Canad. Math. Bull. Vol. **24** (3), 1981

R-PROJECTIVE MODULES OVER A SEMIPERFECT RING

BY
R. D. KETKAR AND N. VANAJA

ABSTRACT. The aim of this paper is to prove the following theorem:

Let R be a semiperfect ring. Let Q be a left R-module satisfying (a) Q is R-projective and (b) $J(Q)$ is small in Q. Then Q is projective.

1. Throughout R denotes an associative ring with unity. By an R-module we mean a unitary left R-module. For further terminology we refer to [1]. Specifically speaking we require the definitions and elementary properties of *R-projective modules, projective covers* and *semiperfect rings.*

2. This section is devoted to the proof of the theorem stated in the abstract. We prove a key fact in

LEMMA. *Let Q be an R-projective module. Suppose $Q=M+N$ where N is cyclic and Q/M has a projective cover $f:P_1\to Q/M$. Then $Q=P\oplus Q_1$ where $P\subseteq N$ and $P\cong P_1$.*

Proof. Let $g:Q\to Q/M$ be the natural map. It is clear that $g_1=g\mid N:N\to Q/M$ is onto. This shows that Q/M and hence P_1 must be cyclic. Since Q is R-projective and P_1 is cyclic there exists a homomorphism $h:Q\to P_1$ such that $f\circ h=g$. Let $h_1=h\mid N$. Then $f\circ h_1=g_1$. Since g_1 is onto and Ker(f) is small in P_1, h_1 splits i.e. there is $j:P_1\to N$ such that $h_1\circ j=1_{P_1}$. Take $P=j(P_1)$, $Q_1=\text{Ker}(h)$.

Now we state the main result

THEOREM 1. *Let R be a semiperfect ring. Let Q be a left R-module satisfying* (a) *Q is R-projective and* (b) *$J(Q)$ is small in Q. Then Q is projective.*

It is known that a semiperfect ring R satisfies a.c.c. on left ideals which are direct summands of R ([3] Theorem 4.3). Hence Theorem 1 will follow immediately from

THEOREM 2. *Let R be a ring satisfying* a.c.c. *on left ideals which are direct summands of R. Let Q be a left R-module satisfying* (1) *every finitely generated*

Received by the editors March 1, 1978 and in revised form May 14, 1979 and March 7, 1980.

https://doi.org/10.4153/CMB-1981-055-x Published online by Cambridge University Press

factor module of Q has a projective cover, (2) *Q is R-projective and* (3) *$J(Q)$ is small in Q. Then Q is a direct sum of cyclic indecomposable projective modules.*

Let $x \in Q$, $x \notin J(Q)$. Then there is a maximal submodule M of Q such that $x \notin M$. Then $Q = Rx + M$. By condition (1), Q/M has a projective cover. Since Q/M is simple, this projective cover is cyclic indecomposable. By the above lemma we can write $Q = P \oplus Q_1$ where $P \subseteq Rx$ and P is a cyclic indecomposable projective module. Then $Rx = Ry_1 \oplus Rx_1$ where $x = y_1 + x_1$, $P = Ry_1$, $Rx_1 = Rx \cap Q_1$. It can be easily checked that Q_1 also satisfies the conditions (1), (2), and (3). Now if $x_1 \notin J(Q_1)$ we can repeat the above process to write $Q_1 = Ry_2 \oplus Q_2$, Ry_2 cyclic indecomposable projective direct summand of Q contained in Rx_1, $x_1 = y_2 + x$, such that $Rx_1 = Ry_2 \oplus Rx_2$ where $Rx_2 = Rx_1 \cap Q_2$. We claim that this process can be repeated only for finitely many times. For otherwise, we obtain an infinite direct sum $Ry_1 \oplus Ry_2 \oplus \cdots \oplus Ry_n \oplus \cdots$ inside Rx such that for each n, $Ry_1 + Ry_2 + \cdots + Ry_n$ is cyclic projective generated by $y_1 + y_2 + \cdots + y_n$. Let $g_n : R \to R(y_1 + \cdots + y_n)$ be the maps defined by $g_n(1) = y_1 + \cdots + y_n$. These maps split and $\text{Ker}(g_n) = \text{Ann}_R(y_1 + \cdots + y_n)$. Therefore, $\text{Ker}(g_1) \supseteq \text{Ker}(g_2) \supseteq \cdots \supseteq \text{Ker}(g_n) \supseteq \cdots$ form a decreasing sequence of summands of R. Hence we can get an increasing sequence $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq \cdots$ of summands of R such that $L_n \cong R(y_1 + \cdots + y_n)$. By a.c.c. on these summands, $L_n = L_{n+1}$ for some n. Hence $Ry_1 \oplus \cdots \oplus Ry_n \cong Ry_1 \oplus \cdots \oplus Ry_{n+1}$. But this cannot happen since each Ry_j is a non-zero indecomposable module. This proves our claim. Now let

$$A = \{y \mid y \in Q,\ y \neq 0,\ Ry \text{ is cyclic indecomposable projective direct summand of } Q\}.$$

