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EQUIVARIANT COMPRESSION OF CERTAIN DIRECT LIMIT GROUPS AND AMALGAMATED FREE PRODUCTS

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Abstract. We give a means of estimating the equivariant compression of a group G in terms of properties of open subgroups $G_i \subset G$ whose direct limit is G. Quantifying a result by Gal, we also study the behaviour of the equivariant compression under amalgamated free products $G_1 *_H G_2$ where H is of finite index in both G_1 and G_2.

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1. Introduction. The Haagerup property, which is a strong converse of Kazhdan's property (T), has translations and applications in various fields of mathematics such as representation theory, harmonic analysis, operator K-theory and so on. It implies the Baum–Connes conjecture and related Novikov conjecture [**7**]. We use the following definition of the Haagerup property.

Definition 1.1. A locally compact second countable group G is said to satisfy the **Haagerup property** if it admits a continuous proper affine isometric action α on some Hilbert space $\mathcal{H}$. Here, proper means that for every $M > 0$, there exists a compact set $K \subset G$ such that $\|\alpha(g)(0)\| \geq M$ whenever $g \in G \setminus K$. We say that the action is continuous if the associated map $G \times \mathcal{H} \to \mathcal{H}, (g, v) \mapsto \alpha(g)(v)$ is jointly continuous.

Convention 1.2. *Throughout this paper, all actions are assumed continuous and all groups will be second countable and locally compact.*

Recall that any affine isometric action α can be written as $\pi + b$ where π is a unitary representation of G and where $b : G \to \mathcal{H}, g \mapsto \alpha(g)(0)$ satisfies

$$\forall g, h \in G : \ b(gh) = \pi(g)b(h) + b(g). \tag{1}$$

In other words, b is a 1-cocycle associated to π.

In [**13**], the authors define *compression* as a means to quantify *how strongly* a finitely generated group satisfies the Haagerup property. More generally, assume that G is a compactly generated group. Denote by S some compact generating subset and equip G with the word length metric relative to S. Using the triangle inequality, one checks easily that any 1-cocycle b associated to a unitary action of G on a Hilbert space is Lipschitz. On the other hand, one can look for the supremum of $r \in [0, 1]$ such that there exists $C, D > 0$ with

$$\forall g \in G : \ \frac{1}{C}|g|^r - D \leq \|b(g)\| \leq C|g| + D.$$

DEFINITION 1.3. The above supremum, denoted $R(b)$, is called the compression of b and taking the supremum over all proper affine isometric actions of G on all Hilbert spaces leads to the **equivariant Hilbert space compression** of G, denoted $\alpha_2^{\#}(G)$. Suppose now that G is no longer compactly generated but still has a proper length function. Then, define $\alpha_2^{\#}(G)$ to be the supremum of $R(b)$ but over all *large-scale Lipschitz* 1-cocycles.

The equivariant Hilbert space compression contains information on the group. First of all, if $\alpha_2^{\#}(G) > 0$, then G is Haagerup. The converse was disproved by T. Austin in [**4**], where the author proves the existence of finitely generated amenable groups with equivariant compression 0. Further, it was shown in [**13**] that if for a finitely generated group $\alpha_2^{\#}(G) > 1/2$, then G is amenable. This result was generalized to compactly generated groups in [**9**] and it provides some sort of converse for the well-known fact that amenability implies the Haagerup property. Much effort has been done to calculate the explicit equivariant compression value of several groups and classes of groups, see e.g. [**2**, **5**, **12**, **19**, **20**].

Given two finitely generated group G and H the group $\bigoplus_H G$ is no longer finitely generated. However, we can view $\bigoplus_H G$ as a subspace of $G \wr H$ and so equip $\bigoplus_H G$ with a natural proper metric. In this article, we are motivated by comparing the compression of $\bigoplus_H G$ with $G \wr H$. We assume that a given group G, equipped with a proper length function l, can be viewed as a direct limit of open (hence closed) subgroups $G_1 \subset G_2 \subset G_3 \subset \ldots \subset G$. We equip each G_i with the subspace metric from G. Our main objective will be to find bounds on $\alpha_2^{\#}(G)$ in terms of properties of the G_i. Note that, as each G_i is a metric subspace of G, we have $\alpha_2^{\#}(G) \leq \inf_{i\in\mathbb{N}} \alpha_2^{\#}(G_i)$. The main challenge is to find a sensible lower bound on $\alpha_2^{\#}(G)$. The key property that we introduce is the (α, l, q) polynomial property, which we shorten to (α, l, q)-PP (see Definition 2.5 below). Precisely, we obtain the following result.

THEOREM 1.4. *Let G be a locally compact, second countable group equipped with a proper length function l. Suppose there exists a sequence of open subgroups $(G_i)_{i\in\mathbb{N}}$, each equipped with the restriction of l to G_i, such that $\varinjlim G_i = G$ and $\alpha = \inf\{\alpha_2^{\#}(G_i)\} > 0$. If $(G_i)_{i\in\mathbb{N}}$ has (α, l, q)-PP, then there are the following two cases:*

$$l \geq q \Rightarrow \alpha_2^{\#}(G) \geq \frac{\alpha}{2l+1},$$

or,

$$l \leq q \Rightarrow \alpha_2^{\#}(G) \geq \frac{\alpha}{l+q+1}.$$

We use this result to obtain a lower bound of the compression of the following examples. Let $F\colon [0, 1] \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ be the function

$$F(\alpha, d) = \begin{cases} d(2\alpha - 1) & \text{if } 2\alpha \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 1.5. *Let G and H be finitely generated groups where H has polynomial growth of degree $d \geq 1$. Then,*

$$\alpha_2^{\#}\left(\bigoplus_H G\right) \geq \frac{\alpha_2^{\#}(G)}{1 + F(\alpha_2^{\#}(G), d) + 2\alpha_2^{\#}(G)(1+d)},$$

where $\bigoplus_H G$ is equipped with the subspace metric from $G \wr H$.

