Can. J. Math., Vol. XXXII, No. 6, 1980, pp. 1306–1310

ARITHMETIC INVARIANTS OF SIMPLICIAL COMPLEXES

M. BROWN AND A. G. WASSERMAN

1. Introduction. What invariants of a finite simplicial complex K can be computed solely from the values $v_0(K), v_1(K), \ldots, v_i(K), \ldots$ where $v_i(K)$ is the number of i-simplexes of K? The Euler chracteristic $\chi(K) = \sum_i (-1)^i v_i(K)$ is a subdivision invariant and a homotopy invariant while the dimension of K is a subdivision invariant and homeomorphism invariant. In [**3**], Wall has shown that the Euler chracteristic is the only linear function to the integers that is a subdivision invariant. In this paper we show that the only subdivision invariants (linear or not) of K are the Euler characteristic and the dimension. More precisely we prove the following theorem.

THEOREM. *Let* $\mathscr{K}$ *be the class of all finite simplicial complexes and* $F: \mathscr{K} \to S$, *a function to a set* S. *If* F *satisfies* i) *and* ii):

i) $v_i(K_i) = v_i(K_2)$ *for all* i *implies* $F(K_1) = F(K_2)$,

ii) K' *is a subdivision of* K *implies* $F(K') = F(K)$

then F *is a function of the Euler characteristic and dimension, i.e., if* $\dim K = \dim L$ *and* $\chi(K) = \chi(L)$, *then* $F(K) = F(L)$.

2. Definitions. We shall use the word complex to denote a finite n-dimensional simplicial complex, where n is held fixed throughout. If Q is a complex, define

$$v(Q) = (v_0(Q), v_1(Q), \ldots, v_n(Q)) \in \mathbf{Z}^{n+1}$$

where $v_i(Q)$ is the number of i-simplexes in Q. If σ is a simplex of Q we denote by Q_σ the *stellar subdivision* of Q along σ, obtained by placing a vertex at the barycenter of σ and constructing all resultant simplexes. (The reader is referred to [**1**], [**2**] for this construction and for such terms as join, link, star, etc.) In particular let $\alpha_i = v(\Delta_i^n) - v(\Delta^n)$ where Δ^n denotes an n-simplex and Δ_i^n denotes the n-simplex Δ^n stellarly subdivided along an i-face Δ^i. It is important to observe that if σ is an i-simplex of Q then $v(Q_\sigma) - v(Q) = \alpha_i$ if and only if link (σ, Q) is an $(n - i - 1)$-simplex. We are thus lead to defining an i-simplex of an n-dimensional complex as *autonomous* if and only if its link is an $(n - i - 1)$-simplex. An n-simplex of K, which has "no link", is for-

Received January 2, 1979 and in revised form January 2, 1980. This research was supported in part by the National Science Foundation.

https://doi.org/10.4153/CJM-1980-100-0 Published online by Cambridge University Press

mally assigned link = 1 = the (−1)-simplex, and is therefore autonomous.

If $v = (v_0, \ldots, v_n) \in \mathbf{Z}^{n+1}$, define

$$\chi(v) = \sum_{i=0}^{n} (-1)^i v_i.$$

Finally, let $\sim$ denote the equivalence relation generated by the relations

$K \sim L$ if $v(K) = v(L)$,

$K \sim L$ if K is a subdivision of L.

3. Construction of autonomous simplexes. Let K be a complex and $\sigma = (a_0 \ldots a_p)$, $(p \geqq 1)$, a simplex of K with link L. As in [**1**], [**2**] we write $K = (\sigma * L) \cup R$. Then K_σ is the stellar subdivision of K along $\sigma = (b * \dot{\sigma} * L) \cup \mathbf{R}$. This can be realized as a two step process. First form $K_1 = K \cup (b * \dot{\sigma} * L)$ which is K with a cone attached to $\dot{\sigma} * L$. Then form $K_\sigma = K_1 - \text{star}(\sigma, K_1)$, i.e., remove from K_1 every simplex of which σ is a face. Observe that

$$\text{link}(ba_0 \ldots a_{p-1}, K_1) = L = \text{link}(\sigma, K_1).$$

Thus if, instead of removing $\text{star}(\sigma, K_1)$ from K_1, we remove star $(b_0a_0 \ldots a_{p-1}, K_1)$ from K_1 then the resulting complex K_σ' has the property that $v(K_\sigma') = v(K_\sigma)$. On the other hand,

$$\text{link}(ba_0 \ldots a_{p-2}, K_1) = (a_{p-1} \cup a_p) * L$$

whereupon

$$\text{link}(ba_0 \ldots a_{p-2}, K_\sigma') = a_p * L.$$

We have proved the following lemma.

