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NATURAL PARTIAL ORDER IN SEMIGROUPS OF TRANSFORMATIONS WITH INVARIANT SET

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Abstract

Let $\mathcal{T}_X$ be the full transformation semigroup on the nonempty set X. We fix a nonempty subset Y of X and consider the semigroup

$$S(X, Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}$$

of transformations that leave Y invariant, and endow it with the so-called natural partial order. Under this partial order, we determine when two elements of $S(X, Y)$ are related, find the elements which are compatible and describe the maximal elements, the minimal elements and the greatest lower bound of two elements. Also, we show that the semigroup $S(X, Y)$ is abundant.

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1. Introduction

Let X be a nonempty set. The semigroup $\mathcal{T}_X$ of full transformations on X consists of the maps from X to X with composition as the semigroup operation. Let $f, g \in \mathcal{T}_X$ and denote by fg the map obtained by performing first g and then f. For $Y \subseteq X$, the semigroup

$$S(X, Y) = \{f \in \mathcal{T}_X : f(Y) \subseteq Y\}$$

of transformations that leave a subset Y invariant is a subsemigroup of $\mathcal{T}_X$. It contains the identity map id_X on X. If $Y = X$, then $S(X, Y) = \mathcal{T}_X$. So we may regard it as a generalisation of $\mathcal{T}_X$. It was investigated in [3, 5, 7, 12]. For example, Symons [12] described the automorphism group of this semigroup. Honyam and Sanwong [3] determined when $S(X, Y)$ is isomorphic to $T(Z)$ for some set Z and proved that every semigroup A can be embedded in $S(A^1, A)$. They also described Green's relations of $S(X, Y)$, its group $\mathcal{H}$-classes, and its ideals.

A semigroup S is regular if for each $a \in S$, $a = axa$ for some $x \in S$. The following result was proved in [7].

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LEMMA 1.1. *The semigroup $S(X, Y)$ is regular if and only if $Y = X$ or $|Y| = 1$.*

In this paper the set X is finite ($|X| \geq 3$) or infinite and the subset $|Y| \geq 2$. We endow $S(X, Y)$ with the so-called natural partial order [6], that is, for $f, g \in S(X, Y)$ $(= S^1(X, Y))$,

$$f \leq g \quad \text{if and only if } f = kg = gh \text{ and } f = kf \text{ for some } k, h \in S(X, Y),$$

and investigate the partial order on $S(X, Y)$. With respect to this partial order, we determine when two elements of $S(X, Y)$ are related, find the elements which are compatible and describe the maximal elements, the minimal elements and the greatest lower bound of two elements. For the study of natural order on transformation semigroups, one may see [4, 8, 10, 11]. Also, we describe the abundance of $S(X, Y)$.

Now we recall some notation which will be useful later. Let $f(X)$ be the image of f. Denote by $\pi(f)$ the partition of X induced by $f \in \mathcal{T}_X$, namely, $\pi(f) = \{f^{-1}(y), y \in X\}$ and call $f^{-1}(y)$ a *ker-class* of f. Also, $\pi_Y(f) = \{f^{-1}(y), y \in Y\}$ and $\pi_Y(f) \subseteq \pi(f)$.

Let $\mathcal{A}, \mathcal{B}$ be two collections of subsets of X. If, for each $A \in \mathcal{A}$, there exists some $B \in \mathcal{B}$ such that $A \subseteq B$, then $\mathcal{A}$ is said to *refine* $\mathcal{B}$.

2. Characterisation of ≤ and compatible elements

In this section, we give a characterisation of this partial order $\leq$ and then find the compatible elements.

THEOREM 2.1. *Let $f, g \in S(X, Y)$. Then $f \leq g$ if and only if the following statements hold:*

(1) *$\pi(g)$ refines $\pi(f)$ and $\pi_Y(g)$ refines $\pi_Y(f)$;*
(2) *if $g(x) \in f(X)$ for some $x \in X$, then $f(x) = g(x)$;*
(3) *$f(X) \subseteq g(X)$ and $f(Y) \subseteq g(Y)$.*

PROOF. Suppose that $f \leq g$. Then there exist some $k, h \in S(X, Y)$ such that

$$f = kg = gh, \quad f = kf.$$

It follows from $f = kg$ that $\pi(g)$ refines $\pi(f)$ and $\pi_Y(g)$ refines $\pi_Y(f)$. Now if $g(x) \in f(X)$ for some $x \in X$, then there exists some $y \in X$ such that $g(x) = f(y)$. So

$$f(x) = kg(x) = kf(y) = f(y) = g(x)$$

and (2) holds. Since $f = gh$, we immediately have $f(X) \subseteq g(X)$ and $f(Y) \subseteq g(Y)$ and so (3) holds.

To show the sufficiency, we assume the conditions hold and define $k, h \in S(X, Y)$ such that

$$f = kg = gh, \quad f = kf.$$

First we define k on X. For each $x \in g(X)$, there exists some $z \in X$ such that $x = g(z)$. Define $k : X \to X$ by

$$k(x) = \begin{cases} f(z) & \text{if } x \in g(X), \\ g(x) & \text{otherwise.} \end{cases}$$

If $x = g(z) = g(z')$ for some distinct $z, z' \in X$, then $f(z) = f(z')$ since $\pi(g)$ refines $\pi(f)$. So k is well defined. We now show $k \in S(X, Y)$. For each $y \in Y$, we have $y \notin g(X)$ or $y \in g(X)$. If $y \notin g(X)$, then $k(y) = g(y) \in Y$. If $y \in g(X)$, then there exists some $x \in X$ such that $y = g(x)$. Noting that $\pi_Y(g)$ refines $\pi_Y(f)$, we have $x \in g^{-1}(y) \subseteq f^{-1}(y')$ for some $y' \in Y$ and $k(y) = f(x) = y' \in Y$. Thus, $k \in S(X, Y)$. It is clear that $f = kg$. To see $f = kf$, note that for each $x \in X$, $f(x) = g(x')$ for some $x' \in X$ (by (3)). It follows from (2) that $f(x') = g(x')$. So

$$kf(x) = kg(x') = f(x') = g(x') = f(x)$$

and $f = kf$ holds.