Our result also allows to consider spaces $\bigoplus_H G_h$ where G_h actually depends on the parameter $h \in H$. For example, we take a collection of finite groups F_i with $F_0 = \{0\}$ and look at $G = \bigoplus_{i \in \mathbb{N}} F_i$. This is the first available lower bound for the equivariant compression of groups of this type.

THEOREM 1.6. *Let $\{F_i\}_{i \in \mathbb{N}}$ be a collection of finite groups. Equip $G = \bigoplus_{i \in \mathbb{N}} F_i$ with the length function $l(g) = \min\left\{n \in \mathbb{N} : g \in \oplus_{i=0}^n F_i\right\}$. Then, $\alpha_2^{\#}(G, l) > 1/3$.*

We give a proof of Theorem 1.4 in Section 2.2 and apply to these concrete examples in Section 2.3. Note that our result can also be viewed as a study of the behaviour of equivariant compression under direct limits. The behaviour of the Haagerup property and the equivariant compression under group constructions has been studied extensively (see e.g. [**11**, **18**], Chapter 6 of [**1**, **7**, **8**]).

In Section 3, we quantify part of [**12**] to study the behaviour of the equivariant compression under certain amalgamated free products $G_1 *_H G_2$ where H is of finite index in both G_1 and G_2. Suppose H is a closed finite index subgroup inside groups compactly generated groups G_1 and G_2 and there exists proper affine isometric actions $\beta_i \colon G_i \to \mathrm{Aff}(V_i)$ on Hilbert spaces V_i. In [**12**], the author shows that if there exists a non-trivial closed subspace $W \subset V_1 \cap V_2$ that is fixed by the restricted actions $\beta_i|_H$ then the product $G_1 *_H G_2$ also admits a proper affine isometric action on a Hilbert space. We quantify this result.

THEOREM 1.7. *With the above assumptions $\alpha_2^{\#}(G_1 *_H G_2) \geq \frac{\alpha_2^{\#}(H)}{2}$*

2. The equivariant compression of direct limits of groups

2.1. Preliminaries and formulation of the main result. Suppose G is a locally compact second countable group equipped with a proper length function l, i.e. closed l-balls are compact. Assume that there exists a sequence of open subgroups $G_i \subset G$ such that $\varinjlim G_i = G$, i.e. G is the direct limit of the G_i. We equip each G_i with the restriction of l to G_i. It will be our goal to find bounds on $\alpha_2^{\#}(G)$ in terms of the $\alpha_2^{\#}(G_i)$. Clearly, as the G_i are subgroups then an upper bound of the equivariant compression is the infimum of the equivariant compressions of the G_i. The challenge is to find a sensible lower bound. The next example will show that it is not enough to only consider the $\alpha_2^{\#}(G_i)$.

EXAMPLE 2.1. Consider the wreath product $\mathbb{Z} \wr \mathbb{Z}$ equipped with the standard word metric relative to $\{(\delta_1, 0), (0, 1)\}$, where δ_1 is the characteristic function of $\{0\}$. Let $\mathbb{Z}^{(\mathbb{Z})} = \{f \colon \mathbb{Z} \to \mathbb{Z} : f \text{ is has finite support}\}$ be equipped with the subspace metric from $\mathbb{Z} \wr \mathbb{Z}$. Consider the direct limit of groups

$$\mathbb{Z} \hookrightarrow \mathbb{Z}^3 \hookrightarrow \mathbb{Z}^5 \cdots \hookrightarrow \mathbb{Z}^{(\mathbb{Z})}$$

where $\mathbb{Z}^{2n+1}$ has the subspace metric from $\mathbb{Z}^{(\mathbb{Z})}$. This metric is quasi-isometric to the standard word metric on $\mathbb{Z}^{2n+1}$ and so each term has equivariant compression 1. So $\mathbb{Z}^{(\mathbb{Z})}$ is a direct limit of groups with equivariant compression 1 but by **[2]** has equivariant compression less than 3/4. On the other hand the sequence

$$\mathbb{Z} \to \mathbb{Z} \to \cdots \to \mathbb{Z},$$

is a sequence of groups with equivariant compression 1 and the equivariant compression of the direct limit is 1. □

Given a sequence of 1-cocycles b_i of G_i, then in order to predict the equivariant compression of the direct limit, it will be necessary to incorporate more information on the growth behaviour of the b_i than merely the compression exponent $R(b_i)$. The growth behaviour of 1-cocycles can be completely caught by so called *conditionally negative definite functions* on the group (See Proposition 2.3 and Theorem 2.4 below).

Definition 2.2. A continuous map $\psi : G \to \mathbb{R}^+$ is called *conditionally negative definite* if $\psi(g) = \psi(g^{-1})$ for every $g \in G$ and if for all $n \in \mathbb{N}$, $\forall g_1, g_2, \dots, g_n \in G$ and all $a_1, a_2, \dots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$, we have

$$\sum_{i,j} a_i a_j \psi(g_i^{-1} g_j) \leq 0.$$

Proposition 2.3 (Example 13, page 62 of **[10]**). *Let $\mathcal{H}$ be a Hilbert space and $b\colon G \to \mathcal{H}$ a 1-cocycle associated to a unitary representation. Then, the map $\psi : G \to \mathbb{R}$, $g \mapsto \|b(g)\|^2$ is a conditionally negative definite function on G.*

Theorem 2.4 (Proposition 14, page 63 of **[10]**). *Let $\psi : G \to \mathbb{R}$ be a conditionally negative definite function on a group G. Then, there exists an affine isometric action α on a Hilbert space $\mathcal{H}$ such that the associated 1-cocycle satisfies $\psi(g) = \|b(g)\|^2$.*

These two results imply that we can pass between conditionally negative definite functions and 1-cocycles associated to unitary actions.

