = \{z, g\}$, a group of order 2. It follows from our investigations so far that these are the only possibilities. Notice that the masculine singly noble bands investigated in Theorems 4.11 and 4.14 have either the dexter or the sinister property.

Before stating the next theorem we require a further definition. Let Y be a semilattice with a single branch-point 0 and let $Y^+ = \{\alpha \in Y : \alpha > 0\}$. A subset C of Y^+ will be called a *positive chain filter* if

$$\left.\begin{array}{ll}\text{(i)} & [\alpha \in C,\ \beta > \alpha] \Rightarrow \beta \in C,\\ \text{(ii)} & [\alpha \in C,\ \beta \text{ not comparable with } \alpha] \Rightarrow \beta \notin C.\end{array}\right\} \tag{5.5}$$

Such subsets exist: the empty set is one such, and so is every set $[\alpha,) = \{\beta \in Y : \beta \geqq \alpha\}$. Let $\mathscr{C}$ denote the set of all positive chain filters in Y.

Theorem 5.6. *Let $B = \mathscr{S}[Y; \{E_\alpha : \alpha \in Y\}]$ be a dexter singly noble band, generated by $B \setminus \{z\}$ and with pivotal $\mathcal{J}$-class E_0. Let N be a null semigroup with zero element z but otherwise disjoint from B. Let*

$$\Phi_r : N \setminus \{z\} \to \mathscr{C}, \qquad \Phi_l : N \setminus \{z\} \to \mathscr{C}$$

be mappings from $N \setminus \{z\}$ into the set $\mathscr{C}$ of positive chain filters of Y. Define a multiplication on $S = B \cup N$ by the rules that

$$\begin{array}{lll} Nb_\alpha = b_\alpha N = \{b_\alpha\} & \text{if} & b_\alpha \in E_\alpha,\quad \alpha < 0,\\ xb_0 = b_0,\ b_0 x = z & \text{if} & x \in N,\quad b_0 \in E_0,\end{array} \tag{5.7}$$

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while if $x \in N \setminus \{z\}$, $b_\alpha \in E_\alpha$, $\alpha > 0$,

$$b_\alpha x = \begin{cases} x & \text{if} \quad \alpha \in \Phi_r(x) \\ z & \text{if} \quad \alpha \notin \Phi_r(x), \end{cases}$$

$$x b_\alpha = \begin{cases} x & \text{if} \quad \alpha \in \Phi_l(x) \\ z & \text{if} \quad \alpha \notin \Phi_r(x). \end{cases}$$

Then S is a dexter singly noble semigroup.

Conversely, every dexter singly noble semigroup is isomorphic to one constructed in this way.

Proof. For the direct part of the theorem the only issue is whether S is a semigroup. Once that is established, the additional properties are clear. The verification of associativity falls into six main cases, labelled (in an obvious notation)

$$BNN,\ NBN,\ NNB,\ BBN,\ BNB,\ NBB.$$

For the first three cases it is helpful to look separately at

$$B^+ = \bigcup \{E_\alpha : \alpha > 0\},\ B^- = \bigcup \{E_\alpha : \alpha < 0\} \text{ and } E_0.$$

Since $N^2 = \{z\}$ and $B^+N, NB^+ \subseteq N$ we see that in cases B^+NN, NB^+N and NNB^+ both products $(pq)r$ and $p(qr)$ must equal z. In case B^-NN the rules give

$$(pq)r = p(qr) = p,$$

and cases NB^-N, NNB^- are equally straightforward. In cases E_0NN and NE_0N it is easy to see that

$$(pq)r = p(qr) = z,$$

while in case NNE_0 we have

$$(pq)r = p(qr) = r.$$

For the case BBN, and indeed in a similar way for each of the remaining cases, it is necessary to consider the products $(bc)x$ and $b(cx)$ (with $b \in E_\beta$, $c \in E_\gamma$, $x \in N$) case by case as follows:

(i) $\beta > \gamma > 0$, (ii) $\beta = \gamma > 0$, (iii) $\gamma > \beta > 0$,
(iv) $\beta > 0 > \gamma$, (v) $\gamma > 0 > \beta$, (vi) $0 > \beta > \gamma$,
(vii) $0 > \beta = \gamma$, (viii) $0 > \gamma > \beta$, (ix) $\beta > \gamma = 0$,
(x) $\gamma > \beta = 0$, (xi) $0 = \beta > \gamma$, (xii) $0 = \gamma > \beta$,
(xiii) $\beta = \gamma = 0$.