Finally, we define h on X. By virtue of $f(Y) \subseteq g(Y)$, for each $x \in Y$, there is some $y \in Y$ such that $f(x) = g(y)$. Moreover, by $f(X) \subseteq g(X)$, for each $x \in X - Y$, there is some $z \in X$ such that $f(x) = g(z)$. Define $h : X \to X$ by

$$h(x) = \begin{cases} y & \text{if } x \in Y, \\ z & \text{if } x \in X - Y. \end{cases}$$

It is routine to show $h \in S(X, Y)$ and $f = gh$. □

In Theorem 2.1, if $Y = X$ (in the case $S(X, Y) = \mathcal{T}_X$), then $f \leq g$ if and only if (1) $\pi(g)$ refines $\pi(f)$, (2) if $g(x) \in f(X)$ for some $x \in X$, then $f(x) = g(x)$ and (3) $f(X) \subseteq g(X)$. The result coincides with that in [4, Proposition 2.3].

As a consequence of Theorem 2.1, we have the following corollary whose proof is omitted.

Corollary 2.2. *Let $f, g \in S(X, Y)$ and $f \leq g$. If $g(X) = f(X)$, then $g = f$.*

An element $h \in S(X, Y)$ is said to be *left compatible* with the partial order if $hf \leq hg$ whenever $f \leq g$. We say that h is *strictly left compatible* if $hf < hg$ whenever $f < g$. *Right compatibility* is defined similarly.

In the full transformation semigroup $\mathcal{T}_X$, if h is injective, then h is left compatible, and if h is surjective, then h is right compatible.

It is clear that a constant map in $S(X, Y)$ is left compatible.

Theorem 2.3. *Let $h \in S(X, Y)$. Then h is strictly left compatible if and only if h is an injection.*

Proof. It is routine to verify the sufficiency and we only show the necessity. If h is not an injection, then $h(a) = h(b)$ for some distinct $a, b \in X$. There are two cases to consider.

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Case 1: $a, b \in X - Y$ or $a, b \in Y$. Define $f, g : X \to X$ by

$$f(x) = \begin{cases} b & \text{if } x = a, \\ x & \text{otherwise,} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{otherwise.} \end{cases}$$

Clearly, $f, g \in S(X, Y)$ and $f < g$ by Theorem 2.1. Thus, $hf < hg$ since h is strictly left compatible, and so $hf(X) \subset hg(X)$ by Corollary 2.2 (where $A \subset B$ means that A is a proper subset of B). However, by the definition of f and g, it follows that $hf(X) = hg(X)$, a contradiction.

Case 2: $a \in X - Y$ and $b \in Y$. Define $f : X \to X$ as in Case 1. Then $f \in S(X, Y)$ and $f < \mathrm{id}_X$. Thus, $hf < h\mathrm{id}_X = h$ and $hf(X) \subset h(X)$. By the definition of f,

$$hf(X) = h(X - \{a\}) = h(X),$$

a contradiction.

In either case, a contradiction will arise, so h is an injection and the necessity follows. □

Theorem 2.4. *Let $h \in S(X, Y)$. Then h is right compatible if and only if either of the following statements holds:*

(1) $h(Y) = h(X) = Y$;
(2) $h(Y) = Y$ *and* $h(X) = X$.

Proof. Suppose that h is right compatible. Now we can assert that $h(Y) = Y$. Indeed, if $h(Y) \subset Y$, take $a \in Y - h(Y)$, $b \in X - Y$ and define $g : X \to X$ by

$$g(x) = \begin{cases} a & \text{if } x \in Y, \\ b & \text{otherwise.} \end{cases}$$

Then $g \in S(X, Y)$ and $g \le \mathrm{id}_X$. So $gh \le h$. By Theorem 2.1(3), $\{a\} = gh(Y) \subseteq h(Y)$, that is, $a \in h(Y)$, a contradiction and the assertion follows. It remains to show that $h(X) = Y$ or $h(X) = X$. For each constant map $\langle x \rangle$ with $x \in Y$, we have $\langle x \rangle \le \mathrm{id}_X$. Thus, $\langle x \rangle h \le \mathrm{id}_X h$ and so $\langle x \rangle \le h$. By Theorem 2.1(3), $x \in h(X)$. Hence, $Y \subseteq h(X)$ which implies that $Y = h(X)$ or $Y \subset h(X)$. If $Y \subset h(X)$, then take $c \in X - Y, d \in Y$ and define $k : X \to X$ by

$$k(x) = \begin{cases} c & \text{if } x \in X - Y, \\ d & \text{otherwise.} \end{cases}$$

It is clear that $k \in S(X, Y)$ and $k \le \mathrm{id}_X$. So $kh \le h$. By Theorem 2.1(3), $\{c, d\} = kh(X) \subseteq h(X)$ and so $c \in h(X)$ which implies that $X - Y \subseteq h(X)$ and $h(X) = X$.

Conversely, let $f, g \in S(X, Y)$ and $f \le g$. We now verify that fh, gh satisfy Theorem 2.1(1)–(3) for the first case $h(X) = h(Y) = Y$. If $gh(x) = gh(y)$ for some distinct $x, y \in X$, then $fh(x) = fh(y)$ since $\pi(g)$ refines $\pi(f)$. Moreover, $gh(x) = gh(y) \in Y$ implies that $fh(x) = fh(y) \in Y$ since $\pi_Y(g)$ refines $\pi_Y(f)$. So fh, gh satisfy

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Theorem 2.1(1). Now if $gh(x) \in fh(X)$ for some $x \in X$, then $gh(x) \in f(Y) \subseteq f(X)$; thus, $fh(x) = gh(x)$ and fh, gh satisfy Theorem 2.1(2). Also, fh, gh satisfy Theorem 2.1(3) since $fh(X) = f(Y) \subseteq g(Y) = gh(X)$ and $fh(Y) = f(Y) \subseteq g(Y) = gh(Y)$. So, $fh \leq gh$. Similarly, for the second case $h(Y) = Y$ and $h(X) = X$, we can also deduce that $fh \leq gh$ and the conclusion follows. □

3. Maximal and minimal elements, greatest lower bound of f, g

For $f, g, h \in S(X, Y)$, if $h \leq f$, $h \leq g$, then h is called a *lower bound* of f, g. Denote by $\inf\{f, g\}$ the greatest lower bound of f, g. In this section, we describe the maximal and minimal elements of $S(X, Y)$, then present a condition for the existence of $\inf\{f, g\}$.