Definition 2.5. Let G be a group equipped with a proper length function l and suppose that $(G_i)_{i\in\mathbb{N}}$ is a normalized nested sequence of open subgroups such that $\varinjlim G_i = G$. Assume that $\alpha := \inf_{i\in\mathbb{N}} \alpha_2^\#(G_i) \in (0, 1]$ and $l, q \geq 0$. The sequence $(G_i)_i$ has the (α, l, q)-*polynomial property* ((α, l, q)-PP) if there exists:

(1) a sequence $(\eta_i)_i \subset \mathbb{R}^+$ converging to 0 such that $\eta_i < \alpha$ for each $i \in \mathbb{N}$,
(2) $(A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$,
(3) a sequence of 1-cocycles $(b_i\colon G_i \to \mathcal{H}_i)_{i\in\mathbb{N}}$, where each b_i is associated to a unitary action π_i of G_i on a Hilbert space $\mathcal{H}_i$

such that

$$\frac{1}{A_i}|g|^{2\alpha-\eta_i} - B_i \leq \|b_i(g)\|^2 \leq A_i|g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}$$

and there is $C, D > 0$ such that $A_i \leq Ci^l$, $B_i \leq Di^q$ for all $i \in \mathbb{N}$.

Note that the only real restrictions are the inequalities $A_i \leq Ci^l$, $B_i \leq Di^q$: we exclude sequences A_i, B_i that grow faster than any polynomial. The intuition is that equivariant compression is a polynomial property (this follows immediately from its

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definition), so that sequences A_i, B_i growing faster than any polynomial would be too dominant and one would lose all hope of obtaining a lower bound on $\alpha_2^{\#}(G)$. On the other hand, if the A_i and B_i grow polynomially, then one can use compression to somehow compensate for this growth. One then obtains a strictly positive lower bound on $\alpha_2^{\#}(G)$ which may decrease depending on how big l and q are. We have the following useful characterisation of (α, l, q)-polynomial property.

LEMMA 2.6. *Let G be a locally compact second countable group and l is a proper length metric. Suppose there exists a sequence of open subgroups* $(G_i)_{i\in\mathbb{N}}$ *such that* $\varinjlim G_i = G$. *If each* G_i *are equipped with the restricted length metric from G then* $(G_i)_{i\in\mathbb{N}}$ *has the* (α, l, q)*-polynomical property if and only if there exists* $C, D > 0$ *such that for all* $\varepsilon > 0$ *there exists*

(1) *a sequence* $(A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$ *such that* $A_i \leq Ci^l$ *and* $B_i \leq Di^q$*;*
(2) *a sequence of 1-cocycles* $(b_i\colon G_i \to \mathcal{H}_i)_{i\in\mathbb{N}}$

such that

$$\frac{1}{A_i}|g|^{2\alpha-\varepsilon} - B_i \leq \|b_i(g)\|^2 \leq A_i|g|^2 + B_i \quad \forall g \in G_i, \forall i \in \mathbb{N}.$$

Proof. The "if" direction is obvious. For the "only if" direction fix $\varepsilon > 0$ and suppose $(G_i)_{i\in\mathbb{N}}$ has the (α, l, q)-polynomial property with respect to sequences $(\eta_i)_{i\in\mathbb{N}}$ and $(b_i\colon G_i \to \mathcal{H}_i)_{i\in\mathbb{N}}$. Choose $N \in \mathbb{N}$ large enough so that $\eta_k < \varepsilon$ for all $k \geq N$. Thus, $b_k\colon G_k \to \mathcal{H}_k$ satisfies the above conditions for all $k \geq N$. For $k \leq N$ we take the restriction of b_N to G_k to obtain the sequence satisfying the above conditions for all $k \in \mathbb{N}$. □

PROPOSITION 2.7. *Let G be a locally compact second countable group and suppose there exists a sequence of open subgroups* $(G_i)_{i\in\mathbb{N}}$ *such that* $\varinjlim G_i = G$. *If* $\alpha := \alpha_2^{\#}(G) > 0$ *then* $(G_i)_{i\in\mathbb{N}}$ *has* $(\alpha, 0, 0)$*-polynomical property.*

Proof. For all $0 < \varepsilon < \alpha$ there exists a 1-cocycle b such that

$$\frac{1}{A}|g|^{\alpha-\varepsilon} - B \leq \|b(g)\| \quad \forall g \in G.$$

The restriction of b to each G_i is a 1-cocycle and gives $(G_i)_{i\in\mathbb{N}}$ the $(\alpha, 0, 0)$-polynomial property. □

Combining this with Theorem 1.4 we have the following consequence which confirms our intuition.

COROLLARY 2.8. *Let G be a locally compact second countable group with a proper length function l. If there exists a sequence of open subgroups* $(G_i)_{i\in\mathbb{N}}$ *such that* $\varinjlim G_i = G$ *then* $(G_i)_{i\in\mathbb{N}}$ *has the* (α, l, q)*-polynomial property for some* $\alpha \in (0, 1]$ *and* $l, q \geq 0$ *if and only if* $\alpha_2^{\#}(G) > 0$.

2.2. The proof of Theorem 1.4

Proof of Theorem 1.4. First, we can assume that l is uniformly discrete. That is there exists a $c > 0$ such that $l(x) > c$ for all $x \in G \setminus \{e\}$. This is because given a length function l one can define a new length function l' such that $l'(x) = 1$ whenever

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$0 < l(x) \leq 1$ and $l'(x) = l(x)$ when $l(x) \geq 1$. Hence l' will be quasi-isometric to l and so will not change the compression of G or G_i.

Take sequences $(\psi_i \colon G_i \to \mathbb{R})_{i\in\mathbb{N}}$, $(\eta_i)_i$ and $(A, B) = (A_i, B_i)_{i\in\mathbb{N}} \subset \mathbb{R}^{>0} \times \mathbb{R}^{\geq 0}$ satisfying the conditions of (α, l, q)-PP (see Definition 2.5). We assume here, without loss of generality, that the sequences $(A_i)_i$, $(B_i)_i$ are non-decreasing.

