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Smooth and proper maps with respect to a fibration

Mathieu Anel[1] and Jonathan Weinberger[2]

[1]Department of Philosophy, Carnegie Mellon University, Pittsburgh, PA, USA and [2]Fowler School of Engineering, Chapman University, Orange, CA, USA
Corresponding author: Mathieu Anel; Email: mathieu.anel@protonmail.com

Abstract

This paper explain how the geometric notions of local contractibility and properness are related to the Σ-types and Π-types constructors of dependent type theory. We shall see how every Grothendieck fibration comes canonically with such a pair of notions—called smooth and proper maps—and how this recovers the previous examples and many more. This paper uses category theory to reveal a common structure between geometry and logic, with the hope that the parallel will be beneficial to both fields. The style is mostly expository, and the main results are proved in external references.

Keywords: Category theory; dependent type theory; fibrations; Beck–Chevalley conditions; smooth maps; proper maps

1. Introduction

Dependent type theory is based on the notion of family of types indexed by a type, and the basic operations are the reindexing and the sums and products of such families: the Σ-types and Π-types constructors. On the other hand, the geometer's toolbox contains methods to study spaces by means of "bundles," that is, using families of spaces indexed by the space of interest (vector bundles, sheaves. . .). There also, bundles can be pulled back along a morphism and sometimes pushed forward in two different ways (additively or multiplicatively). This has suggested an analogy where Σ-types can be understood as total spaces of families, while Π-types can be regarded as spaces of sections of families.

The fact that the pushforwards are not always defined is the source of an interesting feature: it distinguishes the classes of maps along which these pushforwards exist: locally contractible and proper morphisms.

In Section 2, we introduce an abstract notion of *smooth* and *proper maps* associated with any category $\mathbb{C}$. This is done in the setting of fibered/indexed categories over a base category $\mathcal{B}$. The smooth (proper) maps are the maps $u : X \to Y$ in $\mathcal{B}$ along which the base change functor $u^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ admits a left adjoint $u_!$ (a right adjoint u_*) compatible with reindexing/base change (aka the Beck–Chevalley conditions). Then Section 3 details many examples in logic, category theory, topology, and geometry, where we show how our abstract definitions connect to existing notions of smoothness and properness.

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*To André Joyal, on his $(\infty,0)$-th Birthday.

Conventions

The paper is written in the context of ∞-categories (Lurie, 2009; Cisinski, 2019), but we are simplifying the terminology and simply saying "category" for ∞-category and "1-categories" for the truncated notion. We denote by Set (SET) the 1-category of small (large) sets and by Cat (CAT) the (∞-)category of small (large) (∞-)categories. Consequently, we refer to ∞-functors simply as "functors," so that under the (∞-categorical) Grothendieck construction, small fibered categories correspond to functors into Cat. The categories of functors from C to D are denoted $[C, D]$. The arrow category of a category C is denoted $C^{\rightarrow}$.

2. Abstract Setting

Let κ be a cardinal and $\mathsf{Set}^{<\kappa}$ be the category of sets of cardinality strictly smaller than κ. The elementary operations on sets are the sum and product of a family of sets. For I an arbitrary set, an I-indexed family (I-family for short) in $\mathsf{Set}^{<\kappa}$ is a functor $I \rightarrow \mathsf{Set}^{<\kappa}$, and the sum and product of I-families are the left and right adjoint to the constant family functor $\mathsf{Set}^{<\kappa} \rightarrow [I, \mathsf{Set}^{<\kappa}]$. Given κ, one can ask for what sets I the sum and product of I-families of κ-small sets are κ-small. Let σ be a cardinal, such that for any cardinal $\rho < \kappa$, we have $\sigma\rho < \kappa$. Then $\mathsf{Set}^{<\kappa}$ admits sums indexed by objects in $\mathsf{Set}^{<\sigma}$. Let $\Sigma(\kappa)$ be the supremum of all such cardinals σ. We shall call $\Sigma(\kappa)$ the *smooth bound* of κ. The cardinal κ is *regular* if and only if $\kappa = \Sigma(\kappa)$. Similarly, let π be a cardinal, such that for any cardinal $\rho < \kappa$, we have $\rho^{\pi} < \kappa$. Then $\mathsf{Set}^{<\kappa}$ admits products indexed by objects in $\mathsf{Set}^{<\pi}$. Let $\Pi(\kappa)$ be the supremum of all such cardinals π. We shall call $\Pi(\kappa)$ the *proper bound* of κ. The cardinal κ is *inaccessible* if and only if $\kappa = \Pi(\kappa) = \Sigma(\kappa)$.

More generally, we can replace the category $\mathsf{Set}^{<\kappa}$ by any category C and extract the classes of sets $\Sigma(C)$ and $\Pi(C)$ indexing the sums and products that exist in C. We shall call $\Sigma(C)$ the *smooth calibration* of C and $\Pi(C)$ the *proper calibration* of C.[1]

We are going to propose an abstract setting for the definition of smooth and proper calibrations and illustrate it with many examples. The category C will be a fibration over some base category B with finite limits, and the calibrations $\Sigma(C)$ and $\Pi(C)$ will be defined as subfibrations of the codomain fibration of B.

2.1 Calibrations and families

We fix a category $\mathcal{B}$ with finite limits.

Definition 1. ($\mathcal{B}$-category). A functor $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \rightarrow \mathsf{CAT}$ is called a $\mathcal{B}$-category (or simply a category when $\mathcal{B}$ is clear from the context). A ($\mathcal{B}$-)functor between $\mathcal{B}$-categories is simply a natural transformation. There is an equivalence between the category of $\mathcal{B}$-categories and the category of fibrations in categories (aka Grothendieck fibrations, or cartesian fibrations) $\mathcal{C} \rightarrow \mathcal{B}$ (Lurie, 2009, Theorem 3.2.0.1 and Section 3.3.2). We shall treat the two objects $\mathbb{C}$ and $\mathcal{C}$ as equivalent and go back and forth between them. We systematically use the same letter in different fonts $\mathbb{C}/\mathcal{C}$ to denote whether we are considering the functorial (aka indexed) or the fibrational point of view on $\mathcal{B}$-categories.