Theorem 3.1. *Let $f \in S(X, Y)$. Then f is maximal if and only if either of the following statements holds:*

(1) *f is either surjective or injective;*
(2) *$f|_{X-Y}$ is injective, $f(X - Y) \cap Y = \emptyset$ and $f(Y) = Y$.*

Proof. Let f be maximal. Suppose to the contrary that neither (1) nor (2) holds. There are three cases to consider.

Case 1: $f|_{X-Y}$ is not injective, $f(X - Y) \cap Y = \emptyset$ and $f(Y) = Y$. Let $f(x_1) = f(x_2)$ for some distinct $x_1, x_2 \in X - Y$. Since f is not surjective, we take $a \in X - f(X)$. Then $a \notin Y$. Define $g : X \to X$ by

$$g(x) = \begin{cases} a & \text{if } x = x_1 \\ f(x) & \text{otherwise.} \end{cases}$$

Then $g \in S(X, Y)$ and $f < g$, a contradiction.

Case 2: $f|_{X-Y}$ is injective, $f(X - Y) \cap Y \neq \emptyset$ and $f(Y) = Y$. Let $f(x_1) = f(x_2)$ for some $x_1 \in X - Y$, $x_2 \in Y$. Take $a \in X - Y - f(X)$ and define $g : X \to X$ as in Case 1. Then $g \in S(X, Y)$ and $f < g$, a contradiction.

Case 3: $f|_{X-Y}$ is injective, $f(X - Y) \cap Y = \emptyset$ and $f(Y) \subset Y$. Since f is not injective, we have $f(x_1) = f(x_2)$ for some distinct $x_1, x_2 \in Y$. Take $a \in Y - f(Y)$ and define $g : X \to X$ as in Case 1. Then $g \in S(X, Y)$ and $f < g$, a contradiction.

Therefore, the necessity follows. Let $f \leq g$ for some $g \in S(X, Y)$. Then $f(X) \subseteq g(X)$. If f is surjective, then $f(X) = g(X)$. By Corollary 2.2, $f = g$ and f is maximal. If f is injective, then we claim that $f(X) = g(X)$. Indeed, if $f(X) \subset g(X)$, take $y \in g(X) - f(X)$. Let $g(x) = y$ for some $x \in X$. Then $f(x) = g(x')$ for some $x' \in X$ ($x' \neq x$). It follows that $f(x') = g(x')$ from Theorem 2.1(2) which implies that $f(x) = f(x')$, a contradiction. So $f(X) = g(X)$. Thus, $f = g$ and f is also maximal. Now let f satisfy statement (2). By Theorem 2.1(3), $Y = f(Y) \subseteq g(Y) \subseteq Y$ and $g(Y) = Y$. To see $f(X - Y) = g(X - Y)$, by Theorem 2.1(3) again, $f(X) \subseteq g(X)$. As $f(X - Y) \cap f(Y) = \emptyset$, we have $f(X - Y) \subseteq g(X - Y)$. If $f(X - Y) \subset g(X - Y)$, then $g(x) \in g(X - Y) - f(X - Y)$ for some $x \in X - Y$. Thus, $f(x) = g(y)$ for some $y \in X - Y$. We claim that $x \neq y$. Indeed, if $x = y$, then $g(x) = g(y) = f(x) \in f(X - Y)$, a contradiction. By Theorem 2.1(2), $f(y) = g(y) = f(x)$ which contradicts that $f|_{X-Y}$ is injective. Hence, $f(X - Y) = g(X - Y)$ and so $f(X) = g(X)$. Consequently, $f = g$ and f is maximal. □

From Theorem 3.1, we have the following corollary.

COROLLARY 3.2. *Let Y be finite. Then $f \in S(X, Y)$ is maximal if and only if f is surjective or injective.*

EXAMPLE 3.3. Let $X = \{1, 2, \ldots\}$, $Y = \{3, 6, 9, \ldots\}$ and

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \cdots \\ 4 & 5 & 3 & 7 & 8 & 3 & 10 & 11 & 6 & 13 & 14 & 6 & 16 & 17 & 9 & 19 & 20 & 9 & \cdots \end{pmatrix}.$$

One can easily verify that $f \in S(X, Y)$ is neither surjective nor injective and satisfies Theorem 3.1(2), so f is a maximal element.

THEOREM 3.4. *Let $f \in S(X, Y)$. Then f is minimal if and only if f is a constant map.*

PROOF. The sufficiency is clear, so we only show the necessity. If f is not a constant map, then $|f(X)| \geq 2$. Take $a \in f(X) \cap Y$ and define $g(x) = a$ for each $x \in X$. Clearly, $g \in S(X, Y)$ and $g < f$, which leads to a contradiction. □

According to Theorems 3.1 and 3.4, we know that if $Y = X$ ($S(X, Y) = \mathcal{T}_X$), then f is maximal if and only if f is either surjective or injective, and that f is minimal if and only if f is a constant map, which was proved in [4, Theorem 3.1].

We now pay attention to the existence of $\inf\{f, g\}$. If f, g are constant maps, then $\inf\{f, g\}$ does not exist. If f is a constant map and f, g are not comparable, then $\inf\{f, g\}$ also does not exist. If f is a constant map and f, g are comparable, then $\inf\{f, g\} = f$. In what follows, we assume that f, g are not constant maps.

