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Endpoint Regularity of Multisublinear Fractional Maximal Functions

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Abstract. In this paper we investigate the endpoint regularity properties of the multisublinear fractional maximal operators, which include the multisublinear Hardy–Littlewood maximal operator. We obtain some new bounds for the derivative of the one-dimensional multisublinear fractional maximal operators acting on the vector-valued function $\vec{f} = (f_1, \dots, f_m)$ with all f_j being BV-functions.

1 Introduction

During the last several years, considerable attention has been given to investigating the behavior of differentiability under a maximal operator. This program began with Kinnunen [9] who showed that the centered Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$. It was noticed that the $W^{1,p}$-bound for the uncentered Hardy-Littlewood maximal operator denoted by $\widetilde{\mathcal{M}}$ also holds by a simple modification of Kinnunen's arguments or [8, Theorem 1]. This paradigm that an L^p-bound implies a $W^{1,p}$-bound was later extended to a local version in [10], to a fractional version in [11], to a bilinear version in [4], to a multi(sub)linear version in [15], and to a one-sided version in [14]. Other interesting works related to this theory are [1,6,7,16,17]. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler; understanding the endpoint regularity seems to be a deeper issue. A crucial question was posed by Hajlasz and Onninen in [8].

Question 1.1 ([8]) Is the operator $f \mapsto |\nabla \mathcal{M} f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?

A standard dilation argument reveals the true nature of this question: whether the variation of the maximal function is controlled by the variation of the original function, *i.e.,* if we have

$$\|\nabla \mathcal{M} f\|_{L^1(\mathbb{R}^d)} \le C \|\nabla f\|_{L^1(\mathbb{R}^d)}. \tag{1.1}$$

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Question 1.1 has been solved in dimension $d = 1$ and remains open in dimension $d \geq 2$. The first work in this direction is due to Tanaka [19] in 2002 when he obtained the bound (1.1) in dimension $d = 1$ for the noncentered maximal operator $\widetilde{\mathcal{M}}$ with constant $C = 2$. This result was later sharpened by Aldaz and Pérez Lázaro [2] who proved that if $f \in \mathrm{BV}(\mathbb{R})$, then $\widetilde{\mathcal{M}}f$ is absolutely continuous and

$$\mathrm{Var}\left(\widetilde{\mathcal{M}}f\right) \leq \mathrm{Var}(f), \tag{1.2}$$

where $\mathrm{Var}(f)$ denotes the total variation of f and $\mathrm{BV}(\mathbb{R})$ is the set of all functions of bounded variation on $\mathbb{R}$. Observe that inequality (1.2) is sharp. Recently, Liu, Chen, and Wu [13] also gave a simple proof of the bound (1.1) in dimension $d = 1$ for the operator $\widetilde{\mathcal{M}}$ with constant $C = 1$. In the remarkable work [12], Kurka showed that if $f \in \mathrm{BV}(\mathbb{R})$, then

$$\mathrm{Var}(\mathcal{M}f) \leq 240,004\ \mathrm{Var}(f).$$

It was also shown in [12] that if $f \in W^{1,1}(\mathbb{R})$, then $\mathcal{M}f$ is weakly differentiable and (1.1) for $d = 1$ also holds with constant $C = 240,004$.

Very recently, Carneiro and Madrid [5] considered the endpoint regularity of the uncentered fractional maximal operator $\widetilde{\mathcal{M}}_\alpha$ for $0 < \alpha < 1$, which is defined by

$$\widetilde{\mathcal{M}}_\alpha f(x) = \sup_{\substack{r,s\geq 0\\ r+s>0}} \frac{1}{(r+s)^{1-\alpha}} \int_{x-r}^{x+s} |f(y)|dy.$$

For a function $f\colon \mathbb{R} \to \mathbb{R}$ and $1 \leq q < \infty$, motivated by the Riemann sums of a Riemann integrable function, we define its q-variation by

$$\mathrm{Var}_q(f) := \sup_{\mathcal{P}} \Big(\sum_{n=1}^{N-1} \frac{|f(x_{n+1}) - f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \Big)^{1/q},$$

where the supremum is taken over all finite partitions $\mathcal{P} = \{x_1 < x_2 < \cdots < x_N\}$. This is also known as the *Riesz q-variation* of f (see, for instance, the discussion in [3] for the object and its generalizations). In particular, $\mathrm{Var}_q(f) = \mathrm{Var}(f)$ if $q = 1$. We now introduce the result of [5] as follows.