Definition 2. (Universe). Since $\mathcal{B}$ is assumed with finite limits, the codomain functor $\mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$ is a cartesian fibration, and we denote by $\mathbb{B} : \mathcal{B}^{\mathrm{op}} \rightarrow \mathsf{CAT}$ the corresponding functor (sending an object X to the slice category $\mathcal{B}_{/X}$). We shall refer to $\mathbb{B}$ as the universe of $\mathcal{B}$.

Since $\mathcal{B}$ has a terminal object, the terminal functor $1 : \mathcal{B}^{\mathrm{op}} \rightarrow \mathsf{CAT}$ is a subfunctor of the universe $\mathbb{B}$. The corresponding fibration is the identity of $\mathcal{B}$. Equivalently, it corresponds to the subcodomain fibration $\mathcal{B} = \mathcal{B}^{\simeq} \subset \mathcal{B}^{\rightarrow}$ spanned by the isomorphisms.[2]

Definition 3. (Calibration). A calibration is a subuniverse $\mathbb{U} \subset \mathbb{B}$ (i.e., a subfibration $\mathcal{U} \subset \mathcal{B}^{\rightarrow}$). The constant calibration is the calibration $1 \rightarrow \mathbb{B}$. A calibration is pointed if it contains the constant calibration (equivalently, if $\mathcal{U} \subset \mathcal{B}^{\rightarrow}$ contains all isomorphisms). A calibration $\mathbb{U} \subset \mathbb{B}$ is regular if it is pointed, and the corresponding class of maps in $\mathcal{B}$ is closed under composition.

A regular calibration defines a wide subcategory of $\mathcal{B}$ (since it contains all isomorphisms). In fact, regular calibrations are in bijection with wide subcategories of $\mathcal{B}$ whose maps are closed under base change in $\mathcal{B}$.

Definition 4. (Family construction). We fix a $\mathbb{B}$-category $\mathbb{C}$, with associated fibration $\mathcal{C} \rightarrow \mathcal{B}$. Let $\mathcal{U} \subset \mathcal{B}^{\rightarrow} \xrightarrow{cod} \mathcal{B}$ be a subfibration of the codomain fibration. The fibration $\mathsf{Fam}_{\mathcal{U}}(\mathcal{C})$ of $\mathcal{U}$-indexed families of objects in $\mathcal{C}$ is defined as the functor ϕ in the diagram

$$
\begin{array}{ccc}
\mathsf{Fam}_{\mathcal{U}}(\mathcal{C}) & \longrightarrow & \mathcal{C} \\
\downarrow & \ulcorner & \downarrow \\
\mathcal{U} & \xrightarrow{\ dom\ } & \mathcal{B} \\
\downarrow {\scriptstyle cod} & & \\
\mathcal{B} & &
\end{array}
$$

(where the square is a fiber product). Following our typographic convention, we denote by $\mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ the associated functor $\mathcal{B}^{\mathsf{op}} \rightarrow \mathsf{CAT}$ (and call it the $\mathbb{B}$-category of $\mathbb{U}$-indexed families of objects in $\mathbb{C}$). Its value at an object I in $\mathcal{B}$ is the category of pairs $(u : U \rightarrow I \in \mathcal{U}, C \in \mathbb{C}(U))$ (with the obvious notion of morphism).

2.2 Sums and products

The family construction associated with the constant calibration is the identity: $\mathbb{C} = \mathsf{Fam}_1(\mathbb{C})$. If $\mathbb{U}$ is a pointed calibration we get a canonical functor $\Delta_{\mathbb{U}} : \mathbb{C} = \mathsf{Fam}_1(\mathbb{C}) \rightarrow \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$.

Definition 5. Let $\mathbb{U}$ be a pointed calibration on $\mathbb{B}$. A ($\mathbb{B}$-)category $\mathbb{C}$ has $\mathbb{U}$-indexed sums (products) if the canonical functor $\Delta_{\mathbb{U}} : \mathbb{C} \rightarrow \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ has a left (right) adjoint.

The following result is proved for 1-categories in Streicher (2020). The statement extends to ∞-categories as well. So do a range of expected results from Streicher (2020) and Moens (1982) about fibrations with internal sums. This has been developed by Buchholtz and Weinberger (2023) and Weinberger (2024) in the type theory of synthetic ∞-categories (Riehl and Shulman, 2017; Riehl, 2023a) (internally to *any* ∞-topos, see Shulman (2020); Riehl (2023b); Weinberger (2022)). See also the work on internal ∞-categories in an arbitrary ∞-topos by Martini (2022), Martini and Wolf (2023a, 2023b).

Proposition 6. (Bénabou–Streicher). *If $\mathbb{U}$ is a regular calibration, then $\mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ is the free cocompletion of $\mathbb{C}$ for sums indexed by objects in $\mathbb{U}$. A pointed calibration $\mathbb{U}$ is regular if and only if $\mathbb{U}$ has $\mathbb{U}$-indexed sums.*

Remark 7. The notion of regular calibration corresponds in dependent type theory to the notion of a subuniverse closed under Σ-types and containing the terminal type.

Proposition 8. *For every category with finite limits $\mathcal{B}$, the universe $\mathbb{B}$ always has $\mathbb{B}$-indexed sums. A map is proper if and only if all its pullbacks are exponentiable maps in $\mathbb{B}$. In particular, the universe $\mathbb{B}$ has $\mathbb{B}$-indexed products if and only if $\mathcal{B}$ is a locally cartesian closed category.*

Sketch of the proof. The left adjoints $f_!$ always exist and are given by composition with f. The Beck–Chevalley conditions are fulfilled since they amount to the cancellation property for pullback squares. The right adjoint f_* exists for every map f if and only if $\mathcal{B}$ is a locally cartesian closed category. Then the Beck–Chevalley conditions are deduced by adjunction from those for $f_!$.   □

2.3 Smooth and proper maps

We fix a category $\mathcal{B}$ with finite limits and a $\mathcal{B}$-category $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$.

