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A FOUR-DIMENSIONAL RANDOM MOTION AT FINITE SPEED

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Abstract

We consider the random motion of a particle that moves with constant finite speed in the space $\mathbb{R}^4$ and, at Poisson-distributed times, changes its direction with uniform law on the unit four-sphere. For the particle's position, $\boldsymbol{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$, $t > 0$, we obtain the explicit forms of the conditional characteristic functions and conditional distributions when the number of changes of directions is fixed. From this we derive the explicit probability law, $f(\boldsymbol{x}, t)$, $x \in \mathbb{R}^4$, $t \geq 0$, of $\boldsymbol{X}(t)$. We also show that, under the Kac condition on the speed of the motion and the intensity of the switching Poisson process, the density, $p(\boldsymbol{x}, t)$, of the absolutely continuous component of $f(\boldsymbol{x}, t)$ tends to the transition density of the four-dimensional Brownian motion with zero drift and infinitesimal variance $\sigma^2 = \frac{1}{2}$.

Keywords: Random motion; finite speed; random evolution; characteristic function; uniformly distributed directions; Bessel function; multidimensional Brownian motion

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1. Introduction

The processes of diffusion with finite speed of propagation are highly appropriate models for describing various real phenomena in statistical physics, biology, financial markets, hydrodynamics, and other fields. In recent decades a great number of works (including monographs) have appeared dealing with both the theoretical and applied aspects and properties of these processes. The basic feature of such approaches is the interpretation of diffusion with finite speed of propagation as a process generated by a random motion at finite speed.

Random motions at finite speed (also called random evolutions or transport processes) have become the subject of a great deal of research (see, for instance, the monographs by Pinsky (1991) and Korolyuk and Swishchuk (1994), which also contain vast bibliographies on the subject).

We should emphasize a very important peculiarity of random evolutions, namely that such processes are asymptotically Wiener processes as both the speed and the intensity of collisions tend to ∞ in a special way (referred to as the Kac condition; see (3), below). This interesting feature has a simple physical interpretation. As the random environment is filled more and more with obstacles, the free runs of the moving particle become shorter and shorter. In order to compensate for this, the particle's speed must simultaneously increase. In the limit, the sample paths of such a motion take the form of a fractal, that is, continuous but nowhere differentiable lines. These are exactly the properties of Brownian trajectories.

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The differences between random evolutions and Wiener processes are particularly manifest in the difference between the equations governing these motions. It is well known that the transition density of Brownian motion is the fundamental solution to the *parabolic* heat equation. In contrast, random motions at finite speed are governed by *hyperbolic* equations in all cases (very few indeed) that such equations have been obtained. Goldstein (1951) and Kac (1974) have shown that the transition density of one-dimensional random evolution is the fundamental solution to the hyperbolic telegraph equation. The explicit form of the transition density of the Goldstein–Kac telegraph process was given by Orsingher (1990, Theorem 1) and Pinsky (1991, p. 9).

Similar results were obtained for the two-dimensional random evolution by Stadje (1987), Masoliver *et al.* (1993), and Kolesnik and Orsingher (2005). In those works the explicit form of the transition density was obtained using different methods (and in Stadje (1987) it was done for unit speed). It was shown that the transition density of this motion is the fundamental solution to the two-dimensional hyperbolic telegraph equation, as in the one-dimensional case. In Kolesnik and Turbin (1998), the planar random evolution with $n \geq 2$ directions was studied and the governing nth-order hyperbolic partial differential equation for the transition density of the process was explicitly derived. In all these cases we see that hyperbolicity is the distinguishing characteristic of the partial differential equations governing such motions.

However, while the transition density of Brownian motion in *any* dimension is the fundamental solution to a corresponding heat equation, the analogous results for random evolutions are known only in the one- and two-dimensional cases. Although random evolutions have trajectories that are considerably more regular (in fact continuous and differentiable almost everywhere) than the Brownian ones, the direct derivation of governing equations is a very complicated problem. In fact, as we have noted above, this has been done only for the one-dimensional Goldstein–Kac telegraph process and for the planar random evolution in a finite number of directions. The governing telegraph equation has not been directly derived even for the planar random evolution with a continuum of directions. It appeared only 'after the fact', when the explicit form of the transition density was obtained by other methods (see Masoliver *et al.* (1993) and Kolesnik and Orsingher (2005)). The three-dimensional random evolution was studied by Tolubinsky (1969, pp. 35–60) and by Stadje (1989). In those works the transition density of the motion was obtained in the form of fairly complicated integrals which seemingly cannot be evaluated explicitly in terms of elementary functions. However, no governing equation was obtained in this case.

In this paper we study the four-dimensional random evolution. In our main result we present the *explicit* form of the transition density of the motion, which is undoubtedly the most desirable aim of any research in this field. We also show that our motion is asymptotically the Wiener process. We should emphasize that the transition density of the motion is *not* a fundamental solution to a standard four-dimensional telegraph equation, in contrast to the one- and two-dimensional cases. The main tools of our research are the characteristic functions of the distributions involved.