For each G_i, define a sequence of maps $(\varphi_k^i \colon G_i \to \mathbb{R})_{k\in\mathbb{N}}$ by

$$\varphi_k^i(g) = \begin{cases} \exp\left(\frac{-\psi_i(g)}{k}\right) & \text{if } g \in G_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that each φ_k^i is continuous as G_i is open and also closed, being the complement of $\cup_{g\notin G_i} gG_i$. By (α, l, q)-PP, for all $i, k \in \mathbb{N}$, we have

$$\exp\left(\frac{-A_i|g|^2 - B_i}{k}\right) \leq \varphi_k^i(g) \quad \forall g \in G_i, \text{ and}$$

$$\varphi_k^i(g) \leq \exp\left(\frac{-|g|^{2\alpha-\eta_i} + A_iB_i}{A_ik}\right) \quad \forall g \in G.$$

Fix some $p > 0$, set $J(i) = (A_i + B_i)i^{1+p}$ and define $\overline{\psi} \colon G \to \mathbb{R}$ by

$$\overline{\psi}(g) = \sum_{i\in\mathbb{N}} 1 - \Phi_i(g),$$

where $\Phi_i(g) := \varphi_{J(i)}^i(g)$. To check that $\overline{\psi}$ is well defined, choose any $g \in G$ and note that for $i > |g|$, we have $g \in G_i$ and so $\varphi_k^i(g) \geq \exp(\frac{-A_i|g|^2 - B_i}{k})$. Hence

$$\begin{aligned} \sum_{i>|g|} 1 - \Phi_i(g) &\leq \sum_{i>|g|} 1 - \exp\left(\frac{-A_i|g|^2 - B_i}{(A_i + B_i)i^{1+p}}\right) \\ &\leq \sum_{i>|g|} 1 - \exp\left(\frac{-|g|^2}{i^{1+p}}\right) \\ &\leq \sum_{i>|g|} \frac{|g|^2}{i^{1+p}} = |g|^2 \sum_{i>|g|} \frac{1}{i^{1+p}} \end{aligned}$$

As

$$\overline{\psi}(g) = \sum_{i=1}^{|g|} 1 - \Phi_i(g) + \sum_{i>|g|} 1 - \Phi_i(g),$$

we see that $\overline{\psi}$ is well defined and that it can be written as a limit of continuous functions converging uniformly over compact sets. Consequently, it is itself continuous. By Schoenberg's theorem (see [**10**, Theorem 5.16]), all of the maps φ_k^i are positive definite on G_i and hence on G (see [**15**, Section 32.43(a)]). In other words,

$$\forall n \in \mathbb{N},\ \forall a_1, a_2, \ldots, a_n \in \mathbb{R}, \forall g_1, g_2, \ldots, g_n \in G : \sum_{i,j=1}^{n} a_ia_j\varphi_k^i(g_i^{-1}g_j) \geq 0.$$

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Hence, $\overline{\psi}$ is a conditionally negative definite map. Moreover, using that l is uniformly discrete, we can find a constant $E > 0$ such that

$$\overline{\psi}(g) \leq |g| + |g|^2 \sum_{i>|g|} \frac{1}{i^{1+p}} \leq E|g|^2, \tag{2}$$

so the 1-cocycle associated to $\overline{\psi}$ via Theorem 2.4 is large-scale Lipschitz.

Let us now try to find the compression of this 1-cocycle. Set $VI \colon \mathbb{N} \to \mathbb{R}$ to be the function

$$VI(i) = (A_i J(i) \ln(2) + A_i B_i)^{\frac{1}{2\alpha - \eta_i}}.$$

One checks easily that

$$|g| \geq VI(i) \Rightarrow \Phi_i(g) = \varphi^i_{J(i)}(g) \leq \frac{1}{2}. \tag{3}$$

To make the function VI more concrete, let us look at the values of A_i, B_i and $J(i)$. Recall that by assumption, we have $A_i \leq Ci^l, B_i \leq Di^q$. Hence for i sufficiently large, we have $J(i) \leq (Ci^l + Di^q)i^{1+p} \leq Fi^X$ where F is some constant and $X = 1 + p + \max(l, q)$. We thus obtain that there is a constant $K > 0$ such that for every i sufficiently large (say $i > I$ for some $I \in \mathbb{N}_0$),

$$VI(i) \leq Ki^{Y/(2\alpha - \eta_i)},$$

where

$$\begin{aligned} Y &= \max(X + l, l + q), \\ &= \max(1 + p + 2l, 1 + p + l + q). \end{aligned}$$

As the sequence η_i converges to 0, we can choose any $\delta > 0$ and take $I > 0$ such that in addition $\eta_i < \delta$ for $i > I$. We then have for all $i > I$ that

$$VI(i) \leq Ki^{Y/(2\alpha - \delta)}.$$

Together with equation (3), this implies that for $i > I$,

$$|g| \geq Ki^{Y/(2\alpha - \delta)} \Rightarrow \Phi_i(g) = \varphi^i_{J(i)}(g) \leq \frac{1}{2}. \tag{4}$$

For every $g \in G$, set

$$c(g)_{p,\delta} = \sup \left\{ i \in \mathbb{N} \mid Ki^{Y/(2\alpha - \delta)} \leq |g| \right\}.$$

We then have for every $g \in G$ with $|g|$ large enough, that

$$\begin{aligned} \overline{\psi}(g) &\geq \sum_{i=1}^{c(g)_{p,\delta}} 1 - \varphi^i_{J(i)}(g), \\ &\geq \sum_{i=I+1}^{c(g)_{p,\delta}} 1/2 = \frac{c(g)_{p,\delta} - I}{2}. \end{aligned}$$