Definition 9. (Smooth & proper maps). A map $u : X \to Y$ in $\mathcal{B}$ is called left (right) Beck–Chevalley for $\mathbb{C}$ if for any cartesian square

$$
\begin{array}{ccc}
\bar{Y} & \xrightarrow{\;\bar{v}\;} & Y \\
{\scriptstyle \bar{u}}\downarrow & \ulcorner & \downarrow{\scriptstyle u} \\
\bar{X} & \xrightarrow{\;v\;} & X
\end{array}
$$

the maps u^* and $\bar{u}^*$ have left (right) adjoints

$$
\begin{array}{ccc}
\mathbb{C}(\bar{Y}) & \xleftarrow{\;\bar{v}^*\;} & \mathbb{C}(Y) \\
{\scriptstyle \bar{u}_!}\updownarrow\,\!{\scriptstyle \bar{u}_*} & & {\scriptstyle u_!}\updownarrow\,\!{\scriptstyle u_*} \\
\mathbb{C}(\bar{X}) & \xleftarrow{\;v^*\;} & \mathbb{C}(X)
\end{array}
$$

and the corresponding mate natural transformation is invertible

$$
\bar{u}_! \bar{v}^* \xrightarrow{\;\sim\;} v^* u_! \qquad\qquad \left(v^* u_* \xrightarrow{\;\sim\;} \bar{u}_* \bar{v}^* \right).
$$

A map $u : X \to Y$ in $\mathcal{B}$ is called smooth, or stably left Beck–Chevalley (proper, or stably right Beck–Chevalley) if every base change of u is left (right) Beck–Chevalley. The relation of these operations with quantification in logic is recalled in Table 2.

The classes of $\mathbb{C}$-smooth and $\mathbb{C}$-proper maps are closed under base change and define subfibrations of the codomain fibration of $\mathcal{B}$. Equivalently, they define regular calibrations $\Sigma(\mathbb{C}) \subset \mathcal{B} \supset \Pi(\mathbb{C})$ of the universe of $\mathbb{B}$, called the *smooth calibration* and the *proper calibration* of $\mathbb{C}$. The interest of the notions is in the following result.

Proposition 10. *The smooth (proper) calibration of $\mathbb{C}$ is the largest calibration for which $\mathbb{C}$ admits sums (products).*

Proof. Unfolding the condition that $\Delta_{\mathbb{U}} : \mathbb{C} \to \mathsf{Fam}_{\mathbb{U}}(\mathbb{C})$ has a left (right) adjoint, we find that the left adjoint exists if and only if all maps in $\mathcal{U}$ are smooth (proper); see Streicher (2020).   □

Remark 11. When $\mathcal{B}$ is the 1-category of sets or the ∞-category of spaces, the Beck–Chevalley conditions simplify for the functors $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$, which send colimits to limits. In that case, we have $\mathbb{C}(X) = \mathbb{C}(1)^X$, and the compatibility with base change holds for free as soon as the adjoint functors exist.

The characterization of the smooth and proper maps can be quite difficult in practice. In the setting where the category $\mathbb{C}$ has a forgetful functor into the universe $\mathbb{B}$ (typically, when $\mathbb{C}$ classifies objects in $\mathbb{B}$ with an extra structure, notably when $\mathbb{C}$ is a calibration), stricter notions of smooth and proper maps can be defined, which are easier to characterize in practice.

Definition 12. (Strict smoothness/properness). Let $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ be equipped with a natural transformation $U : \mathbb{C} \to \mathbb{B}$ to the universe of $\mathbb{B}$ (typically the inclusion of a subuniverse). For $f : X \to Y$ a map in $\mathcal{B}$, we have a commutative square

$$
\begin{array}{ccc}
\mathbb{C}(Y) & \xrightarrow{\ U_Y\ } & \mathcal{B}_{/Y} \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathbb{C}(X) & \xrightarrow{\ U_X\ } & \mathcal{B}_{/X}\,.
\end{array}
$$

Recall from Proposition 8 that the functor $f^* : \mathcal{B}_{/Y} \to \mathcal{B}_{/X}$ has always a left adjoint $f_!$ given by composition with f and that the right adjoint f_* exists if and only if f is an exponentiable map in $\mathcal{B}$. We shall say that a $\mathbb{C}$-smooth map is strictly smooth if the mate $f_! U_X \to U_Y f_!$ is invertible. We shall say that a $\mathbb{C}$-proper map is strictly proper if it is exponentiable in $\mathcal{B}$ (i.e., if f_* exists) and if the mate $U_Y f_* \to f_* U_X$ is invertible. Essentially, this says that a map f is strictly smooth if $f_!$ can be computed by composition with f in $\mathcal{B}$, and that it is strictly proper if f_* can be computed by exponentiation along f in $\mathcal{B}$.

Remark 13. We shall see that such maps are sometimes easier to characterize in practice. In particular, when $\mathbb{C}$ is a calibration, a map is strictly smooth if the corresponding class $\mathcal{C}$ of maps in $\mathcal{B}$ is closed under composition with every pullback of f. And a map f is strictly proper if it is exponentiable and the functor f_* sends maps in $\mathcal{C}$ to maps in $\mathcal{C}$.

Lemma 14. *If $\mathbb{C} \subset \mathbb{B}$ is a regular calibration, then a $\mathbb{C}$-smooth map is strict if and only if it is in $\mathcal{C} \subset \mathcal{B}^{\to}$.*

Proof. A map $u : X \to Y$ is strict if composition with u sends maps $r : \bar{X} \to X$ in $\mathcal{C}$ to maps $\bar{X} \to Y$ in $\mathcal{C}$. Applied to $r = \mathrm{id}_X$, this implies that u is in $\mathcal{C}$ and that every strict smooth map is in $\mathcal{C}$. The converse is true by regularity of $\mathbb{C}$. $\qquad\square$

A *modality* on the category $\mathcal{B}$ is a (unique) factorization system $(\mathcal{L}, \mathcal{R})$ such that the factorization (or equivalently the left class $\mathcal{L}$) is stable under base change. In this situation, both classes $\mathcal{L}$ and $\mathcal{R}$ define calibrations $\mathbb{L} \subset \mathbb{B} \supset \mathbb{R}$ and $\mathbb{R}$ is even a reflective calibration (with the reflection given by the factorization) (Anel et al., 2020; Rijke et al., 2019).

Lemma 15. *If $\mathbb{C} = \mathbb{R} \subset \mathbb{B}$ is the subuniverse of the right class of a modality on $\mathcal{B}$, then every map in $\mathcal{B}$ is $\mathbb{C}$-smooth.*

Proof. For a map $u : X \to Y$ in $\mathcal{B}$, and $r : \bar{X} \to X$ in $\mathcal{R}$, the left adjoint $u_!(r)$ is given by the right map of the $(\mathcal{L}, \mathcal{R})$-factorization of the map composite map $\bar{X} \to X \to Y$. It satisfies the Beck–Chevalley condition because the factorization is stable under base change. $\qquad\square$

Notice that Lemmas 14 and 15 together provide examples where smooth and strictly smooth maps do not coincide.