The stochastic motion considered here is that of a particle starting at the origin of the space $\mathbb{R}^4$ and moving with a constant finite speed c. Its motion is subject to the control of a homogeneous Poisson process of rate $\lambda > 0$. When Poisson events occur the particle instantaneously takes on a random direction uniformly distributed on the unit four-sphere, S_1. The sample paths therefore appear in the space $\mathbb{R}^4$ as continuous and almost-everywhere-differentiable zigzag lines composed of segments of random length and of orientation uniformly distributed on S_1. Note that the total length of each trajectory at any instant $t > 0$ is ct.

The set of possible positions of the particle at time $t > 0$ is a four-dimensional ball, B_{ct}, of radius ct, and the density of the position, $f(\boldsymbol{x}, t) = f(x_1, x_2, x_3, x_4, t)$, has a singular part concentrated on the surface ∂B_{ct} (corresponding to the case in which the initial direction is maintained until time t). When the number of changes of direction is greater than or equal to 1, the particle is located inside B_{ct} and, since the trajectories are continuous and the motion has finite speed, an absolutely continuous part of the distribution exists.

In Section 2 we obtain the following conditional distribution for the particle's position, $\boldsymbol{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$, $t > 0$, where $N(t)$ is the number of Poisson events that have occurred by time $t > 0$ and $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \operatorname{int} B_{ct}$:

$$\begin{aligned}\Pr\{X_1(t) \in \mathrm{d}x_1,\ X_2(t) \in \mathrm{d}x_2,\ X_3(t) \in \mathrm{d}x_3,\ X_4(t) \in \mathrm{d}x_4 \mid N(t) = n\} \\ = \frac{n(n+1)}{\pi^2 (ct)^4}\left(1 - \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{c^2 t^2}\right)^{n-1} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4, \qquad n \geq 1. \quad (1)\end{aligned}$$

The function (1) is circularly symmetric for all $n \geq 1$ because of the uniform distribution of the choice of direction at each Poisson instant.

Interestingly, when $n = 1$ the conditional distribution (1) becomes uniform in the ball B_{ct}. For $n \geq 2$ the distribution (1) takes a bell-shaped form due to the fact that, the more frequently do changes of the direction of motion occur, the more fragmented become the sample paths and the shorter becomes the distance of the moving particle from the origin.

In Section 3, from the conditional distribution (1) we derive our main result, stating that the absolutely continuous part, $p(\boldsymbol{x}, t) = p(x_1, x_2, x_3, x_4, t)$, of $f(\boldsymbol{x}, t)$ has the fairly simple analytic form

$$p(\boldsymbol{x}, t) = \frac{\lambda t}{\pi^2 (ct)^4}\left[2 + \lambda t\left(1 - \frac{\|\boldsymbol{x}\|^2}{c^2 t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\|\boldsymbol{x}\|^2\right), \qquad \boldsymbol{x} \in \operatorname{int} B_{ct}, \qquad (2)$$

with $\|\boldsymbol{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. We also show that, under the Kac condition

$$c \to \infty, \qquad \lambda \to \infty, \qquad \frac{c^2}{\lambda} \to 1, \qquad (3)$$

the density (2) tends to the transition density of the four-dimensional Brownian motion with zero drift and infinitesimal variance $\sigma^2 = \frac{1}{2}$.

The density (2) takes an especially interesting form in polar coordinates, namely

$$\begin{aligned}\tilde{p}(\rho, \theta_1, \theta_2, \theta_3, t) = \frac{\lambda t \rho^3}{\pi^2 (ct)^4}\left[2 + \lambda t\left(1 - \frac{\rho^2}{c^2 t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\rho^2\right) \sin^2\theta_1 \sin\theta_2, \\ 0 < \rho < ct, \qquad 0 \leq \theta_1, \theta_2 \leq \pi, \qquad 0 \leq \theta_3 \leq 2\pi, \qquad t > 0,\end{aligned}$$

which shows that the radial component is independent of the circular ones. This interesting representation looks like that of the multidimensional Brownian transition function in polar coordinates, where the radial and circular components also are independent.

Our model yields the four-dimensional counterpart of the random motions at finite speed in lower dimensions studied in the works mentioned above. The most important feature of our paper is that the distribution of the process is obtained in an *explicit* form. It seems doubtful that this is possible in more than four dimensions. We should emphasize that explicit forms of distributions appear in the even-dimensional cases only, whereas in the three-dimensional case the distribution was obtained in a form of a fairly complicated integral which seemingly cannot be evaluated explicitly in terms of elementary functions (see Stadje (1989, Equations (1.3) and (4.21)) for details).

The main results of this paper were announced in Kolesnik (2006).

2. Description of motion and conditional distributions

We consider a particle starting its motion from the origin, $x_1 = x_2 = x_3 = x_4 = 0$, of the space $\mathbb{R}^4$ at time $t = 0$. The particle is endowed with constant finite speed c (note that c is treated as the constant norm of the velocity). The initial direction is a four-dimensional random vector with uniform distribution (Riemann–Lebesgue probability measure) on the unit four-sphere

$$S_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At each of these moments it instantaneously takes on a new direction uniformly distributed on S_1, independently of its previous motion.