This is a tedious procedure and we shall record only a couple of cases where there is special interest. First, in case (i) we have

$$(bc)x=cx=\begin{cases} x & \text{if} \quad \gamma\in\Phi_r(x) \\ z & \text{otherwise,}\end{cases}$$

while

$$b(cx)=\begin{cases} bx & \text{if} \quad \gamma\in\Phi_r(x) \\ bz & \text{otherwise}\end{cases}$$

$$=\begin{cases} x & \text{if} \quad \gamma\in\Phi_r(x) \\ z & \text{otherwise,}\end{cases}$$

since $\beta>\gamma$ and $\gamma\in\Phi_r(x)$ gives $\beta\in\Phi_r(x)$ by (5.5). Thus $(bc)x=b(cx)$.

In case (x) we note that $bc\in\{b,z\}$. Hence $(bc)x\in\{bx,zx\}$ and so by (5.7) $(bc)x=z$. For the other product, note that $cx=x$ if $\gamma\in\Phi_r(x)$ and is otherwise equal to z. Hence $b(cx)\in\{bx,bz\}$ and so, by (5.7) and the right zero property of E_0,

$$b(cx)=z.$$

None of the 39 cases is harder than these, and most are easier.

Turning now to the converse part, we know from Theorems 5.3 and 5.4 that if S is a dexter singly noble semigroup generated by $S\backslash\{z\}$ then $S=B\cup N$, where B is a dexter singly noble band with z as pivot, N is a null semigroup with zero element z, and $B\cap N=\{z\}$. The structure of $B=\mathscr{S}[Y;\{E_\alpha:\alpha\in Y\}]$ is given by Theorem 4.14. Denote the pivotal $\mathscr{J}$-class of B by $E_0=\{z,z_2,\ldots\}$. We now show that the multiplication rules in $B\cup N$ must be as listed in the statement of the theorem.

First, let $x\in N\backslash\{z\}$ and suppose that $b_\alpha\in E_\alpha$, $\alpha<0$. Then $xb_\alpha\in\{x,b_\alpha,z\}$. If $xb_\alpha=x$ then

$$z=x^2=x^2b_\alpha=zb_\alpha,$$

in contradiction to the multiplication rules in B; if $xb_\alpha=z$ then again we have a contradiction, since

$$z=xb_\alpha=xb_\alpha^2=(xb_\alpha)b_\alpha=zb_\alpha;$$

hence $xb_\alpha=b_\alpha$. Thus $Nb_\alpha=\{b_\alpha\}$, and similarly $b_\alpha N=\{b_\alpha\}$.

Next, let $x\in N\backslash\{z\}$ and suppose that $b_0\in E_0\backslash\{z\}$. Then $xb_0\in\{x,b_0,z\}$. If $xb_0=x$ then the right zero property of E_0 gives

$$z=x^2=x^2b_0=zb_0=b_0,$$

a contradiction; if $xb_0=z$ then we again have a contradiction, since

$$z=xb_0=xb_0^2=(xb_0)b_0=zb_0=b_0;$$

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hence we must have

$$xb_0 = b_0.$$

Turning now to b_0x, which also belongs to $\{x, b_0, z\}$, we see that $b_0x = x$ gives

$$x = b_0x = (xb_0)x = x(b_0x) = x^2 = z,$$

a contradiction. If we assume $b_0x = b_0$ we obtain

$$b_0 = b_0x = b_0x^2 = b_0z = z,$$

again a contradiction. Hence $b_0x = z$.

Suppose now that $x \in N\backslash\{z\}$ and that $b_\alpha \in E_\alpha$, $\alpha > 0$. Then $b_\alpha x = \{b_\alpha, x, z\}$. If $b_\alpha x = b_\alpha$ then

$$b_\alpha = b_\alpha x = b_\alpha x^2 = b_\alpha z,$$

a contradiction to the multiplication properties of B; hence

$$b_\alpha x = x \quad \text{or} \quad b_\alpha x = z.$$

For a given x in $N\backslash\{z\}$ there are now just two possibilities: either $bx = z$ for all b in $\bigcup\{E_\alpha, \alpha > 0\}$, in which case we define

$$\Phi_r(x) = \emptyset,$$

or there exists $\delta > 0$ and $b_\delta \in E_\delta$ for which $b_\delta x = x$. Suppose that the latter event occurs, and let $c_\delta \in E_\delta$. Then $c_\delta x \in \{x, z\}$. If $c_\delta x = z$ we obtain a contradiction: if $E_\delta \in \mathbf{RZ}$ we have

$$z = c_\delta x = c_\delta(b_\delta x) = (c_\delta b_\delta)x = b_\delta x = x;$$

while if $E_\delta \in \mathbf{LZ}$ we have

$$x = b_\delta x = (b_\delta c_\delta)x = b_\delta(c_\delta x) = b_\delta z = z.$$

Hence $c_\delta x = x$.

If as before we write

$$B^+ = \bigcup\{E_\alpha : \alpha > 0\}$$

We can deduce from the result just proved that the set $\{b \in B^+ : bx = x\}$ is a union

$$\bigcup\{E_\alpha : \alpha \in C\}$$

of $\mathcal{J}$-classes, where $C \subseteq Y^+$. We now show that C is a positive chain filter.