We consider now the case of a *weak* factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$. The class $\mathcal{R}$ is still closed under base change and defines a regular calibration $\mathbb{R} \subset \mathbb{B}$ (but not the class $\mathcal{L}$ in general). The following lemma is a classical trick to recognize proper maps.

Lemma 16. *If $\mathbb{C} = \mathcal{R} \subset \mathcal{B}$ if the regular calibration is associated with the right class of a weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{B}$, then a map u is strictly $\mathbb{C}$-proper if and only if for any base change $\bar{u} \to u$, the functor $\bar{u}^*$ preserves the class $\mathcal{L}$. In particular, if the factorization system $(\mathcal{L}, \mathcal{R})$ is a modality, then every map is strictly proper.*

Proof. A map u is strictly proper if, for any base change $\bar{u} \to u$, the functor $\bar{u}_*$ preserves the maps in $\mathcal{R}$, but this is equivalent to $\bar{u}^*$ preserving the class $\mathcal{L}$. And when $(\mathcal{L}, \mathcal{R})$ is a modality, every $\bar{u}^*$ preserves the class $\mathcal{L}$. $\qquad\square$

Definition 17. (Acyclic and localic maps). A map $u : X \to Y$ in $\mathcal{B}$ is called $\mathbb{C}$-pre-acyclic if $u^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ is an equivalence. A map $u : X \to Y$ in $\mathcal{B}$ is called $\mathbb{C}$-acyclic if every base change of u is $\mathbb{C}$-pre-acyclic. We denote by $A(\mathbb{C})$ (Alpha) the codomain subfibration of $\mathbb{C}$-acyclic maps. Acyclic maps are always both smooth and proper. A map is called $\mathbb{C}$-localic if it is right orthogonal to $\mathbb{C}$-acyclic maps. We denote by $\Lambda(\mathbb{C})$ the codomain subfibration of $\mathbb{C}$-localic maps.

The name "acyclic" is motivated by the following result. The name "localic" is motivated by an application to topos theory (see third example of Section 3.3). In an ∞-topos $\mathcal{E}$, an acyclic class is a class of maps containing all isomorphisms, closed under composition and base change, and under colimits in the arrow category of $\mathcal{E}$ (Anel et al., 2022, Definition 3.2.8). If $(\mathcal{L}, \mathcal{R})$ is a (unique) factorization system on $\mathcal{E}$, the class $\mathcal{L}$ is acyclic if and only if $(\mathcal{L}, \mathcal{R})$ is a modality.

Lemma 18. *If $\mathcal{B}$ is an ∞-topos and if $\mathbb{C} : \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ sends colimits to limits, then the class of $\mathbb{C}$-acyclic maps is an acyclic class.*

Proof. It is easy to see from the definition that $A(\mathbb{C})$ contains all isomorphisms and is closed under composition and base change. It is also closed under small colimits in $\mathcal{B}^{\to}$ since the functor $\mathbb{C}$ send colimits to limits and $A(\mathbb{C})$ is the inverse image of the class $\mathsf{CAT}^{\simeq} \subset \mathsf{CAT}^{\to}$, which is closed under limits. $\qquad\square$

The $\mathbb{C}$-acyclic and $\mathbb{C}$-localic maps sometimes form a factorization system on $\mathcal{B}$. When this the case, it is always a modality since both classes are stable by base change by definition.

3. Examples
3.1 Set theory and type theory
The examples are summarized in Table 1.

Any 1-category C represents a functor $\mathcal{B}^{\mathrm{op}} = \mathsf{Set}^{\mathrm{op}} \to \mathsf{CAT}$, sending I to C^I. In this setting, all conditions on maps can be computed fiberwise. A set I is smooth (proper) if the coproduct $\coprod_I : C^I \to C$ (product functor $\prod_I : C^I \to C$) exists. Notice that in this case, the Beck–Chevalley condition is always fulfilled since everything is determined fiberwise (Remark 11). A map is smooth (proper) if its fibers are smooth (proper) sets. If C is the terminal category, every set is smooth and proper. If C is the initial category, the smooth (proper) maps are the surjections. A set I is acyclic if $C \to C^I$ is an equivalence. If C is the terminal category, every set is acyclic (and every map is acyclic). If C is the empty category, the acyclic sets are the nonempty sets (the acyclic maps are surjections, and the localic maps are the injections). If C is otherwise, the only acyclic sets are the singletons; thus, the acyclic maps are the bijections, and every map is localic.

Similar considerations hold when $\mathcal{B} = \mathcal{S}$ is the ∞-category of spaces and C some ∞-category. In particular, Remark 11 applies, and the Beck–Chevalley conditions are always fulfilled.

The second example is the one detailed in the introduction of Section 2. It is an instance of the previous example if and only if κ is a regular cardinal (use $C = \mathsf{Set}^{<\kappa}$). The example of the