Let $\boldsymbol{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t))$ be the position of the particle at an arbitrary time $t > 0$. In this section we concentrate our attention on the conditional distribution

$$\begin{aligned} &\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} \\ &\qquad = \Pr\{X_1(t) \in \mathrm{d}x_1,\ X_2(t) \in \mathrm{d}x_2,\ X_3(t) \in \mathrm{d}x_3,\ X_4(t) \in \mathrm{d}x_4 \mid N(t) = n\}, \\ &\qquad\qquad n \ge 1, \end{aligned} \tag{4}$$

where (recall) $N(t)$ is the number of Poisson events that have occurred in the interval $(0, t)$ and $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$ is the infinitesimal volume of the space $\mathbb{R}^4$.

At any time $t > 0$, the particle is located with probability 1 in the four-dimensional ball of radius ct,

$$B_{ct} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon x_1^2 + x_2^2 + x_3^2 + x_4^2 \le c^2t^2\}.$$

The distribution $\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x}\}$, $\boldsymbol{x} \in B_{ct}$, $t \ge 0$, consists of two components. The singular component corresponds to the case in which no Poisson event occurs in the interval $(0, t)$, and is concentrated on the sphere

$$S_{ct} = \partial B_{ct} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon x_1^2 + x_2^2 + x_3^2 + x_4^2 = c^2t^2\}.$$

In this case the particle is located on the sphere S_{ct}, and the probability of this event is

$$\Pr\{\boldsymbol{X}(t) \in S_{ct}\} = \mathrm{e}^{-\lambda t}.$$

If one or more Poisson events occur, the particle is located strictly inside the ball B_{ct}, and the probability of this event is

$$\Pr\{\boldsymbol{X}(t) \in \operatorname{int} B_{ct}\} = 1 - \mathrm{e}^{-\lambda t}. \tag{5}$$

The part of the distribution $\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x}\}$ corresponding to this case is concentrated in the interior of B_{ct},

$$\operatorname{int} B_{ct} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \colon x_1^2 + x_2^2 + x_3^2 + x_4^2 < c^2t^2\},$$

and forms its absolutely continuous component.

Therefore, the density, $p(\boldsymbol{x}, t) = p(x_1, x_2, x_3, x_4, t)$, $\boldsymbol{x} \in \operatorname{int} B_{ct}$, $t > 0$, of the absolutely continuous component of the distribution $\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x}\}$ exists. Studying $p(\boldsymbol{x}, t)$ is the main aim of our research.

Our first result concerns the explicit form of the conditional distribution (4).

Theorem 1. *For any $n \geq 1$ and any $t > 0$, the conditional distribution (4) has the form*

$$\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} = \frac{n(n+1)}{\pi^2 (ct)^4} \left(1 - \frac{\|\boldsymbol{x}\|^2}{c^2 t^2}\right)^{n-1} \mathrm{d}\boldsymbol{x}, \qquad n \geq 1, \tag{6}$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \operatorname{int} B_{ct}$, $\|\boldsymbol{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, and $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3\, \mathrm{d}x_4$.

Proof. For $N(t) = n$, the displacement of the particle at any time $t > 0$, $\boldsymbol{X}(t)$, is given by the coordinates

$$X_k(t) = c \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k, \qquad k = 1, 2, 3, 4, \tag{7}$$

where the x_j^k are the components of the independent random vectors $\boldsymbol{x}_j = (x_j^1, x_j^2, x_j^3, x_j^4)$, $j = 1, \dots, n+1$, uniformly distributed on the unit sphere S_1; the s_j, $j = 1, \dots, n$, represent the instants at which Poisson events occur; and $s_0 = 0$ and $s_{n+1} = t$.

We now evaluate the conditional characteristic function

$$H_n(\boldsymbol{\alpha}, t) = \mathrm{E}\{\mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha}, \boldsymbol{X}(t))} \mid N(t) = n\}, \qquad n \geq 1, \tag{8}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ is the real vector of inversion parameters and $(\boldsymbol{\alpha}, \boldsymbol{X}(t))$ denotes the inner product of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{X}(t)$.

By substituting (7) into (8) we obtain

$$H_n(\boldsymbol{\alpha}, t) = \mathrm{E}\left\{\exp\left(\mathrm{i}c \sum_{k=1}^{4} \alpha_k \sum_{j=1}^{n+1} (s_j - s_{j-1}) x_j^k\right)\right\} = \mathrm{E}\left\{\exp\left(\mathrm{i}c \sum_{j=1}^{n+1} (s_j - s_{j-1}) (\boldsymbol{\alpha}, \boldsymbol{x}_j)\right)\right\}.$$