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First, let $\alpha \in C$ and let $\beta > \alpha$. Then for all c_β in E_β

$$c_\beta x = c_\beta(b_\alpha x) \quad (\text{where } b_\alpha \in E_\alpha)$$
$$= (c_\beta b_\alpha)x = b_\alpha x = x.$$

Hence $\beta \in C$.

Next, let $\alpha \in C$ and suppose that β is not comparable with α. Then $E_\alpha E_\beta = \{z\}$ and we must have $c_\beta x = z$ for all c_β in E, since $c_\beta x = x$ (the only other possibility) implies

$$x = b_\alpha x \quad (\text{where } b_\alpha \in E_\alpha)$$
$$= b_\alpha(c_\beta x) = (b_\alpha c_\beta)x = zx = z,$$

a contradiction. Hence $\beta \notin C$.

The conclusion is that for a given x in $N\backslash\{z\}$ either $bx = z$ for all b in B^+, giving

$$\Phi_r(x) = \emptyset$$

as already remarked, or there exists a unique non-empty positive chain filter C such that

$$\{b \in B^+ : bx = x\} = \bigcup \{E_\alpha : \alpha \in C\}.$$

In this case we define $\Phi_r(x) = C$.

In exactly the same way we can consider xb and obtain a mapping $\Phi_l : N\backslash\{z\} \to \mathscr{C}$ such that

$$\{b \in B^+ : xb = x\} = \bigcup \{E_\alpha : \alpha \in \Phi_l(x)\}.$$

This completes the proof.

We also have a dual theorem, in which *dexter* is replaced by *sinister* throughout and (5.7) becomes

$$xb_0 = z,\ b_0 x = b_0 \quad \text{if} \quad x \in N,\ b_0 \in E_0. \tag{5.8}$$

The case of a *feminine* singly noble semigroup is also effectively dealt with: in Theorem 5.5 simply replace dexter by feminine throughout and replace (5.7) by

$$xz = zx = z \quad \text{if} \quad x \in N. \tag{5.9}$$

(In this case $E_0 = \{z\}$.)

The remaining case, that of a *balanced* singly noble semigroup, is now disposed of fairly easily.

Theorem 5.10. *Let $F=B\cup N$, with $B=\mathscr{S}[Y;\{E_\alpha:\alpha\in Y\}]$, be a feminine singly noble semigroup, with pivotal $\mathscr{J}$-class $E_0=\{z\}$, and let g be an element not in F. Define a multiplication on $S=F\cup\{g\}$ by the rules that*

$$g^2=z,\ gN=Ng=\{g\},$$

$$gE_\alpha=E_\alpha g=g \quad \text{if} \quad \alpha>0,$$

$$gb=bg=b \quad \text{if} \quad b\in E_\alpha,\ \alpha<0.$$

Then S is a balanced singly noble semigroup. Conversely, every balanced singly noble semigroup is isomorphic to one of this kind.

Proof. Again the direct part of the proof is a routine associativity verification, but this time the verification is much easier and we omit the details.

As regards the converse part, suppose that S is a balanced singly noble semigroup with $\{g,z\}$ as pivotal $\mathscr{J}$-class. By Theorem 5.3 we have $S=B\cup N\cup\{g\}$, where $F=B\cup N$ is a feminine singly noble semigroup with $\{z\}$ as pivotal $\mathscr{J}$-class. We now show that the multiplication rules on $F\cup\{g\}$ must be as listed in the theorem. We know of course that

$$gz=zg=g,\ g^2=z.$$

Now let $x\in N\backslash\{z\}$. Then $gx\in\{g,x,z\}$. If $gx=x$ then

$$z=x^2=gx^2=gz=g,$$

a contradiction; if $gx=z$ then

$$z=zx=g^2x=g(gx)=gz=g,$$

again a contradiction; hence

$$gx=g,$$

and similarly $xg=g$. Thus

$$gN=Ng=g.$$

Next, let $b\in E_\alpha$, $\alpha>0$. Then since $J_{gb}\leqq J_g=J_z<J_b$ we must have that

$$gb\in\{g,z\}.$$

However, $gb=z$ implies

$$z=z^2=(gb)z=g(bz)=gz=g,$$

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a contradiction, and so in fact $gb=g$. A similar argument establishes that $bg=g$. Thus

$$gE_\alpha=E_\alpha g=\{g\}.$$

Finally, let $b\in E_\alpha$, $\alpha<0$. Then since $gb\in\{g,b,z\}$ and

$$J_{gb}\leqq J_b<J_z=J_g$$

we must in fact have $gb=b$. By the same argument $bg=b$. This completes the proof.

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