Table 1. Examples from set theory and type theory

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$	$A(\mathbb{C})$	$\Lambda(\mathbb{C})$
Cat. of sets	a category C	I for which $\bigsqcup_I$ exists in C	sets I for which $\prod_I$ exists in C	(see text)	
Cat. of sets	maps with κ-small fibers ($\kappa \geq 2$)	maps with Σ-small fibers for $\Sigma = \sup\{\sigma \mid \rho < \kappa \Rightarrow \rho.\sigma < \kappa\}$	maps with Π-small fibers for $\Pi = \sup\{\pi \mid \rho < \kappa \Rightarrow \rho^\pi < \kappa\}$	bijections	all maps
Cat. of sets	subsets ($\kappa = 2$)	all maps	all maps	bijections	all maps
$\mathcal{B}$	hyperdoctrine $\mathcal{B}^{op} \to$ Poset	all maps	all maps	?	?
$\mathcal{B}$	codomain fibration	all maps	stably exp. maps	isomorphisms	all maps
$\mathcal{B}$	π-clan structure $\mathbb{D} \subset \mathcal{B}$	strict smooth = all maps in $\mathcal{D}$	strict proper $\supset$ all maps in $\mathcal{D}$	?	?
A cat. $\mathcal{E}$ with subobject classifier Ω	dominance $\mathcal{O} \subset \Omega$	smooth = overt maps strict smooth = maps in $\mathcal{O}$	proper = proper maps str. proper = ?	(see text)	
A 1-topos $\mathcal{E}$	Grothendieck topology $\Omega_j \subset \Omega$	smooth = all maps strict smooth = closed monos	proper = "quasi-compact" maps	maps inverted by the localization	relative sheaves
$\mathcal{S}$, the ∞-category of ∞-groupoids	subcategory $\mathcal{S}^{<\kappa}$ of κ-small ∞-groupoids	strict : maps with κ-small fibers (if κ regular)	strict : maps with κ-small fibers (if κ inacc.)	?	?
An ∞-topos $\mathcal{E}$	subuniverse $\mathbb{T}_n$ of n-truncated objects	the whole universe (strict smooth = n-truncated maps)	the whole universe (all maps are strictly proper)	$(n+1)$-conn. maps	$(n+1)$-trunc. maps
An ∞-topos $\mathcal{E}$	subuniv. $\mathbb{R} \subset \mathbb{E}$ of modal types (for a modality $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$)	the whole universe (strict maps are those in $\mathbb{R}$)	?	décalage class of $\mathcal{L}$ (Anel et al., 2024)	right class of the décalage modality of Anel et al. (2023)
An ∞-topos $\mathcal{E}$	subuniv. $\mathbb{F} \subset \mathbb{E}$ of sheaves assoc. to a lex loc. $\mathcal{F} = \mathcal{E}[\mathcal{W}^{-1}]$	the whole universe (strict maps are those in $\mathbb{F}$)	?	maps in $\mathcal{W}$	maps in $\mathbb{F}$

fibration of subsets (or of injections) is the case $\kappa = 2$. Every map $u : I \to J$ is smooth and proper. The functors $u_!$ and u_* are the two direct images of subsets, classically related to existential and universal quantifiers ($u_! A = \{j \mid \exists a \in A, a \in u^{-1}(j)\}$, $u_* A = \{j \mid \forall a \in A, a \in u^{-1}(j)\}$). More generally, the setting of sets and subsets could be replaced by a hyperdoctrine in the sense of Lawvere (1970, 2006) and Seely (1983). We recall the correspondence between logical quantifiers and the adjoints to base change in Table 2.

In the case of the codomain fibration, we have seen in Proposition 8 that all maps are smooth and that proper maps are the exponentiable maps whose pullbacks are also exponentiable. All the projections $X \times Y \to X$ are proper if and only if the category $\mathcal{B}$ is cartesian closed, and all the maps are proper if and only if the category $\mathcal{B}$ is locally cartesian closed. In this setting, only the isomorphisms are acylic and every map is localic.

Another example related to type theory is that of a category $\mathcal{B}$ with a π-clan structure in the sense of Joyal (2020). The clan structure distinguishes a class of maps in $\mathcal{D} \subset \mathcal{B}$ closed under base

Table 2. Quantifiers and direct images

	Indexing object	Families	Change of index	Left image	Right image
Predicate logic	variables	predicates	substitution	$\exists$	$\forall$
Dependent type theory	contexts	dependent types	substitution	Σ	Π
Category theory: fibration $C \to B$	object in base	object in fiber	base change u^*	$u_!$	u_*

change and containing all isomorphisms, or equivalently a regular calibration $\mathbb{D} \subset \mathbb{B}$. This implies that strict smooth maps are exactly the maps in $\mathcal{D}$. And the definition of the π-clan structure says that every map in $\mathcal{D}$ is strictly proper (Joyal, 2020, Definition 2.4.1).

If $\mathcal{B}$ is a cartesian closed 1-category with a subobject classifier (e.g., 1-topos), we consider the example of a *dominance*, which is a regular calibration of monomorphisms classified by a subobject $\mathcal{O} \subset \Omega$ of the subobject classifier (Escardó, 2004; Hyland, 1991; Rosolini, 2020). Intuitively, the object $\mathcal{O}$ classifies the subobjects meant to be "open" in the sense of topology, and the exponential $\mathcal{O}^A \subset \Omega^A$ is the lattice of open subobjects sitting within that of all subobjects. The smooth (proper) maps define a notion of open (compact) maps in the sense of topology. The strictly smooth maps are those classified by $\mathcal{O}$ (Lemma 14). The strictly proper maps are those maps for which the direct image can be computed within the posets of all subobjects. Acyclic and localic maps are considered in the literature, but (seemingly) only with the base change property along cartesian projections and not along arbitrary maps. They are called $\mathcal{O}$-*equable* maps and $\mathcal{O}$-*replete* maps in Hyland (1991).

When $\mathcal{B}$ is a 1-topos, any Grothendieck topology defines a dominance $\Omega_j \subset \Omega$ where Ω_j is the classifier of closed monos. In this case, all maps are smooth, the closed monos are the strictly smooth maps. The acyclic maps are the maps inverted in the localization by the topology, and the localic maps are the relative sheaves. A map $A \to 1$ is proper if and only if A it is quasi-compact (every covering family has a covering finite subfamily). General proper maps can be described by a relative version of the same condition.

The ∞-category $\mathcal{S}$ of ∞-groupoids is a higher categorical model for dependent type theory with univalence: precisely, the small subuniverses correspond to univalent maps in $\mathcal{S}$. It seems difficult to describe the smooth, proper, acyclic, and localic maps for an arbitrary subuniverse, but it is easier with strict smooth and proper maps. We shall only consider the subuniverse of κ-small spaces. When κ is regular, the strict smooth maps are the maps with κ-small fibers by Lemma 14. When κ is inaccessible, the strict proper maps are the maps with κ-small fibers by Lemma 16. In particular, if κ is the inaccessible cardinal bounding the size of small objects, all maps are smooth and proper. In this case, the acyclic maps are reduced to equivalences and all maps are localic.

In the example of an ∞-topos $\mathcal{E}$, we denote the universe by $\mathbb{E}$.[3] By definition of an ∞-topos, the functor $\mathbb{E} : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$ sends colimits of $\mathcal{E}$ to limits in CAT, and the interesting subuniverses are those satisfying a similar condition (corresponding to local classes of maps (Lurie, 2009, Proposition 6.1.3.7)). For $n \geq -2$, we denote by $\mathcal{C}_n$ and $\mathcal{T}_n$ the classes of n-connected and n-truncated maps. For example, the class $\mathcal{C}_{-1}$ and $\mathcal{T}_{-1}$ are the classes of surjections and monomorphisms. Every map can be factored uniquely into an n-connected map followed by an n-truncated maps, and this factorization is stable under base change (it is a modality). The class $\mathcal{T}_n$ contains all isomorphisms, is closed under composition, and is local. It defines a regular calibration $\mathbb{T}_n \subset \mathbb{E}$, which is also a reflective subuniverse (where the reflection is given by the factorization). Every map is smooth for $\mathbb{T}$, and the strict smooth maps are those in $\mathbb{T}$. Every map is strictly proper (hence proper) for $\mathbb{T}$ because dependent products preserve the truncation level of objects.