Computing the expectation in this last equality, we obtain

$$H_n(\boldsymbol{\alpha}, t) = \frac{n!}{t^n} \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \left\{\prod_{j=1}^{n+1} \frac{1}{\operatorname{mes}(S_1)} \int_{S_1} \mathrm{e}^{\mathrm{i}c(s_j - s_{j-1})(\boldsymbol{\alpha}, \boldsymbol{x}_j)}\, \mathrm{d}\boldsymbol{x}_j\right\},$$

where $\operatorname{mes}(\cdot)$ denotes Lebesgue measure. The surface integral over the unit sphere S_1 in the last equality can easily be computed by changing to four-dimensional polar coordinates, and is found to be

$$\int_{S_1} \mathrm{e}^{\mathrm{i}c(s_j - s_{j-1})(\boldsymbol{\alpha}, \boldsymbol{x}_j)}\, \mathrm{d}\boldsymbol{x}_j = (2\pi)^2 \frac{J_1(c(s_j - s_{j-1})\|\boldsymbol{\alpha}\|)}{c(s_j - s_{j-1})\|\boldsymbol{\alpha}\|},$$

where $\|\boldsymbol{\alpha}\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2}$ and $J_1(x)$ is the Bessel function of first order with real argument. Since $\operatorname{mes}(S_1) = 2\pi^2$, we obtain

$$H_n(\boldsymbol{\alpha}, t) = \frac{2^{n+1} n!}{t^n} \int_0^t \mathrm{d}s_1 \int_{s_1}^t \mathrm{d}s_2 \cdots \int_{s_{n-1}}^t \mathrm{d}s_n \left\{\prod_{j=1}^{n+1} \frac{J_1(c(s_j - s_{j-1})\|\boldsymbol{\alpha}\|)}{c(s_j - s_{j-1})\|\boldsymbol{\alpha}\|}\right\}. \tag{9}$$

We now show that the conditional characteristic function $H_n(\boldsymbol{\alpha}, t)$ has the form

$$H_n(\boldsymbol{\alpha}, t) = 2^{n+1}(n+1)!\, \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n+1}}, \qquad n \geq 1. \tag{10}$$

To do so, in view of (9) it is sufficient to prove the equality

$$\begin{aligned}
I_n := &\int_0^t \mathrm{d}s_1 \Bigg\{ \frac{J_1(cs_1\|\boldsymbol{\alpha}\|)}{s_1} \int_{s_1}^t \mathrm{d}s_2 \Bigg\{ \frac{J_1(c(s_2-s_1)\|\boldsymbol{\alpha}\|)}{s_2-s_1} \int_{s_2}^t \mathrm{d}s_3 \Bigg\{ \frac{J_1(c(s_3-s_2)\|\boldsymbol{\alpha}\|)}{s_3-s_2} \cdots \\
&\quad \times \int_{s_{n-2}}^t \mathrm{d}s_{n-1} \Bigg\{ \frac{J_1(c(s_{n-1}-s_{n-2})\|\boldsymbol{\alpha}\|)}{s_{n-1}-s_{n-2}} \\
&\qquad \times \int_{s_{n-1}}^t \mathrm{d}s_n \Bigg\{ \frac{J_1(c(s_n-s_{n-1})\|\boldsymbol{\alpha}\|)}{s_n-s_{n-1}} \frac{J_1(c(t-s_n)\|\boldsymbol{\alpha}\|)}{t-s_n} \Bigg\}\Bigg\} \cdots \Bigg\}\Bigg\}\Bigg\} \\
= &\frac{n+1}{t} J_{n+1}(ct\|\boldsymbol{\alpha}\|).
\end{aligned} \tag{11}$$

Our proof is based on successive integrations on the left-hand side of (11). Consider the integral with respect to s_n. Making the substitution $\xi = c(s_n - s_{n-1})\|\boldsymbol{\alpha}\|$, we obtain

$$\begin{aligned}
&\int_{s_{n-1}}^t \frac{J_1(c(s_n-s_{n-1})\|\boldsymbol{\alpha}\|)}{s_n-s_{n-1}} \frac{J_1(c(t-s_n)\|\boldsymbol{\alpha}\|)}{t-s_n}\, \mathrm{d}s_n \\
&\qquad = c\|\boldsymbol{\alpha}\| \int_0^{c(t-s_{n-1})\|\boldsymbol{\alpha}\|} \frac{J_1(\xi)}{\xi} \frac{J_1(c(t-s_{n-1})\|\boldsymbol{\alpha}\| - \xi)}{c(t-s_{n-1})\|\boldsymbol{\alpha}\| - \xi}\, \mathrm{d}\xi.
\end{aligned} \tag{12}$$

This integral can be computed by means of Gradshteyn and Ryzhik (1980, Formula 6.533(2)), namely

$$\int_0^z \frac{J_p(x)}{x} \frac{J_q(z-x)}{z-x}\, \mathrm{d}x = \left(\frac{1}{p} + \frac{1}{q}\right) \frac{J_{p+q}(z)}{z}, \qquad \operatorname{Re} p > 0,\ \operatorname{Re} q > 0. \tag{13}$$