The subset X' of X is said to be *complete* with respect to $f, g \in S(X, Y)$ if it is both a union of ker-classes of f and a union of ker-classes of g. In general, the subset $f^{-1}(z) \cup g^{-1}(z)$ is not complete for $z \in f(X) \cap g(X)$. Denote by $\mathcal{K}_z(f, g)$ the smallest complete subset of X that contains $f^{-1}(z) \cup g^{-1}(z)$. (It is easy to show that such a subset exists.) The following example shows that, generally speaking, $\mathcal{K}_z(f, g) \cap \mathcal{K}_u(f, g) \neq \emptyset$ for all distinct $z, u \in f(X) \cap g(X)$.

EXAMPLE 3.5. Let $X = \{1, 2, \ldots, 9\}$, $Y = \{3, 6, 9\}$ and

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 6 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 6 & 4 & 4 & 6 & 4 & 4 & 3 \end{pmatrix}.$$

Then $f, g \in S(X, Y)$ and

$$f(X) \cap g(X) = \{1, 3, 4, 6\}.$$

Obviously, $\mathcal{K}_3(f, g) = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $\mathcal{K}_4(f, g) = \{2, 3, 4, 5, 6, 7, 8\}$. Thus,

$$\mathcal{K}_3(f, g) \cap \mathcal{K}_4(f, g) = \{2, 3, 4, 5, 6, 7, 8\}.$$

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A necessary condition for the existence of $\inf\{f, g\}$ is given in the following lemma.

LEMMA 3.6. *Let $f, g \in S(X, Y)$. Write $Z = f(X) \cap g(X) \neq \emptyset$ and $W = f(Y) \cap g(Y) \neq \emptyset$ with $|W| \geq 2$. If $h = \inf\{f, g\}$ exists, then the following statements hold:*

(1) *$h(X) = f(X) \cap g(X)$ and $h(Y) = f(Y) \cap g(Y)$;*

(2) *if $f(x), g(x) \in Z$ for some $x \in X$, then $f(x) = g(x)$;*

(3) *$f^{-1}(z) \cup g^{-1}(z) \subseteq P \in \pi(h)$ for each $z \in Z$ and $h(f^{-1}(z) \cup g^{-1}(z)) = z$; consequently, $h(\mathcal{K}_z(f, g)) = z$ for each $z \in Z$;*

(4) *for all distinct $z, u \in Z$, $f^{-1}(z)$ and $f^{-1}(u)$ are not contained in the same ker-class of h; similarly, $g^{-1}(z)$ and $g^{-1}(u)$ are also not contained in the same ker-class of h;*

(5) *for all distinct $z, u \in Z$, $f^{-1}(z) \cup g^{-1}(z)$ and $f^{-1}(u) \cup g^{-1}(u)$ are not contained in the same ker-class of h.*

PROOF. (1) By Theorem 2.1(3), $h(X) \subseteq f(X)$ and $h(X) \subseteq g(X)$. So $h(X) \subseteq f(X) \cap g(X)$. Similarly, $h(Y) \subseteq f(Y) \cap g(Y)$. To see $f(Y) \cap g(Y) \subseteq h(Y)$, take $a \in f(Y) \cap g(Y)$ and define $h_a(x) = a$ for each $x \in X$. Clearly, $h_a \in S(X, Y)$. Then $h_a \leq f$, $h_a \leq g$ and $h_a \leq h$ which implies that $\{a\} \subseteq h(Y)$, that is, $a \in h(Y)$. Since $a \in f(Y) \cap g(Y)$ is arbitrary, we have $f(Y) \cap g(Y) \subseteq h(Y)$. Therefore, $h(Y) = f(Y) \cap g(Y)$ and $h(\mathcal{K}_w(f, g)) = w$ for each $w \in W$ (by Theorem 2.1(2)). Now we show that $f(X) \cap g(X) \subseteq h(X)$. There are two cases to consider.

Case 1: $f(Y) \cap g(Y) = f(X) \cap g(X)$. Of course, $f(X) \cap g(X) = h(X)$.

Case 2: $f(Y) \cap g(Y) \subset f(X) \cap g(X)$. Take $a \in f(X) \cap g(X) - f(Y) \cap g(Y)$, $b \in f(Y) \cap g(Y)$ and define $h_{a,b} : X \to X$ by

$$h_{a,b}(x) = \begin{cases} a & \text{if } x \in \mathcal{K}_a(f, g), \\ b & \text{otherwise.} \end{cases}$$

To see $h_{a,b} \in S(X, Y)$, we need show that $\mathcal{K}_a(f, g) \cap Y = \emptyset$. Indeed, if $\mathcal{K}_a(f, g) \cap Y \neq \emptyset$, then $h(\mathcal{K}_a(f, g)) \in Y$. It follows from $h(Y) = f(Y) \cap g(Y)$ that $h(\mathcal{K}_a(f, g)) \in f(Y) \cap g(Y)$. Then there exists some $w \in f(Y) \cap g(Y)$ such that $h(\mathcal{K}_w(f, g)) = h(\mathcal{K}_a(f, g)) = w$. Take $w' \in f(Y) \cap g(Y) (w' \neq w)$ and define $h' : X \to X$ by

$$h'(x) = \begin{cases} w' & \text{if } x \in \mathcal{K}_a(f, g), \\ h(x) & \text{otherwise.} \end{cases}$$

Then $h' \in S(X, Y)$ and $h' \neq h$. Observing that,

$$h(\mathcal{K}_w(f, g) \cup \mathcal{K}_a(f, g)) = w, \quad h(\mathcal{K}_{w'}(f, g)) = w',$$
$$h'(\mathcal{K}_w(f, g)) = w, \quad h'(\mathcal{K}_{w'}(f, g) \cup \mathcal{K}_a(f, g)) = w',$$

we have $h'(X) = h(X)$. We now show that $h' \leq f$. Since $\mathcal{K}_a(f, g), \mathcal{K}_w(f, g), \mathcal{K}_{w'}(f, g)$ are all a union of ker-classes of f, we know that $\pi(f)$ refines $\pi(h')$. To see that $\pi_Y(f)$ refines $\pi_Y(h')$, let $f(P) \in Y$. If $P \subseteq \mathcal{K}_a(f, g)$, then $h'(P) = w' \in Y$.