LEMMA 1. *Let K be a complex and σ_p a p-simplex of K, $(p \geqq 1)$, with* link L. *Then there is a complex K_σ' and a $(p - 1)$-simplex σ_{p-1}' in K_σ' such that $v(K_\sigma') = v(K_\sigma)$ and* $\text{link}(\sigma_{p-1}', K_\sigma')$ *is (isomorphic to) the cone on L.*

Addendum. If $L = 1$, i.e., σ has no link, then the cone on L is a vertex. If σ_p is autonomous then σ_{p-1}' is autonomous.

LEMMA 2. *Let K be a complex and let $0 \leqq N \leqq n - 1$. Then there is a complex $\hat{K}$ such that*

$$v(\hat{K}) = v(K) + \alpha_n + \ldots + \alpha_{N+1},$$

$\hat{K}$ contains an autonomous N-simplex, and $\hat{K} \sim K$.

Proof. The proof is merely a repeated application of Lemma 1 and its addendum beginning with an n-simplex σ_n of K which is autonomous by

https://doi.org/10.4153/CJM-1980-100-0 Published online by Cambridge University Press

definition. Its link is 1 so we get a complex K_{σ_n}' with an autonomous simplex σ_{n-1}' such that

$$v(K_{\sigma_n}') = v(K_{\sigma_n}) = v(K) + \alpha_n.$$

Apply the lemma now to K_{σ_n}' and σ_{n-1}', etc.

LEMMA 3. *Let L be a complex with an autonomous N-simplex σ_N and let a_N be a positive integer. Then there is a subdivision L^* of L such that*

$$v(L^*) = v(L) + a_N\alpha_N.$$

Proof. If $a_N = 1$ then L^* is the stellar subdivision of L along σ_N. It is easy to see that this new complex again has an autonomous N-simplex. Thus we may perform a_N successive subdivisions along autonomous N-simplexes. (Note: In forming the first stellar subdivision, simplexes of some dimensions other than N may lose their autonomy. Fortunately, we can circumvent that difficulty.)

4. The goal of this section is Lemma 5. As in Section 2,

$$\alpha_i = v(\Delta_i^n) - v(\Delta^n) = v(b * \dot{\Delta}^i * \Delta^{n-i}) - v(\Delta^n).$$

Let us regard $v(\Delta^i)$ as a vector in $\mathbf{Z}^{n+1}$. (e.g.: $\Delta^3 = [4, 6, 4, 1, 00 \ldots 0]^{\text{tr}}$).

LEMMA 4. *For $1 \leqq i \leqq n$,*

$$\alpha_i = iv(\Delta^n) - \binom{i+1}{2}v(\Delta^{n-1}) + \ldots + (-1)^i\binom{i+1}{i+1}v(\Delta^{n-i}).$$

Proof. $\dot{\Delta}^i$ is the boundary of Δ^i which is the union of $(i+1)$ $(i-1)$-simplexes such that the intersection of any p of them is an $(n-p)$-simplex. Hence $b * \dot{\Delta}^i * \Delta^{n-i-1}$ is the union of $(i+1)$ n-simplexes $K_0^n, \ldots, K_i^n$ such that the intersection of any p of them is an $(n-p)$-simplex. Thus

$$v(b * \dot{\Delta}^i * \Delta^{n-i-1}) = v\left(\bigcup_{j=0}^{i} K_j\right) = \sum_j v(K_j^n) - \sum_{j<k} v(K_j^n \cap K_k^n)$$
$$+ \sum_{i<k<l} v(K_j^n \cap K_k^n \cap K_l^n) - \ldots = \binom{i+1}{1}v(\Delta^n)$$
$$- \binom{i+1}{2} v(\Delta^{n-1}) + \ldots + (-1)^i\binom{i+1}{i+1}v(\Delta^{n-i}).$$

Subtracting $v(\Delta^n)$ produces the required equation.