Application of (13) to the integral on the right-hand side of (12) yields

$$\int_{s_{n-1}}^t \frac{J_1(c(s_n-s_{n-1})\|\boldsymbol{\alpha}\|)}{s_n-s_{n-1}} \frac{J_1(c(t-s_n)\|\boldsymbol{\alpha}\|)}{t-s_n}\, \mathrm{d}s_n = 2\frac{J_2(c(t-s_{n-1})\|\boldsymbol{\alpha}\|)}{t-s_{n-1}}.$$

The integral with respect to s_{n-1} in (11) is computed in the same way to be

$$2\int_{s_{n-2}}^t \frac{J_1(c(s_{n-1}-s_{n-2})\|\boldsymbol{\alpha}\|)}{s_{n-1}-s_{n-2}} \frac{J_2(c(t-s_{n-1})\|\boldsymbol{\alpha}\|)}{t-s_{n-1}}\, \mathrm{d}s_{n-1} = 3\frac{J_3(c(t-s_{n-2})\|\boldsymbol{\alpha}\|)}{t-s_{n-2}}.$$

Continuing this integration process yields, after the $(n-1)$th step,

$$I_n = n\int_0^t \frac{J_1(cs_1\|\boldsymbol{\alpha}\|)}{s_1} \frac{J_n(c(t-s_1)\|\boldsymbol{\alpha}\|)}{t-s_1}\, \mathrm{d}s_1.$$

Making the substitution $\xi = cs_1\|\boldsymbol{\alpha}\|$ in this integral and applying (13) once more, we finally obtain

$$I_n = \frac{n+1}{t} J_{n+1}(ct\|\boldsymbol{\alpha}\|),$$

proving (11). Thus, the explicit form of the conditional characteristic function (10) is valid.

In order to prove the statement of the theorem, we need to show that inverse Fourier transformation of the conditional characteristic function $H_n(\boldsymbol{\alpha}, t)$ leads to the conditional distribution (6). However, this requires complex calculations, so instead we will check that, conversely, the Fourier transform of function (6) in the ball B_{ct} coincides with the conditional characteristic function (10) for any $n \geq 1$.

For our further analysis we need the following formula concerning the Fourier transform of the function identically equal to 1 in the four-dimensional ball, B_r, of radius $r > 0$:

$$\begin{aligned}\int_{B_r} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x} &= \iiiint_{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\le r} \exp(\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4))\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 \\ &= (2\pi r)^2 \frac{J_2(r\|\boldsymbol{\alpha}\|)}{\|\boldsymbol{\alpha}\|^2}. \end{aligned} \tag{14}$$

This can be easily checked by changing to four-dimensional polar coordinates.

Let us compute the Fourier transform of function (6) in the ball B_{ct}. First we note that, for $n = 1$, in view of (14) we have

$$\begin{aligned}&\int_{B_{ct}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = 1\} \\ &\quad= \frac{2}{\pi^2 (ct)^4} \iiiint_{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\le ct} \exp(\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4))\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4 \\ &\quad= 8\frac{J_2(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^2},\end{aligned}$$

which coincides with (10) for $n = 1$.

Now let $n \geq 2$. Then

$$\begin{aligned}&\int_{B_{ct}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} \\ &\quad= \frac{n(n+1)}{\pi^2 (ct)^4} \iiiint_{\sqrt{x_1^2+x_2^2+x_3^2+x_4^2}\le ct} \exp(\mathrm{i}(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha_4 x_4)) \\ &\qquad\qquad \times \left(1 - \frac{x_1^2+x_2^2+x_3^2+x_4^2}{c^2t^2}\right)^{n-1} \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4. \end{aligned} \tag{15}$$

The integral on the right-hand side of (15) can be evaluated by changing to four-dimensional polar coordinates. However, this also requires complex calculations. Instead, we prefer a more simple way of evaluation based on the application of the multidimensional Catalan theorem of classical analysis (see, for instance, Gradshteyn and Ryzhik (1980, Theorem 4.645) or Fichtenholtz (1970, p. 407, Equation (12))). Applying this Catalan theorem to our case and taking into account (14), the four-dimensional integral on the right-hand side of (15) can be reduced to a Stieltjes integral, and we have

$$\begin{aligned}&\int_{B_{ct}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} \\ &\quad= \frac{n(n+1)}{\pi^2 (ct)^4} \int_0^{ct} \left(1 - \frac{u^2}{c^2t^2}\right)^{n-1} \mathrm{d}\left[(2\pi u)^2 \frac{J_2(u\|\boldsymbol{\alpha}\|)}{\|\boldsymbol{\alpha}\|^2}\right].\end{aligned}$$

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By integrating by parts and taking into account the fact that the free term vanishes for $n \geq 2$, we easily obtain

$$\begin{aligned}\int_{B_{ct}} & \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} \\ &= \frac{8n(n-1)(n+1)}{(ct\|\boldsymbol{\alpha}\|)^2} \int_0^1 u^3(1-u^2)^{n-2} J_2(ct\|\boldsymbol{\alpha}\|u)\,\mathrm{d}u.\end{aligned}$$

By now applying Gradshteyn and Ryzhik (1980, Formula 6.567(1)), we finally have

$$\begin{aligned}\int_{B_{ct}} & \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = n\} \\ &= \frac{8n(n-1)(n+1)}{(ct\|\boldsymbol{\alpha}\|)^2} 2^{n-2}\Gamma(n-1)(ct\|\boldsymbol{\alpha}\|)^{-n+1} J_{n+1}(ct\|\boldsymbol{\alpha}\|) \\ &= 2^{n+1}(n+1)!\,\frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n+1}},\end{aligned}$$

which coincides with (10). The theorem is thus completely proved.