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If $P \cap \mathcal{K}_a(f, g) = \emptyset$, then $h'(P) = h(P) \in Y$ which implies $\pi_Y(f)$ refines $\pi_Y(h')$. So h', f satisfy Theorem 2.1(1). Let $f(x) \in h'(X) = h(X)$ for some $x \in X$. If $f(x) = w$, that is, $x \in f^{-1}(w) \subseteq \mathcal{K}_w(f, g)$, then $h'(x) = h(x) = f(x)$. If $f(x) \in h'(X) - \{w\}$, that is, $x \notin f^{-1}(w)$, then we assert that $x \notin \mathcal{K}_a(f, g)$. Indeed, if $x \in \mathcal{K}_a(f, g)$, then $h(x) = f(x) = w$, a contradiction, which means that $x \notin \mathcal{K}_a(f, g)$. Thus, $h'(x) = h(x) = f(x)$. So h', f satisfy Theorem 2.1(2). Moreover, $h'(Y) = h(Y) \subseteq f(Y)$. Hence, $h' \leq f$. Similarly, $h' \leq g$. Thus, $h' \leq h$. However, $h'(X) = h(X)$, by Corollary 2.2, $h' = h$, a contradiction. Therefore, we deduce that $h_{a,b} \in S(X, Y)$. It is routine to verify that $h_{a,b} \leq f$ and $h_{a,b} \leq g$. So $h_{a,b} \leq h$ and

$$\{a, b\} = h_{a,b}(X) \subseteq h(X),$$

that is, $a \in h(X) - h(Y)$. Noting that $a \in f(X) \cap g(X) - f(Y) \cap g(Y)$ is arbitrary, we have $f(X) \cap g(X) - f(Y) \cap g(Y) \subseteq h(X) - h(Y)$ and $f(X) \cap g(X) \subseteq h(X)$.

Consequently, $h(X) = f(X) \cap g(X)$ and $h(Y) = f(Y) \cap g(Y)$.

(2) According to (1), we have $f(x), g(x) \in h(X)$. By Theorem 2.1(2), $h(x) = f(x) = g(x)$.

(3) This follows from (1) and Theorem 2.1(2).

(4) By (3), $h(f^{-1}(z)) = z$ and $h(f^{-1}(u)) = u$, so $f^{-1}(z) \subseteq h^{-1}(z)$ and $f^{-1}(u) \subseteq h^{-1}(u)$. The argument for g is the same.

(5) This follows from (4). □

We now present a necessary and sufficient condition for the existence of $\inf\{f, g\}$.

Theorem 3.7. *Let $f, g \in S(X, Y)$. Write $Z = f(X) \cap g(X) \neq \emptyset$ and $W = f(Y) \cap g(Y) \neq \emptyset$ with $|W| \geq 2$. Then $h = \inf\{f, g\}$ exists if and only if the following statements hold:*

(1) $\bigcup \mathcal{K}_{z \in Z}(f, g) = X$;
(2) $\mathcal{K}_z(f, g) \cap \mathcal{K}_u(f, g) = \emptyset$ *for all distinct* $z, u \in Z$;
(3) $Y \subseteq \bigcup \mathcal{K}_{w \in W}(f, g)$;
(4) *for* $P \in \pi_Y(f)$ *or* $P \in \pi_Y(g)$, *if* $P \subseteq \mathcal{K}_w(f, g)$ *for some* $w \in Z$, *then* $w \in W$;
(5) *if* $f(x), g(x) \in Z$ *for some* $x \in X$, *then* $f(x) = g(x)$.

Proof. Suppose that $h = \inf\{f, g\}$ exists.

(1) If $M = X - \bigcup \mathcal{K}_{z \in Z}(f, g) \neq \emptyset$, then by virtue of Lemma 3.6(1), $h(M) \subseteq f(X) \cap g(X)$. There are two cases to consider.

Case 1: $|h(M - Y)| = 1$. Say $h(M - Y) = \{z\}$. Take $w \in W (w \neq z)$ and define $h' : X \to X$ by

$$h'(x) = \begin{cases} w & \text{if } x \in M - Y, \\ h(x) & \text{otherwise.} \end{cases}$$

Then $h' \in S(X, Y)$ and $h' \neq h$. Noting that

$$h(\mathcal{K}_z(f, g) \cup (M - Y)) = z, \quad h'(\mathcal{K}_z(f, g)) = z \quad \text{and} \quad h'(M - Y) = w \in W \subseteq Z,$$

we have $h'(X) = h(X)$. We now show that $h' \leq f$. Since $\pi(f)$ refines $\pi(h)$ and $\mathcal{K}_z(f, g) \cup (M - Y) \in \pi(h)$, we know that $M - Y$ is also a union of ker-classes of f.

By the definition of h', we have $\mathcal{K}_z(f,g), \mathcal{K}_w(f,g) \cup (M-Y) \in \pi(h')$. Thus, $\pi(f)$ refines $\pi(h')$. To see that $\pi_Y(f)$ refines $\pi_Y(h')$, let $f(P) \in Y$, that is, $P \in f^{-1}(Y)$. If $P \subseteq M-Y$, then $h'(P) = w \in Y$. If $P \cap (M-Y) = \emptyset$, then $h'(P) = h(P) \in Y$ which implies that $\pi_Y(f)$ refines $\pi_Y(h')$. So h', f satisfy Theorem 2.1(1). Let $f(x) \in h'(X) = f(X) \cap g(X)$ for some $x \in X$; then $x \in f^{-1}(u)$ for some $u \in Z$. By Lemma 3.6(3), $h(x) = f(x)$. So $h'(x) = h(x) = f(x)$ and h', f satisfy Theorem 2.1(2). Moreover, $h'(Y) = h(Y) \subseteq f(Y)$. Hence, $h' \leq f$. Similarly, $h' \leq g$. Thus, $h' \leq h$. However, $h'(X) = h(X)$, by Corollary 2.2, $h' = h$, a contradiction.