Table 3. Examples from category theory

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$
Cat. of categories	all functors	all functors	Conduché fib.
Cat. of categories	left fibrations	smooth = ? strictly smooth = left fib.	proper = ? strictly proper = Lurie-proper functor (= Grothendieck-smooth)
Cat. of categories	right fibrations	smooth = ? strictly smooth = right fib.	proper = ? strictly proper = Lurie-smooth functors (= Grothendieck-proper)
Cat. of categories	cocartesian fibrations	smooth = ? strictly smooth = cocart. fib.	proper = ? strictly proper = Cond. + u^* pres. ff left adj.
Cat. of categories	cartesian fibrations	smooth = ? strictly smooth = cart. fib.	proper = ? strictly proper = Cond. + u^* pres. ff right adj.

The $\mathbb{T}_n$-acyclic maps and $\mathbb{T}_n$-localic maps can be characterized as the $(n+1)$-connected maps and $(n+1)$-truncated maps (see below).

More generally, a modality on $\mathcal{E}$ is a (unique) factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{E}$ such that both classes are stable under base change. Then both classes $\mathcal{L}$ and $\mathcal{R}$ are local by Anel et al. (2020, Proposition 3.6.5) and define subuniverses $\mathbb{L}, \mathbb{R} \subset \mathbb{E} : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$. We shall consider the case $\mathbb{C} = \mathbb{R}$. Since the class $\mathcal{R}$ is closed under composition and base change, Lemma 14 shows that the strict smooth maps are all maps in $\mathcal{R}$. But the subuniverse actually admits sums indexed by all maps in $\mathcal{E}$. Given a map $u : X \to Y$ in $\mathcal{E}$ and a map $r : \bar{X} \to X$ in $\mathcal{R}$, the $(\mathcal{L}, \mathcal{R})$-factorization of the composite map $\bar{X} \to X \to Y$ gives maps $\bar{X} \to \bar{Y}$ in $\mathcal{L}$ and $r' : \bar{Y} \to Y$ in $\mathcal{R}$. The image of r by $u_!$ is the map $\bar{Y} \to Y$. The Beck–Chevalley condition holds because the factorization is stable under base change. A more conceptual way to see this is to say that $\mathbb{R}$ is in fact a reflective subuniverse (where the reflection is given by the factorization) and therefore complete under all sums existing in $\mathbb{E}$.

In this example, it seems difficult to describe the proper maps without further assumptions on $\mathcal{R}$. However, the acyclic maps are exactly the maps in the *décalage* of the class $\mathcal{L}$ (Anel et al., 2024, 2.2.7) (they are also the "fiberwise $\mathbb{R}$-equivalences" of Anel et al. (2022, Def. 3.3.1)). In particular, they form an acyclic class in $\mathcal{E}$ (Anel et al., 2022, Theorem 3.3.9). In this context, acyclic and localic maps define a modality on $\mathcal{E}$, which is detailed in Anel et al. (2023, Theorem 2.3.32).

Finally, any left-exact localization of ∞-topoi $q^* : \mathcal{E} \to \mathcal{E}[\mathcal{W}^{-1}]$ provides a modality $(\mathcal{W}, \mathcal{F})$ on $\mathcal{E}$, where $\mathcal{W}$ is the class if maps inverted by q^* and $\mathcal{F}$ are the class of "relative sheaves" (Rijke et al., 2019; Anel et al., 2022). Such a modality is left-exact in the sense that the factorization of a map preserves finite limits (in the arrow category). In fact, there is a bijection between left-exact localizations and left-exact modalities (Anel et al., 2022). Let $\mathbb{W} \subset \mathbb{E} \supset \mathbb{F}$ be the corresponding subuniverses. If $\mathbb{E}' : \mathcal{E}[\mathcal{W}^{-1}]^{\mathrm{op}} \to \mathsf{CAT}$ is the universe of the ∞-topos $\mathcal{E}[\mathcal{W}^{-1}]$, one can show that $\mathbb{F} = \mathbb{E}' \circ q^* : \mathcal{E}^{\mathrm{op}} \to \mathsf{CAT}$. Since left-exact modalities are fixed by décalage (Anel et al., 2024, Lemma 2.4.6 (1)), the acyclic (localic) maps coincides with the class $\mathcal{W}$ ($\mathcal{F}$).

3.2 Category theory

We now consider examples from category theory. The study of fibrations of categories was the motivation of Grothendieck for introducing his notion of smooth and proper functors. The examples are summarized in Table 3.

The first example is that of the universe of $\mathcal{B} = \mathsf{Cat}$. It will be the base of all the other examples. We denote by $\mathbb{C}\mathsf{at}$ the universe of Cat (which can be thought of either as the codomain fibration, or as the slice functor $C \mapsto \mathsf{Cat}_{/C}$). The category Cat is cartesian closed but not locally cartesian

closed. The exponentiable functors are the Conduché fibrations (see Ayala and Francis (2020, Lemma 1.11) for a higher categorical account). The acyclic maps are the equivalences of categories and every functor is localic.

The following examples will study various kinds of "fibrations" conditions on functors. This will lead to consider functors $\mathbb{C} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$ with a natural forgetful morphism $\mathbb{C} \to \mathbb{C}\mathsf{at}$ into the universe of Cat (or equivalently a morphism between the corresponding fibrations over Cat).