Remark 1. For $n = 1$, the conditional density (6) becomes

$$\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x} \mid N(t) = 1\} = \frac{2\,\mathrm{d}\boldsymbol{x}}{\pi^2(ct)^4}, \qquad \boldsymbol{x} \in \operatorname{int} B_{ct},\ t > 0,$$

which is exactly the uniform distribution in the ball B_{ct}. This interesting fact means that, after the first change of direction, the particle is located (in the ball B_{ct}) near the border or near the origin with the same probability. For $n \geq 2$, the conditional distributions have a bell-shaped form centred at the origin. The reason for the appearance of the bell-shaped form is that, when a sufficient number of changes of direction have been recorded (the minimal number clearly being two), the sample paths are so fragmented that the particle can barely leave the neighbourhood of the origin. This is similar to the one-dimensional case, in which the conditional distribution is uniform for $n = 1$ and the bell-shaped densities appear for $n \geq 2$. In contrast, in both the two- and three-dimensional cases the conditional density corresponding to a single change of direction has an infinite discontinuity on the boundary of the diffusion area and the uniform distribution appears only after the second change of direction (see Kolesnik and Orsingher (2005, Remark 1) and Stadje (1989, second terms of Equations (1.3) and (4.21)) for comparison).

3. The density of the process and its connection with four-dimensional Brownian motion

Now we are able to derive the absolutely continuous component of the distribution of the motion. From Theorem 1 we can straightforwardly infer our main result.

Theorem 2. *The absolutely continuous part of the distribution of* $\boldsymbol{X}(t)$, $t > 0$, *has the form*

$$\Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x}\} = \frac{\lambda t}{\pi^2(ct)^4}\left[2 + \lambda t\left(1 - \frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right]\exp\left(-\frac{\lambda}{c^2 t}\|\boldsymbol{x}\|^2\right)\mathrm{d}\boldsymbol{x}, \tag{16}$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \operatorname{int} B_{ct}$, $\|\boldsymbol{x}\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$, *and* $\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$.

Proof. In view of Theorem 1 we have

$$
\begin{aligned}
\Pr\{\boldsymbol{X}(t)\in \mathrm{d}\boldsymbol{x}\} &= \sum_{n=1}^{\infty}\Pr\{\boldsymbol{X}(t)\in \mathrm{d}\boldsymbol{x}\mid N(t)=n\}\Pr\{N(t)=n\}\\
&= \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\frac{n(n+1)}{\pi^2(ct)^4}\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)^{n-1}\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t\mathrm{e}^{-\lambda t}}{\pi^2(ct)^4}\sum_{n=0}^{\infty}\frac{(\lambda t)^n(n+2)}{n!}\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)^{n}\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t\mathrm{e}^{-\lambda t}}{\pi^2(ct)^4}\left[2+\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{(n-1)!}\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)^{n}+2\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)^{n}\right]\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t\mathrm{e}^{-\lambda t}}{\pi^2(ct)^4}\left[2+\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\exp\left(\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right)\right.\\
&\qquad\left.+2\left(\exp\left(\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right)-1\right)\right]\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t\mathrm{e}^{-\lambda t}}{\pi^2(ct)^4}\left[2+\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right]\exp\left(\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right)\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t}{\pi^2(ct)^4}\left[2+\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right]\exp\left(-\frac{\lambda}{c^2t}\|\boldsymbol{x}\|^2\right)\mathrm{d}\boldsymbol{x},
\end{aligned}
$$

proving (16).

It remains to check that, according to (5), for any $t>0$,

$$\int_{B_{ct}}\Pr\{\boldsymbol{X}(t)\in \mathrm{d}\boldsymbol{x}\} = 1-\mathrm{e}^{-\lambda t}. \tag{17}$$