Case 2: $|h(M-Y)| \geq 2$. Take some distinct $z, u \in h(M-Y)$. Then $z, u \in Z$. Let

$$M_1 = \{x \in M-Y : h(x) = z\}, M_2 = \{x \in M-Y : h(x) = u\}.$$

If $z, u \in Z - W$ or $z, u \in W$, then define $h' : X \to X$ by

$$h'(x) = \begin{cases} u & \text{if } x \in M_1, \\ z & \text{if } x \in M_2, \\ h(x) & \text{otherwise.} \end{cases}$$

If $z \in W$, $u \in Z - W$, then take $w \in W (w \neq z)$ and define $h' : X \to X$ by

$$h'(x) = \begin{cases} w & \text{if } x \in M_1, \\ h(x) & \text{otherwise,} \end{cases}$$

which also leads to a contradiction.

(2) If $\mathcal{K}_z(f,g) \cap \mathcal{K}_u(f,g) \neq \emptyset$ for some distinct $z, u \in Z$, then, by Lemma 3.6(3),

$$z = h(\mathcal{K}_z(f,g)) = h(\mathcal{K}_u(f,g)) = u,$$

a contradiction. Therefore, $\mathcal{K}_z(f,g) \cap \mathcal{K}_u(f,g) = \emptyset$.

(3) There are two cases to consider.

Case 1: $W = f(Y) \cap g(Y) = f(X) \cap g(X) = Z$. Then by (1), $\bigcup \mathcal{K}_{z\in Z}(f,g) = X$. So $Y \subseteq X = \bigcup \mathcal{K}_{z\in Z}(f,g) = \bigcup \mathcal{K}_{z\in W}(f,g)$.

Case 2: $W = f(Y) \cap g(Y) \subset f(X) \cap g(X) = Z$. For each $x \in Y$, by (1), $x \in \mathcal{K}_w(f,g)$ for some $w \in Z$. We claim that $w \in W$. Indeed, if $w \in Z - W$, then

$$h(x) = h(\mathcal{K}_w(f,g)) = w \in Z - W \subseteq X - Y$$

which implies that $h \notin S(X,Y)$, a contradiction. Hence, $Y \subseteq \bigcup \mathcal{K}_{w\in W}(f,g)$.

(4) Since $\pi_Y(f)$ refines $\pi_Y(h)$, we have $h(P) = h(\mathcal{K}_w(f,g)) = w \in Y$. Then $w \in W$. The argument for g is the same.

(5) This follows from Lemma 3.6(2).

Conversely, suppose that (1)–(5) hold. Define $h : X \to X$ by $h(x) = z$ for each $x \in \mathcal{K}_z(f,g)$ and $z \in Z$. Since

$$\mathcal{K}_z(f,g) \cap \mathcal{K}_u(f,g) = \emptyset \quad \text{for all distinct } z, u \in Z,$$

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we see that h is well defined. Also note that $h(X) = f(X) \cap g(X)$ and $h(Y) = f(Y) \cap g(Y)$. By (3), we know that $h \in S(X, Y)$. Now we verify $h \leq f$. Let $P \in \pi(f)$; then, by (1), $P \subseteq \mathcal{K}_z(f, g)$ for some $z \in Z$. By the definition of h, $\mathcal{K}_z(f, g)$ is mapped to z by h. So $\pi(f)$ refines $\pi(h)$. Let $f(P') \in Y$, namely, $P' \in f^{-1}(Y)$. Then by (1) and (4), $P' \subseteq \mathcal{K}_w(f, g)$ for some $w \in W$. So $h(\mathcal{K}_w(f, g)) = h(P') = w \in Y$ which implies that $\pi_Y(f)$ refines $\pi_Y(h)$ and h, f satisfy Theorem 2.1(1). Let $z = f(x) \in h(X) = f(X) \cap g(X)$ for some $x \in X$, then $x \in f^{-1}(z) \subseteq \mathcal{K}_z(f, g)$ and $h(x) = h(\mathcal{K}_z(f, g)) = z = f(x)$ which means that h, f satisfy Theorem 2.1(2). As seen above, h, f satisfy Theorem 2.1(3). Hence, $h \leq f$. Similarly, $h \leq g$. Therefore, $h \leq \inf\{f, g\}$. It follows from $h(X) = f(X) \cap g(X) = \inf\{f, g\}(X)$ and Corollary 2.2 that $h = \inf\{f, g\}$. □

We allow Y to be X and have the following corollary by a modification in Lemma 3.6 and Theorem 3.7.

Corollary 3.8. *Let $f, g \in \mathcal{T}_X$ and $Z = f(X) \cap g(X) \neq \emptyset$ ($|Z| \geq 2$). Then $h = \inf\{f, g\}$ exists if and only if the following statements hold:*

(1) $\bigcup \mathcal{K}_{z\in Z}(f, g) = X$;
(2) $\mathcal{K}_z(f, g) \cap \mathcal{K}_u(f, g) = \emptyset$ *for all distinct* $z, u \in Z$;
(3) *if* $f(x), g(x) \in Z$ *for some* $x \in X$, *then* $f(x) = g(x)$.