Theorem 1.2 ([5]) *Let $0 < \alpha < 1$ and $q = 1/(1-\alpha)$. Let $f \in \mathrm{BV}(\mathbb{R})$ such that $\widetilde{\mathcal{M}}_\alpha f \not\equiv \infty$. Then $\widetilde{\mathcal{M}}_\alpha f$ is absolutely continuous, and its derivative satisfies*

$$\|(\widetilde{\mathcal{M}}_\alpha f)'\|_{L^q(\mathbb{R})} = \mathrm{Var}_q(\widetilde{\mathcal{M}}_\alpha f) \leq 8^{1/q}\,\mathrm{Var}(f).$$

The main purpose of this paper is to investigate the endpoint regularity of the multisublinear fractional maximal functions. More precisely, let m be a positive integer and $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. For $0 \leq \alpha < md$, we define the centered m-sublinear fractional maximal operator $\mathfrak{M}_\alpha$ by

$$\mathfrak{M}_\alpha(\vec{f})(x) = \sup_{r>0} |B_r(x)|^{\alpha/d-m} \prod_{j=1}^{m} \int_{B_r(x)} |f_j(y)|dy$$

for any $x \in \mathbb{R}^d$, where $B_r(x)$ is the open ball in $\mathbb{R}^d$ centered at x with radius r and $|B_r(x)|$ denotes the volume of $B_r(x)$. Observe that the centered Hardy–Littlewood maximal operator $\mathcal{M}$ corresponds to the special case of $\mathfrak{M}_\alpha$ for $m = 1$ and $\alpha = 0$. Meanwhile, the centered fractional maximal operator denoted by $\mathcal{M}_\alpha$ corresponds to the special case of $\mathfrak{M}_\alpha$ for $m = 1$ and $0 < \alpha < d$. It was shown in [15] that $\mathfrak{M}_\alpha\colon W^{1,p_1}(\mathbb{R}^d)\times\cdots\times W^{1,p_m}(\mathbb{R}^d) \to W^{1,q}(\mathbb{R}^d)$ is bounded, where $1 < p_1, \dots, p_m < \infty$ and $\sum_{i=1}^m 1/p_i - \alpha/d = 1/q \le 1$. For the endpoint regularity of $\mathfrak{M}_\alpha$, there exists a constant $C = C(\alpha, m, d) > 0$ such that

$$\|\nabla \mathfrak{M}_\alpha(\vec{f})\|_{L^{\frac{d}{m(d-1)-\alpha+1}}(\mathbb{R}^d)} \le C \prod_{j=1}^m \|\nabla f_j\|_{L^1(\mathbb{R}^d)}, \tag{1.3}$$

if $d \ge 2$, $1 \le \alpha < m(d-1)+1$, $\alpha > (m-1)(d-1)$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,1}(\mathbb{R}^d)$ for $i = 1, 2, \dots, m$. The same results hold for the uncentered version of $\mathfrak{M}_\alpha$. To see this, let us consider, for instance, the centered case. By [15, Theorem 2.3(ii)], there exists a constant $C = C(\alpha, m, d) > 0$ such that

$$\Big\|\frac{\partial \mathfrak{M}_\alpha(\vec{f})}{\partial x_l}\Big\|_{L^q(\mathbb{R}^d)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^d)}. \tag{1.4}$$

for all $1 \le l \le d$, $1 \le \alpha < md$, $1 < p_1, p_2, \dots, p_m \le \infty$, and

$$0 < \frac{1}{q} = \sum_{i=1}^m \frac{1}{p_i} - \frac{(\alpha-1)}{d}.$$

Taking $p_i = d/(d-1)$ for all $1 \le i \le m$, (1.4) leads to

$$\|\nabla \mathfrak{M}_\alpha(\vec{f})\|_{L^{\frac{d}{m(d-1)-\alpha+1}}(\mathbb{R}^d)} \le C \prod_{j=1}^m \|f_j\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}$$

if $d \ge 2$, $\alpha \ge 1$ and $(m-1)(d-1) < \alpha < m(d-1)+1$. Combining this inequality with the Sobolev embedding theorem gives (1.3). Specifically, if $m = 1$, $d \ge 2$, $1 \le \alpha < d$, and $f \in W^{1,1}(\mathbb{R}^d)$, then $\mathcal{M}_\alpha f$ is weakly differentiable, and there exists a constant $C = C(\alpha, d) > 0$ such that

$$\|\nabla \mathcal{M}_\alpha f\|_{L^{\frac{d}{d-\alpha}}(\mathbb{R}^d)} \le C\|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

The same results hold for the uncentered version of $\mathcal{M}_\alpha$.