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As $c(g)_{p,\delta} \geq (\frac{|g|}{K})^{(2\alpha-\delta)/Y} - 1$, we conclude that $R(b) \geq \frac{2\alpha-\delta}{2\max(1+p+2l,1+p+l+q)}$. As this is true for any small $p, \delta > 0$, we can take the limit for $p, \delta \to 0$ to obtain $\alpha_2^\#(G) \geq \frac{\alpha}{\max(1+2l,1+l+q)}$. Hence, we have the following two cases:

$$l \geq q \Rightarrow \alpha_2^\#(G) \geq \frac{\alpha}{1+2l},$$

or,

$$l \leq q \Rightarrow \alpha_2^\#(G) \geq \frac{\alpha}{l+q+1}.$$

□

2.3. Examples

Let $F\colon [0,1] \times \mathbb{R}^{\geq 0} \to \mathbb{R}$ be the function

$$F(\alpha, d) = \begin{cases} d(2\alpha - 1) & \text{if } 2\alpha \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 2.9. *Let G and H be finitely generated groups where H has polynomial growth of degree $d \geq 1$. Then,*

$$\alpha_2^\#\left(\bigoplus_H G\right) \geq \frac{\alpha_2^\#(G)}{1 + F(\alpha_2^\#(G), d) + 2\alpha_2^\#(G)(1+d)},$$

where $\bigoplus_H G$ is equipped with the subspace metric from $G \wr H$.

REMARK 2.10. Theorem 1.3. from [**17**] provides a lower bound to the compression of $G \wr H$. Under the assumptions in Theorem 2.9, Theorem 1.3. in [**17**] gives a lower bound $\alpha_2^\#(G \wr H) \geq \alpha_1^\#(G)/2$. As this bound is in terms of L^1-compression, this makes comparison between the bound in Theorem 2.9 and [**17**, Theorem 1.3.] difficult. However, it is known that $\alpha_2^\#(G) \leq \alpha_1^\#(G) \leq 2\alpha_2^\#(G)$ for all finitely generated groups G, see the proof of Theorem 1.1. and Theorem 1.3. in [**17**] and [**18**, Lemma 2.3.].

We use this to show that under some circumstances the above lower bound is larger than the bound provided in [**17**, Theorem 1.3.]. Suppose that $\alpha_1^\#(G)/2 < \alpha_2^\#(G)$. Then, there exists a $c > 0$ such that $\frac{2\alpha_2^\#(G)}{\alpha_1^\#(G)} > 1 + c$. If $\alpha_2^\#(G) \leq \min\left\{\frac{c}{2(1+d)}, 1/2\right\}$ then by Theorem 2.9

$$\alpha_2^\#(\oplus_H G) \geq \frac{\alpha_2^\#(G)}{1+c} > \frac{\alpha_1^\#(G)}{2}.$$

Unfortunately, the values of $\alpha_2^\#$ are not so well understood and at the time of writing the only know values for $\alpha_2^\#$ are 1, 1/2, 0 and $\frac{1}{2-2^{1-k}}$ for $k \in \mathbb{N}$ [**2**, **4**, **18**]. In the non-equivariant case any value for compression can be achieved [**3**]. It is likely that there exists groups such that $\alpha_2^\#$ takes values strictly between 0 and 1/2 in which case our theorem can be applied to provide larger lower bounds than $\alpha_1^\#(G)/2$.

Proof. We consider $\bigoplus_H G$ to be the group of functions $\mathbf{f}\colon H \to G$ that have finite support. Let $\mathbf{f} \in \bigoplus_H G$ and let $\mathrm{Supp}(\mathbf{f}) = \{h_1, \ldots, h_n\} \subset H$. Set the length of $\mathbf{f}$ as

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follows

$$|\mathbf{f}|_{G\wr H} = \inf_{\sigma \in S_n} \left(d_H(1, h_{\sigma(1)}) + \sum_{i=1}^{n} d_H(h_{\sigma(i)}, h_{\sigma(i+1)}) + d_H(h_{\sigma(n)}, 1) \right) + \sum_{h \in H} |\mathbf{f}(h)|_G.$$

This is the induced length metric from $G \wr H$ and so this is a proper length function on $\bigoplus_H G$. Consider the following group

$$G_i = \{\mathbf{f} \colon H \to G : \operatorname{Supp}(\mathbf{f}) \subset B(1, i)\},$$

and set $n_i = |B(1, i)|$. Each G_i is finitely generated and the restricted wreath metric to G_i is proper and left invariant so the wreath metric and the word metric are quasi-isometric. In particular

$$|\mathbf{f}|_{G\wr H} - 2i|B(1, i)| \leq \sum_{h \in B(1,i)} |\mathbf{f}(h)|_G \leq |\mathbf{f}|_{G\wr H},$$

for all $\mathbf{f} \in G_i$. By [**14**, Proposition 4.1. and Corollary 2.13.] it follows that $\alpha_2^{\#}(G_i) = \alpha_2^{\#}(G)$ for all $i \in \mathbb{N}$. Set $0 < \alpha < \alpha_2^{\#}(G)$ and consider a 1-cocyle $b \colon G \to \mathcal{H}$ such that

$$\frac{1}{C}|g|_G^{2\alpha} \leq \|b(g)\|^2 \leq C|g|_G^2.$$

Enumerate $B(1, i)$ so that $\{h_1, \ldots, h_{n_i}\} = B(1, i)$ and define a 1-cocycle $b_i \colon G_i \to \mathcal{H}^{n_i}$, where $b_i(\mathbf{f}) = (b(\mathbf{f}(h_1)), \ldots, b(\mathbf{f}(h_{n_i})))$. If $|\mathbf{f}|_{G\wr H} > 4i|B(1, i)|$, then

$$\|b_i(\mathbf{f})\|_{1/\alpha} = \left(\sum_{j=1}^{i} \left\| b(\mathbf{f}(h_{n_j})) \right\|^{1/\alpha} \right)^{\alpha} \geq \frac{1}{C^{1/\alpha}} \left(\sum_{j=1}^{i} |\mathbf{f}(h_{n_j})|_G \right)^{\alpha} \geq \frac{1}{C^{1/\alpha}} \left(|\mathbf{f}|_{G\wr H} - 2i|B(1, i)| \right)^{\alpha} \geq \frac{1}{2C^{1/\alpha}} |\mathbf{f}|_{G\wr H}^{\alpha}.$$