Example 3.9. Let $X = \{1, 2, \ldots, 24\}$ and $Y = \{6, 12, 18, 24\}$. Choose $f, g \in S(X, Y)$ to be

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 1 & 2 & 3 & 3 & 3 & 12 & 5 & 5 & 9 & 8 & 6 & 12 & 13 & 13 & 15 & 15 & 15 & 18 & 17 & 17 & 19 & 19 & 19 & 24 \end{pmatrix},$$

and

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 1 & 4 & 2 & 2 & 4 & 12 & 7 & 7 & 5 & 7 & 14 & 12 & 12 & 14 & 14 & 14 & 16 & 18 & 20 & 20 & 17 & 17 & 20 & 12 \end{pmatrix}.$$

Obviously, $f, g \in S(X, Y)$ and

$$f(X) \cap g(X) = \{1, 2, 5, 12, 17, 18\}, \quad f(Y) \cap g(Y) = \{12, 18\}.$$

Also

$$\begin{array}{ll} f^{-1}(1) \cup g^{-1}(1) = \{1\}, & \text{complete} \\ f^{-1}(2) \cup g^{-1}(2) = \{2, 3, 4\}, & \text{not complete} \\ f^{-1}(5) \cup g^{-1}(5) = \{7, 8, 9\}, & \text{not complete} \\ f^{-1}(12) \cup g^{-1}(12) = \{6, 12, 13, 24\}, & \text{not complete} \\ f^{-1}(17) \cup g^{-1}(17) = \{19, 20, 21, 22\}, & \text{not complete} \\ f^{-1}(18) \cup g^{-1}(18) = \{18\}, & \text{complete.} \end{array}$$

Then

$$\mathcal{K}_2(f, g) = \{2, 3, 4, 5\}, \quad \mathcal{K}_5(f, g) = \{7, 8, 9, 10\},$$
$$\mathcal{K}_{12}(f, g) = \{6, 11, 12, 13, 14, 15, 16, 17, 24\}, \quad \mathcal{K}_{17}(f, g) = \{19, 20, 21, 22, 23\}$$

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and

$$\mathcal{K}_1(f,g)\cup\mathcal{K}_2(f,g)\cup\mathcal{K}_5(f,g)\cup\mathcal{K}_{12}(f,g)\cup\mathcal{K}_{17}(f,g)\cup\mathcal{K}_{18}(f,g)=X,$$
$$Y\subseteq\mathcal{K}_{12}(f,g)\cup\mathcal{K}_{18}(f,g).$$

Let

$$h=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\1&2&2&2&2&12&5&5&5&5&12&12&12&12&12&12&12&18&17&17&17&17&17&12\end{pmatrix}.$$

Then $h=\inf\{f,g\}$.

Remark. In Lemma 3.6 and Theorem 3.7, the condition $|W=f(Y)\cap g(Y)|\geq 2$ cannot be omitted. When $\inf\{f,g\}$ exists and $|W|=1$ (say $W=\{w\}$), we see that $h(X)=f(X)\cap g(X)$, $\bigcup\mathcal{K}_{z\in Z}(f,g)=X$ and $Y\subseteq\mathcal{K}_w(f,g)$ may not be true.

Example 3.10. Let $X=\{1,2,\ldots,24\}$ and $Y=\{6,12,18,24\}$. Choose $f,g\in S(X,Y)$ to be

$$f=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\8&8&8&8&8&6&6&7&7&10&10&12&10&10&7&7&7&12&7&7&7&7&7&12\end{pmatrix},$$

and

$$g=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\7&7&7&7&7&18&7&7&7&9&9&12&10&10&7&7&7&12&7&7&7&7&7&12\end{pmatrix}.$$

Obviously,

$$f(X)\cap g(X)=\{7,10,12\},\qquad f(Y)\cap g(Y)=\{12\}$$

and

$$\mathcal{K}_7(f,g)=\{1,2,3,4,5,6,7,8,9,15,16,17,19,20,21,22,23\}(\cap Y=\{6\}),$$
$$\mathcal{K}_{10}(f,g)=\{10,11,13,14\},\qquad \mathcal{K}_{12}(f,g)=\{12,18,24\}.$$

Let

$$h=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\12&12&12&12&12&12&12&12&12&10&10&12&10&10&12&12&12&12&12&12&12&12&12&12\end{pmatrix}.$$

Then $h=\inf\{f,g\}$. However,

$$h(X)=\{10,12\}\subset f(X)\cap g(X)\quad\text{and}\quad\mathcal{K}_{12}(f,g)=\{12,18,24\}\subset Y.$$

Choose $f',g'\in S(X,Y)$ to be

$$f'=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\8&8&8&8&8&12&7&7&7&10&10&12&10&10&7&7&7&12&6&6&6&6&6&12\end{pmatrix},$$

and

$$g'=\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15&16&17&18&19&20&21&22&23&24\\7&7&7&7&7&12&7&7&7&9&9&12&10&10&7&7&7&12&18&18&18&18&18&12\end{pmatrix}.$$

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Obviously,

$$f'(X) \cap g'(X) = \{7, 10, 12\}, \quad f'(Y) \cap g'(Y) = \{12\}$$

and

$$\mathcal{K}_7(f', g') = \{1, 2, 3, 4, 5, 7, 8, 9, 15, 16, 17\}, \quad \mathcal{K}_{10}(f', g') = \{10, 11, 13, 14\},$$
$$\mathcal{K}_{12}(f', g') = \{6, 12, 18, 24\}.$$

Let

$$h' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 7 & 7 & 7 & 7 & 7 & 12 & 7 & 7 & 7 & 10 & 10 & 12 & 10 & 10 & 7 & 7 & 7 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \end{pmatrix}.$$

Then $h' = \inf\{f', g'\}$. However,

$$\mathcal{K}_7(f', g') \cup \mathcal{K}_{10}(f', g') \cup \mathcal{K}_{12}(f', g') = X - \{19, 20, 21, 22, 23\} \subset X.$$