Changing to four-dimensional polar coordinates, we obtain

$$
\begin{aligned}
\int_{B_{ct}}\Pr\{\boldsymbol{X}(t)\in \mathrm{d}\boldsymbol{x}\} &= \int_{B_{ct}}\frac{\lambda t}{\pi^2(ct)^4}\left[2+\lambda t\left(1-\frac{\|\boldsymbol{x}\|^2}{c^2t^2}\right)\right]\exp\left(-\frac{\lambda}{c^2t}\|\boldsymbol{x}\|^2\right)\mathrm{d}\boldsymbol{x}\\
&= \frac{\lambda t}{\pi^2(ct)^4}\int_0^{ct}\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi}\left[2+\lambda t\left(1-\frac{r^2}{c^2t^2}\right)\right]\exp\left(-\frac{\lambda}{c^2t}r^2\right)\\
&\qquad\times r^3\sin^2\varphi_1\sin\varphi_2\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_3\,\mathrm{d}r\\
&= \frac{2\lambda t}{(ct)^4}\int_0^{ct}r^3\left[2+\lambda t\left(1-\frac{r^2}{c^2t^2}\right)\right]\exp\left(-\frac{\lambda}{c^2t}r^2\right)\mathrm{d}r\\
&= \frac{2\lambda t}{(ct)^4}\left[\int_0^{c^2t^2}z\exp\left(-\frac{\lambda}{c^2t}z\right)\mathrm{d}z+\frac{\lambda t}{2}\int_0^{c^2t^2}z\left(1-\frac{z}{c^2t^2}\right)\exp\left(-\frac{\lambda}{c^2t}z\right)\mathrm{d}z\right]\\
&= 2\lambda t\left[\int_0^1\xi\mathrm{e}^{-\lambda t\xi}\,\mathrm{d}\xi+\frac{\lambda t}{2}\int_0^1\xi(1-\xi)\mathrm{e}^{-\lambda t\xi}\,\mathrm{d}\xi\right]\\
&= 2\lambda t\left[\left(1+\frac{\lambda t}{2}\right)\int_0^1\xi\mathrm{e}^{-\lambda t\xi}\,\mathrm{d}\xi-\frac{\lambda t}{2}\int_0^1\xi^2\mathrm{e}^{-\lambda t\xi}\,\mathrm{d}\xi\right]\\
&= 1-\mathrm{e}^{-\lambda t},
\end{aligned}
$$

where, in the last step, we have used integration by parts. Thus, (17) holds and the theorem is completely proved.

The 'missing' part of the probability in (17) (namely $\mathrm{e}^{-\lambda t}$) pertains to the singular component of the distribution and is concentrated on the boundary $S_{ct} = \partial B_{ct}$.

Remark 2. From (16), by means of simple but tedious calculations, we can show that

$$\begin{aligned}
\Pr\{\boldsymbol{X}(t) \in B_r\} &= \int_{B_r} \Pr\{\boldsymbol{X}(t) \in \mathrm{d}\boldsymbol{x}\} \\
&= \frac{\lambda t}{\pi^2 (ct)^4} \int_{B_r} \left[2 + \lambda t\left(1 - \frac{\|\boldsymbol{x}\|^2}{c^2 t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\|\boldsymbol{x}\|^2\right) \mathrm{d}\boldsymbol{x} \\
&= 1 - \left(1 + \frac{\lambda}{c^2 t} r^2 - \frac{\lambda}{c^4 t^3} r^4\right) \exp\left(-\frac{\lambda}{c^2 t} r^2\right), \qquad 0 \le r \le ct, \qquad (18)
\end{aligned}$$

which is the probability of the particle lying within any ball B_r of radius r, $0 \le r \le ct$. It is clear that, for $r = ct$, (18) becomes (17).

Remark 3. The complete density, $f(\boldsymbol{x}, t)$, $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in B_{ct}$, $t \ge 0$, of the distribution of $\boldsymbol{X}(t)$ can be written in terms of generalized functions as

$$\begin{aligned}
f(\boldsymbol{x}, t) &= \frac{\mathrm{e}^{-\lambda t}}{2\pi^2 (ct)^3} \delta(c^2 t^2 - \|\boldsymbol{x}\|^2) \\
&\quad + \frac{\lambda t}{\pi^2 (ct)^4} \left[2 + \lambda t\left(1 - \frac{\|\boldsymbol{x}\|^2}{c^2 t^2}\right)\right] \exp\left(-\frac{\lambda}{c^2 t}\|\boldsymbol{x}\|^2\right) \Theta(ct - \|\boldsymbol{x}\|), \qquad (19)
\end{aligned}$$

where $\delta(\cdot)$ is the Dirac delta-function and $\Theta(\cdot)$ is the Heaviside function.

Remark 4. By changing to four-dimensional polar coordinates it is easy to show that the conditional characteristic function related to the case in which no changes of direction occur has the form

$$H_0(\boldsymbol{\alpha}, t) = \mathrm{E}\{\mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha}, \boldsymbol{X}(t))} \mid N(t) = 0\} = \frac{1}{2\pi^2 (ct)^3} \int_{S_{ct}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha}, \boldsymbol{x})} \, \mathrm{d}\boldsymbol{x} = 2 \frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|};$$

therefore, (10) is valid also for $n = 0$. In view of (10), the characteristic function of the random vector $\boldsymbol{X}(t)$ then has the following series representation:

$$\begin{aligned}
H(\boldsymbol{\alpha}, t) &= \mathrm{E}\{\mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha}, \boldsymbol{X}(t))}\} \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} H_n(\boldsymbol{\alpha}, t) \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} 2^{n+1} (n+1)! \frac{J_{n+1}(ct\|\boldsymbol{\alpha}\|)}{(ct\|\boldsymbol{\alpha}\|)^{n+1}} \\
&= \frac{2\mathrm{e}^{-\lambda t}}{ct\|\boldsymbol{\alpha}\|} \sum_{n=0}^{\infty} \left(\frac{2\lambda}{c\|\boldsymbol{\alpha}\|}\right)^n (n+1) J_{n+1}(ct\|\boldsymbol{\alpha}\|). \qquad (20)
\end{aligned}$$

Clearly, series (20) represents the Fourier transform of the density (19) in the ball B_{ct}.