If $2\alpha < 1$ then $\|b_i(\mathbf{f})\|_2 \geq \|b_i(\mathbf{f})\|_{1/\alpha}$ for all $\mathbf{f} \in G_i$ and so it follows that

$$\frac{1}{4C^{2/\alpha}} |\mathbf{f}|_{G\wr H}^{2\alpha} - \frac{i^{2\alpha}}{C} |B(1, i)|^{2\alpha} \leq \|b_i(\mathbf{f})\|_2^2,$$

for all $\mathbf{f} \in G_i$. Hence $(G_i)_{i \in \mathbb{N}}$ has the $(\alpha, 0, 2\alpha(1 + d))$ polynomial property.

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If $2\alpha \geq 1$ then by Hölder's inequality $\|b_i(\mathbf{f})\|_2 \geq n_i^{\frac{1-2\alpha}{2}} \|b_i(\mathbf{f})\|_{1/\alpha}$ for all $\mathbf{f} \in G_i$ and so it follows that

$$\frac{1}{4C^{2/\alpha}|B(1,i)|^{2\alpha-1}}|\mathbf{f}|^{2\alpha}_{G\wr H} - \frac{i^{2\alpha}}{C}|B(1,i)|^{2\alpha} \leq \|b_i(\mathbf{f})\|_2^2 .$$

for all $\mathbf{f} \in G_i$. Hence $(G_i)_{i\in\mathbb{N}}$ has the $(\alpha, d(2\alpha-1), 2\alpha(1+d))$ polynomial property. Thus by Theorem 1.4 and that $\alpha, d \geq 0$ it follows that

$$\alpha_2^{\#}(\bigoplus_H G) \geq \frac{\alpha}{1+F(\alpha,d)+2\alpha(1+d)},$$

for all $\alpha < \alpha_2^{\#}(G)$ and so the statement of the theorem holds. □

THEOREM 2.11. *Let* $\{F_i\}_{i\in\mathbb{N}}$ *be a collection of finite groups such that* $F_0 = \{1\}$. *Let* $G = \bigoplus_{i\in\mathbb{N}} F_i$ *be equip with the proper length function* $l(g) = \min\left\{n \in \mathbb{N} : g \in \oplus_{i=0}^n F_i\right\}$. *Then* $\alpha_2^{\#}(G) \geq 1/3$.

Proof. Set $G_i = \bigoplus_{j=0}^i F_j$ and observe that $\alpha_2^{\#}(G_i) = 1$ as G_i is finite for all $i \in \mathbb{N}$. Define $f_i \colon G_i \to \mathbb{R}$ to be the 0-map. This is clearly a 1-cocycle and satisfies

$$\forall g \in G_i : \ l(g)^2 - i^2 \leq |f_i(g)|^2 \leq l(g)^2 + i^2.$$

Hence $(G_i)_{i\in\mathbb{N}}$ has the (1,0,2)-polynomial property. Thus $\alpha_2^{\#}(G) \geq 1/3$. □

EXAMPLE 2.12. We will use [**3**] to provide an example of a sequence that does not have (α, l, q)-polynomial property for any $\alpha \in (0,1]$ and $l, q > 0$. Let Π_k, $k \geq 1$ be a sequence of Lafforgue expanders that do not embed into any uniformly convex Banach space [**16**]. These are finite factor groups M_k of a lattice Γ of $\mathrm{SL}_3(F)$ for a local field F.

For every $\alpha \in [0,1]$ there exists a finitely generated group G and a sequence of scaling constants λ_k such that $\lambda_k\Pi_k$ has compression α and G is quasi-isometric to $\lambda_k\Pi_k$. Furthermore, G contains the free product $*_k M_k$ as a subgroup. Let $\alpha = 0$ and let G and the scaling constants λ_k be such that G has compression 0. We can equip $*_k M_k$ with a proper left invariant metric coming from G. Hence we have a sequence

$$M_1 \hookrightarrow M_1 * M_2 \hookrightarrow \cdots \hookrightarrow *_{k=1}^n M_k \hookrightarrow \cdots \hookrightarrow *_k M_k.$$

For each $n > 0$, $*_{k=1}^n M_k$ has equivariant compression $1/2$ [**11**, Theorem 1.4.] however the limit group $*_k M_k$ contains a quasi-isometric copy of $\lambda_k\Pi_k$ and so has compression 0. Thus, this sequence cannot have the (α, l, q)-polynomial property for any $\alpha \in (0,1]$ and $l, q > 0$.

3. The behaviour of compression under free products amalgamated over finite index subgroups.

It is known that the Haagerup property is not preserved under amalgamated free products. Indeed, $(SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2, \mathbb{Z}^2)$ has the relative property (T). So $SL_2(\mathbb{Z}) \rtimes \mathbb{Z}^2 = (\mathbb{Z}_6 \rtimes \mathbb{Z}^2) *_{(\mathbb{Z}_2 \rtimes \mathbb{Z}^2)} (\mathbb{Z}_4 \rtimes \mathbb{Z}^2)$ is not Haagerup. In [**12**], S.R. Gal proves the following result.

THEOREM 3.1. *Let* G_1 *and* G_2 *be finitely generated groups with the Haagerup property that have a common finite index subgroup* H. *For each* $i = 1, 2$, *let* β_i *be a proper affine isometric action of* G_i *on a Hilbert space* $V_i (= l^2(\mathbb{Z}))$. *Assume that* $W < V_1 \cap V_2$ *is*

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invariant under the actions $(\beta_i)_{|H}$ and moreover that both these (restricted) actions coincide on W. Then, $G_1 *_H G_2$ is Haagerup.