4. Abundant semigroups $S(X, Y)$

Let S be a semigroup and $a, b \in S$. We say that a, b are $\mathcal{L}^*$-related if they are $\mathcal{L}$-related in a semigroup T such that S is a subsemigroup of T. We have the dual definition of the relation $\mathcal{R}^*$. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations. They were studied by Fountain [2] and others. A semigroup S is called *abundant* if any $\mathcal{L}^*$-class and any $\mathcal{R}^*$-class contains an idempotent of S. The word abundant comes from the fact that such a semigroup has a plentiful supply of idempotents. Umar [13] proved that the semigroup S_n^- of nonbijective, order-decreasing transformations on the set $X = \{1 < 2 < \cdots < n\}$ is abundant but not regular. Araujo and Konieczny [1] proved that the semigroup

$$T_E(X, R) = \{f \in \mathcal{T}_X : f(R) \subseteq R \text{ and } (x, y) \in E \Rightarrow (f(x), f(y)) \in E\}$$

where E is an equivalence relation on X and R is a cross-section of the partition X/E induced by E, is abundant if and only if it is regular. Pei and Zhou [9] proved that the semigroup

$$T_E(X) = \{f \in \mathcal{T}_X : (x, y) \in E \Rightarrow (f(x), f(y)) \in E\}$$

is abundant and not regular when the equivalence relation E is simple or 2-bounded. In this section, we prove that each semigroup $S(X, Y)$ is abundant.

The following lemma gives a characterisation of $\mathcal{L}^*$ and $\mathcal{R}^*$ in $S(X, Y)$.

LEMMA 4.1. *Let $f, g \in S(X, Y)$. Then the following statements hold:*

(1) *$(f, g) \in \mathcal{L}^*$ if and only if $\pi(f) = \pi(g)$;*
(2) *$(f, g) \in \mathcal{R}^*$ if and only if $f(X) = g(X)$.*

PROOF. (1) Certainly if $\pi(f) = \pi(g)$, then $(f, g) \in \mathcal{L}(\mathcal{T}_X)$ and so $(f, g) \in \mathcal{L}^*(S(X, Y))$.

Conversely, if $(f, g) \in \mathcal{L}^*$, then for all $h, k \in S(X, Y)$, $fh = fk$ if and only if $gh = gk$. Let $f(x) = f(y)$ for some distinct $x, y \in X$. There are two cases to consider.

Case 1: $x, y \in X - Y$ or $x, y \in Y$. Define $k_{x,y} : X \to X$ by

$$k_{x,y}(z) = \begin{cases} y & \text{if } z = x, \\ z & \text{otherwise.} \end{cases}$$

Clearly, $k_{x,y} \in S(X, Y)$ and $fk_{x,y} = f\mathrm{id}_X$. Then $gk_{x,y} = g\mathrm{id}_X$ and so $g(x) = g(y)$. Thus, $\pi(f)$ refines $\pi(g)$. By symmetry, $\pi(g)$ refines $\pi(f)$. Thus, $\pi(f) = \pi(g)$.

Case 2: $x \in X - Y, y \in Y$. We also define $k_{x,y}$ as in Case 1 and $\pi(f) = \pi(g)$.

(2) If $f(X) = g(X)$, then $(f, g) \in \mathcal{R}(\mathcal{T}_X)$ and so $(f, g) \in \mathcal{R}^*(S(X, Y))$. We now suppose $(f, g) \in \mathcal{R}^*$. Take $x \notin f(X)$ and $y \in Y \cap f(X)$. Define $h_{x,y} : X \to X$ by

$$h_{x,y}(z) = \begin{cases} y & \text{if } z = x, \\ z & \text{otherwise.} \end{cases}$$

Clearly, $h_{x,y} \in S(X, Y)$ and $h_{x,y}f = \mathrm{id}_X f$. So $h_{x,y}g = \mathrm{id}_X g$. We can deduce that $x \notin g(X)$. Indeed, if $g(x') = x$ for some $x' \in X$, then $h_{x,y}g(x') = \mathrm{id}_X g(x')$ and $y = x$, a contradiction. Hence, $g(X) \subseteq f(X)$. Similarly, $f(X) \subseteq g(X)$. Consequently, $f(X) = g(X)$. □

THEOREM 4.2. *Let $f \in S(X, Y)$. Then the following statements hold:*

(1) *$(e, f) \in \mathcal{L}^*$ for some idempotent $e \in S(X, Y)$;*
(2) *$(e', f) \in \mathcal{R}^*$ for some idempotent $e' \in S(X, Y)$.*

Consequently, the semigroup $S(X, Y)$ is abundant.

PROOF. (1) We use the notation

$$f(x) = \begin{pmatrix} X_i & Y_j \\ x_i & y_j \end{pmatrix}$$

to mean that $f \in S(X, Y)$ and take as understood that the subscripts i, j belong to some (unmentioned) index sets I, J, respectively, the abbreviations $\{x_i\}, \{y_j\}$ denote $\{x_i : i \in I\}, \{y_j : j \in J\}$, respectively, and that $Y \subseteq \bigcup Y_j$, $f(X_i) = x_i \in X - Y$, $f(Y_j) = y_j \in Y$, $f^{-1}(x_i) = X_i$, $f^{-1}(y_j) = Y_j$.

Take $a_i \in X_i$ and $b_j \in Y_j \cap Y$ ($b_j \in Y_j$ if $Y_j \cap Y = \emptyset$). Define $e : X \to X$ by

$$e(x) = \begin{cases} a_i & \text{if } x \in X_i, \\ b_j & \text{if } x \in Y_j. \end{cases}$$

Then $e \in S(X, Y)$ is an idempotent and $\pi(e) = \pi(f)$. By Lemma 4.1(1), we have $(e, f) \in \mathcal{L}^*$.

(2) Take $a \in f(X) \cap Y$ and define $e' : X \to X$ by

$$e'(x) = \begin{cases} x & \text{if } x \in f(X), \\ a & \text{otherwise.} \end{cases}$$

Then $e' \in S(X, Y)$ is an idempotent and $e'(X) = f(X)$. By Lemma 4.1(2), we have $(e', f) \in \mathcal{R}^*$. □

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