We can also give the integral form of the characteristic function $H(\boldsymbol{\alpha}, t)$:

$$\begin{aligned}H(\boldsymbol{\alpha}, t) &= 2\mathrm{e}^{-\lambda t}\frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|} \\ &\quad + \frac{\lambda t\mathrm{e}^{-\lambda t}}{\pi^2(ct)^4}\int_{B_{ct}} \mathrm{e}^{\mathrm{i}(\boldsymbol{\alpha},\boldsymbol{x})}\bigg[2 + \lambda t\bigg(1 - \frac{\|\boldsymbol{x}\|^2}{c^2t^2}\bigg)\bigg]\exp\bigg(-\frac{\lambda}{c^2t}\|\boldsymbol{x}\|^2\bigg)\,\mathrm{d}\boldsymbol{x} \\ &= 2\mathrm{e}^{-\lambda t}\frac{J_1(ct\|\boldsymbol{\alpha}\|)}{ct\|\boldsymbol{\alpha}\|} \\ &\quad + \frac{4\lambda t\mathrm{e}^{-\lambda t}}{(ct)^4\|\boldsymbol{\alpha}\|}\int_0^{ct} r^2\bigg[2 + \lambda t\bigg(1 - \frac{r^2}{c^2t^2}\bigg)\bigg]\exp\bigg(-\frac{\lambda}{c^2t}r^2\bigg)J_1(r\|\boldsymbol{\alpha}\|)\,\mathrm{d}r.\end{aligned}$$

It seems unlikely that this integral can be expressed in terms of elementary functions.

Finally, we are able to establish a result concerning the limiting behaviour of the process $\boldsymbol{X}(t)$ under the Kac condition (3). The following theorem states that our stochastic motion is asymptotically a Wiener process.

Theorem 3. *Under the Kac condition (3), the density of the absolutely continuous component of the distribution of* $\boldsymbol{X}(t)$*, given by (16), converges to the transition density of the four-dimensional Brownian motion with zero drift and infinitesimal variance* $\sigma^2 = \frac{1}{2}$.

Proof. Rewriting the density in (16) in the form

$$p(\boldsymbol{x}, t) = \frac{1}{\pi^2 t^3}\frac{\lambda}{c^2}\bigg[\frac{2}{c^2} + \frac{\lambda}{c^2}t\bigg(1 - \frac{\|\boldsymbol{x}\|^2}{c^2t^2}\bigg)\bigg]\exp\bigg(-\frac{\lambda}{c^2}\frac{\|\boldsymbol{x}\|^2}{t}\bigg)$$

and taking into account the fact that, under the Kac condition (3), the factor in square brackets tends to t, we see that

$$p(\boldsymbol{x}, t) \to u(\boldsymbol{x}, t) := \frac{1}{\pi^2 t^2}\mathrm{e}^{-\|\boldsymbol{x}\|^2/t}, \qquad \boldsymbol{x} \in \operatorname{int} B_{ct},\ t > 0. \tag{21}$$

The function $u(\boldsymbol{x}, t)$ on the right-hand side of (21) is exactly the transition density of the four-dimensional Brownian motion with zero drift and infinitesimal variance $\sigma^2 = \frac{1}{2}$. The theorem is thus proved.

It is well known that the limiting function $u \equiv u(x_1, x_2, x_3, x_4, t)$ on the right-hand side of (21) is the fundamental solution to the four-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\Delta u,$$

where Δ is the four-dimensional Laplacian, such that

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2}.$$

This means that the limiting process is the four-dimensional Brownian motion with generator $\frac{1}{4}\Delta$. This exactly coincides with the previous limiting result of Pinsky (1976, Proposition 4.8) for our four-dimensional case.

Remark 5. Returning now to the discussion concerning the relationships between random evolutions and partial differential equations, we can check that the density of our motion, given by (16), is *not* a fundamental (or even partial) solution to the four-dimensional hyperbolic telegraph equation of standard form. This allows us to assume that random evolutions in spaces of higher dimension (greater than two) are, apparently, driven by other sorts of equation (at least with other coefficients). This confirms our previous conjecture concerning the type of equations governing random evolutions in higher dimensions (see Kolesnik and Turbin (1998, p. 69)). The form, $\frac{1}{4}\Delta$, of the generator of the limiting Wiener process can give some hints about the possible structure of such equations. On the other hand, since random evolutions develop at finite speed, we expect that such governing equations also must be of hyperbolic type. This interesting problem is still open for spaces of dimension greater than two.

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