LEMMA 5. *Let $y = (y_0, \ldots, y_n) \in \mathbf{Z}^{n+1}$ with $\chi(y) = 0$. Then there exist integers a_i such that*

$$y = \sum_{i=1}^{n} a_i\alpha_i.$$

https://doi.org/10.4153/CJM-1980-100-0 Published online by Cambridge University Press

Proof. First note that $\{v(\Delta^0), v(\Delta^1), \ldots, v(\Delta^n)\}$ is a basis for $\mathbf{Z}^{n+1}$ since the matrix with these vectors as columns is upper triangular with ones along the diagonal. Next we show that $(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1, v(\Delta^n))$ is a basis for $\mathbf{Z}^{n+1}$. This follows from Lemma 4, since the matrix with these vectors as columns is upper triangular with ± 1 on the diagonal. Therefore there exist integers z, a_i such that

$$y = \sum_{i=1}^{n} a_i\alpha_i + zv(\Delta^n).$$

Finally

$$0 = \chi(y) = \sum a_i\chi(\alpha_i) + z\chi(\Delta^n) = z.$$

Proof of theorem. Let K, L be complexes (still n-dimensional) and suppose $\chi(v(K)) = \chi(v(L))$. Then

$$\chi(v(K)) - \chi(v(L)) = \chi(v(K) - v(L)) = 0.$$

Hence by Lemma 5 there exist integers a_1 such that

$$v(K) = v(L) + \sum_{i=1}^{n} a_i\alpha_i.$$

If $a_i = 0$ for all i then $v(K) = v(L)$ so $F(v(K)) = F(v(L))$ and we are done. The rest of the proof is by induction on the largest integer N such that $a_N \neq 0$. Suppose

$$v(K) = v(L) + a_N\alpha_N + \sum_{i<N} a_i\alpha_i,$$

where we assume without loss of generality that $a_N > 0$. If $N = n$ then we need merely to subdivide L along an n-simplex a_N successive times to produce a subdivision L' of L such that $v(L') = v(L) + a_n\alpha_N$, (this is really a case of Lemma 3) and by condition (2) of the Theorem, $F(L') = F(L)$. Thus

$$v(K) = v(L') + \sum_{i<N} a_i\alpha_i.$$

By the induction hypothesis $F(K) = F(L')$. If $N < n$ then we apply Lemma 2 to get complexes $\hat{K} \sim K$ and $\hat{L} \sim L$ such that

(i) $v(\hat{K}) = v(K) + \alpha_n + \ldots + \alpha_{N+1}$
$v(\hat{L}) = v(L) + \alpha_n + \ldots + \alpha_{N+1}$

(ii) $\hat{K}$, $\hat{L}$ each have an autonomous N-simplex.

Thus $F(K) = F(\hat{K})$, $F(L) = F(\hat{L})$ and

$$v(\hat{K}) = v(\hat{L}) + a_N\alpha_N + \sum_{i<N} a_i\alpha_i.$$

By Lemma 3 there is a subdivision $\hat{L}^*$ of $\hat{L}$ such that

$$v(\hat{L}^*) = v(\hat{L}) + a_N\alpha_N$$

https://doi.org/10.4153/CJM-1980-100-0 Published online by Cambridge University Press

and (since it is a subdivision) $F(\hat{L}^*) = F(\hat{L})$. Thus

$$v(\hat{K}) = v(\hat{L}^*) + \sum_{i<N} a_i \alpha_i$$

where

$$F(\hat{K}) = F(K) \text{ and } F(\hat{L}^*) = F(L).$$

By the induction hypothesis $F(\hat{K}) = F(\hat{L}^*)$ and the proof of the theorem is complete.

Remark. As a consequence of the theorem, and since topological invariants are subdivision invariants, the only topological invariants that can be computed from $v(K)$ are $\chi(K)$ and dim K.

The only homotopy invariant that can be computed from $v(K)$ is $\chi(K)$. For if $\chi(K) = x(L)$ then $F(K) = F(K \vee \Delta^N)$ and $F(L) = F(L \vee \Delta^N)$ and $\dim(K \vee \Delta^N) = \dim(L \vee \Delta^N)$ if N is chosen very large. $\chi(K \vee \Delta^N) = \chi(L \vee \Delta^N)$ so $F(K \vee \Delta^N) = F(L \vee \Delta^N)$ since homotopy invariants are subdivision invariants.

An argument similar to (but easier than) the proof of the theorem shows that the only topological invariants that can be computed from the numbers $c_i(f)$ (the number of non-degenerate critical points of index i of a Morse function $f: M \to \mathbf{R}$ on a compact manifold M) are $\chi(M)$ and dim M.

References

1. L. Glaser, *Geometrical combinatorial topology* (Van Nostrand, 1970).
2. T. B. Rushing, *Topological embeddings* (Academic Press, N.Y., 1973).
3. C. T. C. Wall, *Arithmetic properties of simplicial complexes*, Can. J. Math. *18* (1966), 92–96.

The University of Michigan,
Ann Arbor, Michigan

https://doi.org/10.4153/CJM-1980-100-0 Published online by Cambridge University Press