Based on the above, it is then natural to consider the extension of Theorem 1.2 to the multisublinear fractional case. This is the main motivation for this work. Here we give a positive answer to above question. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R})$. For $0 \le \alpha < m$, we define the uncentered m-sublinear fractional maximal operator $\widetilde{\mathfrak{M}}_\alpha$ by

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \sup_{\substack{r,s\ge 0\\ r+s>0}} \frac{1}{(r+s)^{m-\alpha}} \prod_{j=1}^m \int_{x-r}^{x+s} |f_j(y)|dy.$$

Obviously, $\widetilde{\mathfrak{M}}_\alpha = \widetilde{\mathcal{M}}_\alpha$ when $m = 1$. Our main results can be listed as follows.

Theorem 1.3 (i) *Let $0 < \alpha < 1$ and $q_1 = 1/(1-\alpha)$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in \mathrm{BV}(\mathbb{R})$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f}) \not\equiv \infty$. Then $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ is absolutely continuous and its derivative satisfies*

$$\big\|\big(\widetilde{\mathfrak{M}}_\alpha(\vec{f})\big)'\big\|_{L^{q_1}(\mathbb{R})} = \mathrm{Var}_{q_1}\big(\widetilde{\mathfrak{M}}_\alpha(\vec{f})\big) \le 8^{1/q_1} \sum_{l=1}^m \mathrm{Var}(f_l) \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})}.$$

(ii) *Let $m-1<\alpha<m$ and $q_2=1/(m-\alpha)$. Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in\mathrm{BV}(\mathbb{R})\cap L^1(\mathbb{R})$. Then $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ is absolutely continuous and its derivative satisfies*

$$\big\|\big(\widetilde{\mathfrak{M}}_\alpha(\vec{f})\big)'\big\|_{L^{q_2}(\mathbb{R})}=\mathrm{Var}_{q_2}\big(\widetilde{\mathfrak{M}}_\alpha(\vec{f})\big)\le 8^{1/q_2}\sum_{l=1}^m \mathrm{Var}(f_l)\prod_{1\le j\ne l\le m}\|f_j\|_{L^1(\mathbb{R})}.$$

Theorem 1.4 (i) *Let $\alpha=0$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in\mathrm{BV}(\mathbb{R})$. Then*

$$\mathrm{Var}\big(\widetilde{\mathfrak{M}}_\alpha(\vec{f})\big)\le 8\sum_{l=1}^m \mathrm{Var}(f_l)\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}.$$

(ii) *Let $\alpha=m-1$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in\mathrm{BV}(\mathbb{R})\cap L^1(\mathbb{R})$. Then*

$$\mathrm{Var}(\widetilde{\mathfrak{M}}_\alpha(\vec{f}))\le 8\sum_{l=1}^m \mathrm{Var}(f_l)\prod_{1\le j\ne l\le m}\|f_j\|_{L^1(\mathbb{R})}.$$

Remark Theorem 1.3 extends Theorem 1.2, which corresponds to the case $m=1$. The problem of finding the corresponding results for $m\ge 3$ and $1\le\alpha<m-1$ is certainly an interesting one. Another inviting possibility is the investigation of the validity of Theorem 1.3(i) for the case $\alpha=0$ and Theorem 1.3(ii) for the case $\alpha=m-1$.

The rest of the paper is organized as follows. After presenting some preliminary lemmas in Section 2, we prove Theorems 1.3 and 1.4 in Section 3. We remark that the main ideas in our proofs are greatly motivated by [5], but our methods and techniques are more delicate and direct than those in [5]. The main ingredients of our proofs that bounds the q-variation of the maximal functions on monotone intervals by the variation (times some L^∞-norms) of the original functions on comparable intervals are very interesting and technically difficult (see Lemmas 2.3–2.4). We expect the centered case of our main results to hold as well. However, our methods here do not adapt to the centered case. In what follows, we use the conventions $\prod_{j\in\varnothing}a_j=1$ and $\sum_{j\in\varnothing}a_j=0$. For convenience, we denote by $\mathrm{Lip}(\mathbb{R})$ the set of all Lipshitz functions on $\mathbb{R}$.