Under the same conditions as above, we want to give estimates on $\alpha_2^\#(G_1 *_H G_2)$ in terms of the equivariant Hilbert space compressions of G_1, G_2 (see Theorem 3.3 below). Note that the following lemma shows that $\alpha_2^\#(G_1) = \alpha_2^\#(H) = \alpha_2^\#(G_2)$ when H is of finite index in both G_1 and G_2. We are indebted to Alain Valette for this lemma and its proof. The notation $\alpha_p^\#$ refers to the equivariant L_p**-compression** for some $p \geq 1$. It is defined in exactly the same way as $\alpha_2^\#$ except that one considers affine isometric actions on L_p-spaces instead of L_2-spaces.

LEMMA 3.2. *Let G be a compactly generated, locally compact group, and let H be an open, finite-index subgroup of G. Then, $\alpha_p^\#(H) = \alpha_p^\#(G)$.*

Proof. As H is embedded H-equivariantly, quasi-isometrically in G, we have $\alpha_p^\#(H) \geq \alpha_p^\#(G)$. To prove the converse inequality, we may assume that $\alpha_p^\#(H) > 0$. Let S be a compact generating subset of H. Let $A(h)v = \pi(h)v + b(h)$ be an affine isometric action of H on L^p, such that for some $\alpha < \alpha_p^\#(H)$ we have $\|b(h)\|_p \geq C|h|_S^\alpha$, for every $h \in H$. Now, we induce up the action A from H to G, as on p. 91 of **[6]**[1]. The affine space of the induced action is

$$E := \{f : G \to L^p : f(gh) = A(h)^{-1}f(g),\ \forall h \in H \text{ and almost every } g \in G\},$$

with distance given by $\|f_1 - f_2\|_p^p = \sum_{x\in G/H} \|f_1(x) - f_2(x)\|_p^p$. The induced affine isometric action $\tilde{A}$ of G on E is then given by $(\tilde{A}(g))f(g') = f(g^{-1}g')$, for $f \in E$, $g, g' \in G$.

A function $\xi_0 \in E$ is then defined as follows. Let $s_1 = e, s_2, , ..., s_n$ be a set of representatives for the left cosets of H in G. Set $\xi_0(s_ih) = b(h^{-1})$, for $h \in H,\ i = 1, \ldots, n$. Define the 1-cocycle $\tilde{b}$ on G by $\tilde{b}(g) = \tilde{A}(g)\xi_0 - \xi_0$, for $g \in G$. For an $h \in H$, we then have:

$$\|\tilde{b}(h)\|_p^p = \sum_{i=1}^{n} \|\xi_0(h^{-1}s_i) - \xi_0(s_i)\|_p^p = \sum_{i=1}^{n} \|\xi_0(h^{-1}s_i)\|_p^p \geq \|\xi_0(h^{-1})\|_p^p = \|b(h)\|_p^p.$$

Set $K = \max_{1\leq i\leq n} \|\tilde{b}(s_i)\|_p$. Take $T = S \cup \{s_1, \ldots, s_n\}$ as a compact generating set of G. For $g \in G$, write $g = s_ih$ for $1 \leq i \leq n,\ h \in H$. Then,

$$\begin{aligned}\|\tilde{b}(g)\|_p &\geq \|\tilde{b}(h)\|_p - K \geq \|b(h)\|_p - K \geq C|h|_S^\alpha - K \geq C|h|_T^\alpha - K\\ &\geq C(|g|_T - 1)^\alpha - K \geq C'|g|_T^\alpha - K'.\end{aligned}$$

So the compression of the 1-cocycle $\tilde{b}$ is at least α, hence $\alpha_p^\#(G) \geq \alpha_p^\#(H)$. □

The following proof uses a construction by S.R. Gal, see page 4 of **[12]**.

THEOREM 3.3. *Let V_1 and V_2 be closed subspaces of a Hilbert space. Suppose H is a finite index subgroup of G_1 and G_2 and suppose there are proper affine isometric actions β_i (with compression α_i) of each G_i on V_i. Assume that $W < V_1 \cap V_2$ is invariant under*

[1] We seize this opportunity to correct a misprint in the definition of the vector ξ_0 in that construction in p. 91 of **[6]**.

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the actions $(\beta_i|_H)$ *and moreover that both these (restricted) actions coincide on* W. *Then,* $\alpha_2^{\#}(G_1 *_H G_2) \geq \frac{\min(\alpha_1,\alpha_2)}{2}$. *In particular,* $\alpha_2^{\#}(G_1 *_H G_2) \geq \frac{\alpha_2^{\#}(H)}{2}$.

Proof. Following [**12**], let us build a Hilbert space W_Γ on which $\Gamma = G_1 *_H G_2$ acts affinely and isometrically. Let ω be a finite alternating sequence of 1's and 2's and suppose π is a linear action of H on some Hilbert space denoted $\mathcal{H}_\omega$. One can induce up the linear action from H to G_i, obtaining a Hilbert space

$$V := \left\{f\colon G_i \to \mathcal{H}_\omega \mid \forall h \in H,\ f(gh) = \pi(h^{-1})f(g)\right\}$$

and an orthogonal action $\pi_i\colon G_i \to \mathcal{O}(V)$ defined by $\pi_i(g)f(g') = f(g^{-1}g')$. The subspace

$$\left\{f\colon G_i \to \mathcal{H}_\omega \mid \forall h \in H,\ f(h) = \pi(h^{-1})f(1), f_{|G_i\setminus H} = 0\right\},$$

can be identified with $\mathcal{H}_\omega$ by letting an element f correspond to $f(1)$. It is clear that the action π_i restricted to H coincides with the original linear action π via this identification.

So, starting from any linear H-action on a Hilbert space $\mathcal{H}_\omega$, we can obtain a linear action of say G_1 on a Hilbert space that can be written as $\mathcal{H}_\omega \oplus \mathcal{H}_{1\omega}$ for some $\mathcal{H}_{1\omega}$. We can restrict this action to a linear H-action on $\mathcal{H}_{1\omega}$ and we can lift this to an action of G_2 on a space $\mathcal{H}_{1\omega} \oplus \mathcal{H}_{21\omega}$ and so on, repeating the process indefinitely. Here, we will execute this infinite process twice.