Let us start with the the functor $\mathsf{LFib} : \mathsf{Cat}^{\mathrm{op}} \to \mathsf{CAT}$ sending a category C to its category $\mathsf{LFib}(C) = [C, \mathcal{S}]$ of (small) left fibrations. This functor is (up to size issues) representable by the category $\mathcal{S}$. The Grothendieck construction provides a natural transformation $\mathsf{LFib}(C) \to \mathsf{Cat}_{/C}$, and we can talk about strict notions. Recall that the left fibrations are the right class of a (unique) factorization system on Cat, whereas the left class is that of initial functors. The characterization of smooth and proper maps is open. The previous factorization system is not stable under base change, and we cannot apply Lemma 15. But the strict smooth maps are exactly the left fibrations themselves. The class of proper functors contains right fibrations And we can use Lemma 16 to characterize the strict proper maps as the exponentiable functors $u : C \to D$ such that for any base change $\bar{u} \to u$, the pullback along $\bar{u}$ preserves the class of initial functors. This is exactly the notion of proper functor of Lurie (2009, dual of Definition 4.1.2.9) and the notion of smooth functor of Grothendieck in Pursuing Stacks and also in Joyal (2008, 21.1) and Cisinski (2019, dual of Definition 4.4.1).[4] In particular, right fibrations are strict proper functors for LFib. The previous characterization of strict proper functors shows that they are also left-final functors in the sense of Ayala–Francis since they satisfy the condition of Ayala and Francis (2020, Lemma 4.1.1 (a)). However, the condition to be left-final seems strictly weaker. In the dual situation of right fibrations, the notion of smooth and proper are reversed.

Finally, we can look at the example of (small) cocartesian (and cartesian) fibrations. The functor of interest is $C \mapsto \mathsf{CcFib}(C) = [C, \mathsf{Cat}]$. Up to size issues, it is represented by Cat itself. The class of cocartesian fibrations is almost the right class of a weak factorization system on Cat (in 1-categories, it is the right class of an algebraic factorization system, Bourke and Garner (2014)). The corresponding left class is the class of fully faithful left adjoint functor. Any functor $f : C \to D$ admits a factorization $C \to f \downarrow D \to D$ into a fully faithful left adjoint functor followed by a cocartesian fibration (where $f \downarrow D$ if the fiber product of $C \to D \xleftarrow{\mathit{dom}} D^{\to}$). Again, the smooth and proper maps are difficult to find. The strict smooth maps are the cocartesian fibrations themselves. The previous factorization is enough to apply Lemma 16 to characterize the strictly proper maps as the exponentiable functors $u : C \to D$ such that u^* preserves fully faithful left adjoint functors. This class includes that of cartesian fibrations. In the dual situation of cartesian fibrations, the notion of smooth and proper are reversed.

In all the "fibration" examples, the acyclic maps are the equivalences and every functor is localic. To see this, recall that a functor $u : C \to D$ induces an equivalence $u_! : \mathcal{S}^C \rightleftarrows \mathcal{S}^D : u^*$ if and only if it is a Morita equivalence. The acyclic maps are Morita equivalences, which are stable under base change. This implies that they must be (essentially) surjective functors. Since $u_!$ is an equivalence, u is fully faithful and is therefore an actual equivalence.

3.3 Topology and geometry

We now consider examples where the base category $\mathcal{B}$ is a category of topological or geometrical objects. We are going to see a tight connection between direct images functors and classical topological conditions. The examples are summarized in Table 4.

In the first example, $\mathcal{B} = \mathsf{Locale}$ is the category of locales[5] with the codomain fibration. Every map is smooth, and the proper maps are the exponentiable ones. The exponentiable locales are the locally compact ones (i.e., the locales such that $\mathrm{Op}\,(X)$ is a continuous lattice) (Johnstone, 1982, Theorem 4.11). Acyclic maps are the isomorphisms, and every map is localic.

Table 4. Examples from topology and geometry

$\mathcal{B}$	$\mathbb{C}$	$\Sigma(\mathbb{C})$	$\Pi(\mathbb{C})$	$A(\mathbb{C})$	$\Lambda(\mathbb{C})$
Cat. of locales	all maps	all maps	exponentiable maps	homeo-morphisms	all maps
Cat. of locales	open immersions (rep. by Sierpiński space)	open maps (strict: open immersions)	proper maps (strict: locally compact and proper)	homeo-morphisms	all maps
Cat. of 1-topoi	open immersions (rep. by Sierpiński topos)	open maps (Johnstone, 2002)	proper maps (Moerdijk and Vermeulen, 2000) (strict: exponentiable + proper)	hyperconnected morphisms	localic morphisms
Cat. of 1-topoi	étale maps (rep. by object classifier)	locally connected maps (Johnstone, 2002)	tidy maps (Moerdijk and Vermeulen, 2000)	equivalences	all morphisms
Cat. of ∞-topoi	étale maps (rep. by object classifier)	locally contractible maps Martini (2023)	proper maps (Martini and Wolf, 2023)	equivalences	all morphisms
Cat. of ∞-topoi	n-tr. étale maps (rep. by n-truncated object classifier)	locally n-connected maps (by methods similar to Martini (2023))	n-proper maps (by methods similar to Martini and Wolf (2023))	hyper- $(n+1)$/con-nected mor.	$(n+1)$-localic morphisms
Cat. of schemes	torsion étale sheaves	smooth map ⊃ smooth morphisms (Freitag and Kiehl, 1988, Thm 7.3)	proper map ⊃ proper morphisms (Freitag and Kiehl, 1988, Thm 6.1)	?	?

The next example is that of the functor Locale$^{\text{op}}$ → CAT sending a locale X to its frame Op (X) of open domains. This functor is represented by the Sierpiński space. Let us say that a map is an *open immersion* if it is isomorphism to the inclusion of an open. The fibration corresponding to Op $(-)$ is the subfibration of the codomain fibration of Locale spanned by open immersions. This shows that the notion of strict smooth and proper maps makes sense.

The smooth morphisms are almost by definition the open morphisms of topology (this can be shown directly, or deduced from the similar result for topoi (Johnstone, 2002, Lemma C.3.1.10), see also Escardó (2004)). By Lemma 14, the strict smooth maps are the open immersions that are also open morphisms, but that is the case of every open immersion. The proper morphisms are the proper morphisms of topology (i.e., the universally closed morphisms, this can be seen directly by considering left adjoints on the posets Op $(X)^{\text{op}}$, or deduced from topos theory (Johnstone, 2002, Lemma C.3.2.81), see also Escardó (2004, 2020)). The characterization of strict proper maps is open. An open immersion is proper if and only if it is isomorphic to the inclusion of a clopen. The computation of acyclic and localic maps is straighforward.

In the third example, $\mathcal{B}$ is the category of 1-topoi and geometric morphisms (Johnstone, 2002); Anel and Joyal, 2021)). If we look at the codomain fibration, the only nontrivial class of maps is that of exponentiable maps (Johnstone and Joyal, 1982).