Then the preceding arguments together with the fact that $J(Q)$ is small in Q show that $Q = \sum_{y \in A} Ry$. Let $\mathscr{A}$ be the family of subsets B of A satisfying the conditions: (a) $\sum_{y \in B} Ry$ is a direct sum and (b) for $y_1, \ldots, y_n \in B$, $Ry_1 + \cdots + Ry_n$ is a direct summand of Q. Clearly $\mathscr{A}$ is non-empty and Zorn's lemma is applicable (where the partial order in $\mathscr{A}$ is given by the usual inclusion relation). Let B_0 be a maximal element in $\mathscr{A}$. Then $P = \sum_{y \in B_0} Ry = \oplus \sum_{y \in B_0} Ry$ is projective. We *claim that* $P = Q$. For this it is sufficient to prove that $A \subseteq P + J(Q)$ since $Q = \sum_{y \in A} Ry$ and $J(Q)$ is small in Q. Let $y \in A$. We consider two cases:

CASE 1. $P \cap Ry = 0$.

Then $B_0 \subsetneqq B_0 \cup \{y\} \subseteq A$. By maximality of B_0 we can find $y_1, \ldots, y_n$ in B_0 such that $Ry_1 \oplus \cdots \oplus Ry_n \oplus Ry$ is not a direct summand of Q. By condition (b) on B_0, we can write $Q = (Ry_1 \oplus \cdots \oplus Ry_n) \oplus Q_1$. Then $Ry_1 \oplus \cdots \oplus Ry_n \oplus Ry = (Ry_1 \oplus \cdots \oplus Ry_n) \oplus ((Ry_1 \oplus \cdots \oplus Ry_n \oplus Ry) \cap Q_1)$. This implies $(Ry_1 \oplus \cdots \oplus Ry_n \oplus Ry) \cap Q_1 \cong Ry$. Let $(Ry_1 \oplus \cdots \oplus Ry_n \oplus Ry) \cap Q_1 = Rz$. Then Rz is cyclic

https://doi.org/10.4153/CMB-1981-055-x Published online by Cambridge University Press

indecomposable submodule of Q_1 and Rz cannot be a direct summand of Q_1. Hence it is clear from the previous arguments that $z \in J(Q_1) \subseteq J(Q)$. It follows that $y \in P + J(Q)$.

CASE 2. $P \cap Ry \neq 0$.

If $y \in P$ we are through. Assume that $y \notin P$. Let $0 \neq sy = x \in P \cap Ry$. Since Ry is non-zero projective, Ann(y) is a direct summand of R. Let Ann(y) $= Rt$. Choose a finite subset $B \subseteq B_0$ such that $x \in \sum_{z \in B} Rz$. Then $\sum_{z \in B} Rz$ is a direct summand of Q. Let $h: Q \to \sum_{z \in B} Rz$ be the natural projection. Let $y' = h(y)$. Then $t(y-y') = 0$. We have also that $s(y-y') = 0$ since $sy' = sh(y) = h(sy) = h(x) = x = sy$. Thus $\text{Ann}(y) \subsetneq \text{Ann}(y-y')$. We claim that $R(y-y')$ does not contain any non-zero projective summand. If possible, let N be such a summand of $R(y-y')$. Since $\text{Ann}(y) \subseteq \text{Ann}(y-y')$, $y \to (y-y')$ defines an epimorphism $f: Ry \to R(y-y')$. Let $g: R(y-y') \to N$ be the natural projection map. Then $g \circ f: Ry \to N$ is an epimorphism which splits. Since Ry is indecomposable this means that $g \circ f$ is an isomorphism. This would imply $\text{Ann}(y-y') \subseteq \text{Ann}(y)$, a contradiction. This proves our claim. It follows that $y - y' \in J(Q)$. Hence $y \in P + J(Q)$. This completes the proof of Theorem 2.

Note. A ring is called left perfect if every left R-module has a projective cover. It is well known that the radical of every left module over a left perfect ring is small. Hence from Theorem 1 and the Proof of Theorem 2 we get

COROLLARY 1. (Sandomierski [4]). *Any R-projective left R-module over a left perfect ring R is projective.*

COROLLARY 2. (H. Bass [2]). *Let P be a projective left R-module over a left perfect ring R. Then P is a direct sum of cyclic indecomposable modules.*

The authors are thankful to the referee for his helpful criticisms which improved the proof of Theorem 2.

REFERENCES

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*. Springer-Verlag (1974).

2. H. Bass, *Finitistic Dimension and a Homological Generalization of Semiprimary Rings*. Trans. Ann. Math. Soc. **95** (1960) 466–488.

3. W. K. Nicholson, *I-Rings*, Trans. Amer. Math. Soc. **207** (1975), 361–373.

4. P. L. Sandomierski, Ph.D. Thesis, Penn State University (1974).

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF BOMBAY,
BOMBAY 400 098.