The first linear H-action on which we apply the process is obtained as follows. As $\beta_i(H)(W) = W$ for each $i = 1, 2$, the restriction to H of β_1, gives naturally a linear H-action on $\mathcal{H}_1 := V_1/W$. The second linear H-action is obtained by similarly noting that the restriction to H of β_2 gives a linear H-action on $\mathcal{H}_2 := V_2/W$. We then apply the above process indefinitely.

$$\mathcal{H}_1^\bullet := \overbrace{\mathcal{H}_1 \oplus \underbrace{\mathcal{H}_{21}}^{G_2\curvearrowright} \oplus \mathcal{H}_{121}}_{G_1\curvearrowright} \oplus \overbrace{\mathcal{H}_{2121} \oplus \cdots}^{G_2\curvearrowright}, \qquad \mathcal{H}_2^\bullet := \overbrace{\mathcal{H}_2 \oplus \underbrace{\mathcal{H}_{12}}^{G_1\curvearrowright} \oplus \mathcal{H}_{212}}_{G_2\curvearrowright} \oplus \overbrace{\mathcal{H}_{1212} \oplus \cdots}^{G_1\curvearrowright},$$

where for ω a sequence of alternating 1's and 2's, G_i acts on $\mathcal{H}_\omega \oplus \mathcal{H}_{i\omega}$. Note that there are two H-actions on $\mathcal{H}_1^\bullet$ as H acts on the first term $\mathcal{H}_1$. One can verify that both H-actions coincide (this fact is also mentioned in [**12**],page 4). The same is true for $\mathcal{H}_2^\bullet$.

Denote $\mathcal{H}_1^\circ = \mathcal{H}_1^\bullet \ominus \mathcal{H}_1$ and similarly, set $\mathcal{H}_2^\circ = \mathcal{H}_2^\bullet \ominus \mathcal{H}_2$. We denote

$$W_\Gamma = W \oplus \mathcal{H}_1^\bullet \oplus \mathcal{H}_2^\bullet = V_1 \oplus \mathcal{H}_1^\circ \oplus \mathcal{H}_2^\bullet = V_2 \oplus \mathcal{H}_2^\circ \oplus \mathcal{H}_1^\bullet.$$

The above formula, which writes W as a direct sum in three distinct ways, shows that both G_1 and G_2 act on W_Γ. As mentioned before, the actions coincide on H and so we obtain an affine isometric action of Γ on W_Γ. Note that the corresponding 1-cocycle, when restricted to G_1 (or G_2), coincides with the 1-cocycle of β_1(or β_2).

We inductively define a length function $\psi_T\colon \Gamma \to \mathbb{N}$ by $\psi_T(h) = 0$ for all $h \in H$ and $\psi_T(\gamma) = \min\{\psi_T(\eta) + 1 \mid \gamma = \eta g,\ \text{where } g \in G_1 \cup G_2\}$. By applying Proposition 2 in [**10**] to the Bass–Serre tree of $G_1 *_H G_2$, we see that this map is conditionally negative definite and thus the normed square of a 1-cocycle associated to an affine isometric action of Γ on a Hilbert space.

Let ψ_Γ be the conditionally negative definite function associated to the action of Γ on W_Γ. We now find the compression of the conditionally negative definite map

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$\psi = \psi_\Gamma + \psi_T$. First set

$$M = \max\left\{|t^i_j|_{G_i} : i = 1, 2 \text{ and } 1 \leq j \leq [G_i : H]\right\},$$

where t^i_j are right coset representatives of H in G_i such that $t^i_1 = 1_{G_i}$ for $i = 1, 2$.

Denote $\alpha = \min(\alpha_1, \alpha_2)$ and fix some $\varepsilon > 0$ arbitrarily small. Let $\gamma \in \Gamma$ and suppose in normal form $\gamma = gt^{i_1}_{j_1} \cdots t^{i_k}_{j_k}$, where $g \in G_i$ for some $i = 1, 2$. Assume first that $\psi_T(\gamma) \geq \frac{|\gamma|^{\alpha-\varepsilon}}{M}$. In that case, $\psi(\gamma) \geq \frac{|\gamma|^{\alpha-\varepsilon}}{M}$. Else, we have that $\psi_T(\gamma) < \frac{|\gamma|^{\alpha-\varepsilon}}{M}$ and so for all $\gamma \in \Gamma$ such that $|\gamma|$ is sufficiently large, we have

$$\begin{aligned}\psi(\gamma) &\geq \psi_\Gamma(\gamma) = \|\gamma \cdot 0\|^2 \\ &\geq (\|g \cdot 0\| - \psi_T(\gamma)M)^2 \\ &\gtrsim ((|\gamma| - \psi_T(\gamma)M)^{\alpha-\varepsilon/2} - \psi_T(\gamma)M)^2 \\ &\geq ((|\gamma| - |\gamma|^{\alpha-\varepsilon})^{\alpha-\varepsilon/2} - |\gamma|^{\alpha-\varepsilon})^2 \\ &\gtrsim |\gamma|^{2\alpha-\varepsilon},\end{aligned}$$

where $\gtrsim$ represents inequality up to a multiplicative constant; we use here that one can always assume, without loss of generality, that the 1-cocycles associated to β_1 and β_2 satisfy $\|b_i(g_i)\| \gtrsim |g_i|^{\alpha-\varepsilon}$ (see Lemma 3.4 in [**1**]).

So now, by the first case, $\psi(\gamma) \geq |\gamma|^{\alpha-\varepsilon}$ for all $\gamma \in \Gamma$ that are sufficiently large. Hence, we obtain the lower bound $\alpha^\#_2(\Gamma) \geq \alpha^\#_2(H)/2$. □

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