Another important fibration is the one of open sub-1-topoi (or open immersions), sending a 1-topos X to the poset of subterminal objects in its category of sheaves Sh(X) (its dual *1-logos* in

the sense of Anel and Joyal (2021)). This fibration is represented by the Sierpiński topos (dual to the 1-logos Set$^{\rightarrow}$). The identification of smooth maps as open morphisms of 1-topoi is done in Johnstone (2002, Theorem C.3.1.28), and that of proper maps as proper morphisms of 1-topoi is done in Moerdijk and Vermeulen (2000, Corollary I.5.9) and in Johnstone (2002, Theorem C.3.1.28). Both references use a "weak" version of Beck–Chevalley conditions (where the mate transformation is only monic) for arbitrary sheaves. But restricted to subterminal objects, this condition recovers the one from Definition 9, and this can be shown to be equivalent to the weak condition (see Johnstone, 2002, Proposition A.4.1.17, and Moerdijk and Vermeulen, 2000, Proposition 3.2).

Open immersions can be composed and define a regular calibration on Topos. Then, Lemma 14 shows that the strict smooth maps are exactly the open immersion of 1-topoi. The characterization of strict proper maps is open. An open immersion of 1-topoi is proper if it is also the inclusion of a closed sub-1-topoi (corresponding to a decidable subterminal object). Interestingly, in this example, the notion of acyclic and localic maps recover the hyperconnected and localic morphisms of topoi (this is in fact the motivation for the name *localic*). The proof is straightforward for hyperconnected maps, and the fact that the right orthogonal to hyperconnected morphisms are the localic morphisms is Johnstone (2002, Lemma A.4.6.4).

The next example is the case of the fibration of étale maps over the category of 1-topoi, sending a 1-topos X to its category of sheaves of sets (its dual 1-logos) Sh(X). This fibration is representable by the "object classifier" or the "topos line" (which we shall denote A, its 1-logos of sheaves is [finset, Set]) and the existence of sums and products for the étale fibration can be interpreted as sums and products structures on the 1-topos A. In this case, the smooth maps are the locally connected morphisms of 1-topoi (Johnstone, 2002, Corollary C.3.3.16), and the proper maps are the tidy morphisms of 1-topoi (see Moerdijk and Vermeulen, 2000, Corollary III.4.9, or Johnstone, 2002, Corollary C.3.4.11). The strict smooth maps are all the étale morphisms. The characterization of strict proper maps is open. In this setting, the acyclic maps are the equivalences, and every map is localic.

Interestingly, not every étale map is proper: the intersection of the two classes is the class of finite maps. This means that the "topos line" A does not have arbitrary products indexed by itself, but only finite products. This might be surprising since categories of sheaves are always locally cartesian closed, but the dependent products are not preserved by geometric morphisms, only finite products are.

The generalization of the previous results to ∞-topoi is quite rich! For the case of the codomain fibration, exponentiable ∞-topoi have been characterized in Lurie (2017, Theorem 21.1.6.12) and Anel and Lejay (2019). The case of the fibration of étale morphisms has been studied recently by Martini and Wolf. If $\mathcal{B} = $ Topos$_{\infty}$ is the category of ∞-topoi and $\mathbb{C}(X) = $ Sh$_{\infty}(X)$ is the category of higher sheaves (the ∞-logos dual to X), then the smooth maps are the locally contractible morphisms of ∞-topoi Martini (2023), and the proper maps are the proper morphisms of ∞-topoi (Lurie, 2009; Martini and Wolf 2023a, 2023b) The acyclic maps are equivalences, and every map is localic.

It is interesting to restrict the fibration of higher sheaves (aka higher etale morphisms) to *n-truncated* sheaves only (the case $n = -1$ recovers the fibration of open immersions). In this case, it is easy to adapt the results of Martini and Wolf to characterize the smooth maps as some *locally n-connected* morphisms of ∞-topoi,[6] and the proper maps are the *n-proper* morphisms of ∞-topoi.[7] The strict smooth maps are the *n*-truncated étale morphisms of topoi. The corresponding acyclic and localic maps define notions of hyper-$(n + 1)$-connected morphisms and $(n + 1)$-localic morphisms, and every geometric morphism should factor into a hyper-$(n + 1)$-connected morphism followed by a $(n + 1)$-localic morphism. Moreover, an ∞-topos X should be *n*-localic in the sense of Lurie (2009, Definition 6.4.5.8) if and only if the morphism $X \rightarrow 1$ is *n*-localic in the previous sense.

Étale maps define a regular calibration of the universe of Topos_∞, and the notions of strictly smooth and strictly proper maps can be defined. By Lemma 14, the strict smooth maps are exactly the etale maps. The strict proper maps are exponentiable maps u whose direct image u_* preserves etale maps. Their characterization is open.

The last example of Table 4 is in algebraic geometry, where the base category is the category Sch of (noetherian) schemes, and the functor $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{CAT}$ is the one sending a scheme X to its category of torsion étale sheaves (Freitag and Kiehl, 1988). The characterization of smooth and proper maps in general is open but Freitag and Kiehl (1988, Theorems 6.1 and 7.3) show that smooth morphisms of schemes are smooth maps (if the torsion is prime to the characteristic) and that proper morphisms of schemes are proper maps. This setting is the one that inspired Grothendieck to name his notions of smooth and proper functors (Maltsiniotis, 2005). The characterization of acyclic and localic maps is an open question.

Notes

1 The names *smooth* and *proper* are taken from Grothendieck in Pursuing Stacks (Maltsiniotis, 2005) and the example of left fibrations of categories. The name *calibration* is borrowed from Bénabou (1975), even if his notion is slightly different.

2 We do not distinguish equivalent categories here.

3 The same example could be presented in the setting of 1-topoi, but the formalism of ∞-topoi is just easier.

4 The reversal of names is due to a preference for covariant or contravariant functors in the definitions. Grothendieck's convention uses presheaves, we have preferred to use covariant functors.

5 Using locales instead of topological spaces simplify the computations. Similar considerations are true for sober spaces.

6 Morphisms u for which the inverse image u^* has a (local) left adjoint $u_!$ when restricted to the subcategories of n-truncated sheaves.

7 Morphisms u whose direct image u_* preserve (internal) filtered colimits of n-truncated objects.

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