2 Preliminary Lemmas

Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^1_{\mathrm{loc}}(\mathbb{R})$. In what follows, for any $x\in\mathbb{R}$, we set

$$\mathcal{A}_{r,s}(\vec{f})(x)=\frac{1}{(r+s)^{m-\alpha}}\prod_{j=1}^m\int_{x-r}^{x+s}|f_j(y)|dy$$

for any r, $s\ge 0$ with satisfying $r+s>0$, and $\mathcal{A}_{0,0}(\vec{f})(x)=\limsup_{r,s\to 0^+}\mathcal{A}_{r,s}(\vec{f})(x)$. Observe that $\mathcal{A}_{0,0}(\vec{f})\equiv 0$ if each f_j is locally bounded and $\alpha>0$.

Let us begin with the following preliminary lemma.

Lemma 2.1 *Let $0\le\alpha<1$ and $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^\infty(\mathbb{R})$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})\not\equiv\infty$.*

(i) *Then we have $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)<\infty$ for all $x\in\mathbb{R}$.*

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(ii) *If* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y)$ *for some* $x, y \in \mathbb{R}$ *with satisfying* $x > y$*, then there exist* $r, s \geq 0$ *such that* $r < x - y$ *and* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$*. In particular, if each* $f_j \in \mathrm{Lip}(\mathbb{R})$*, then*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \leq \min\{r+s, x-y-r\}(r+s)^{\alpha-1} \sum_{l=1}^{m} \int_{y}^{x+s} |f_l'(z)| dz \prod_{1 \leq j \neq l \leq m} \|f_j\|_{L^\infty(\mathbb{R})}. \tag{2.1}$$

(iii) *If* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y)$ *for some* $x, y \in \mathbb{R}$ *with satisfying* $x < y$*, then there exist* $r, s \geq 0$ *such that* $s < y - x$ *and* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$*. In particular, if each* $f_j \in \mathrm{Lip}(\mathbb{R})$*, then*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \leq \min\{r+s, y-x-s\}(r+s)^{\alpha-1} \sum_{l=1}^{m} \int_{x-r}^{y} |f_l'(z)| dz \prod_{1 \leq j \neq l \leq m} \|f_j\|_{L^\infty(\mathbb{R})}.$$

Proof (i) Suppose that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \infty$ for some $x \in \mathbb{R}$. Then there exists a sequence (r_j, s_j) such that $r_j, s_j \geq 0$, $r_j + s_j \to \infty$ and $\mathcal{A}_{r_j,s_j}(\vec{f})(x) \to \infty$ as $j \to \infty$. For any $y \in \mathbb{R}$ and $j \in \mathbb{Z}$ with $r_j + s_j > |x - y|$, one can easily check that

$$\left|\mathcal{A}_{r_j,s_j}(\vec{f})(y) - \mathcal{A}_{r_j,s_j}(\vec{f})(x)\right| \leq 2m|y-x|(r_j+s_j)^{\alpha-1} \prod_{i=1}^{m} \|f_i\|_{L^\infty(\mathbf{R})}. \tag{2.2}$$

Taking $j \to \infty$, (2.2) yields $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) = \infty$, a contradiction.

(ii) We only prove (ii), as (iii) is analogous. Assume that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)$ is not attained by any average $\mathcal{A}_{r,s}(\vec{f})(x)$ with $r, s \geq 0$. Then there exists a sequence (r_j, s_j) such that $r_j, s_j \geq 0$, $r_j + s_j \to \infty$ and $\mathcal{A}_{r_j,s_j}(\vec{f})(x) \to \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)$ as $j \to \infty$. Combining these facts with (2.2) yield

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \leq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) + \lim_{j\to\infty} 2m|y-x|(r_j+s_j)^{\alpha-1} \prod_{i=1}^{m} \|f_i\|_{L^\infty(\mathbb{R})} \leq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y),$$

which gives a contradiction. Thus, there exist $r, s \geq 0$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$. If $r = x - y + t_0$ for some $t_0 \geq 0$, then

$$\begin{aligned}\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x) &= \frac{1}{(x-y+s+t_0)^{m-\alpha}} \prod_{j=1}^{m} \int_{y-t_0}^{y+x-y+s} |f_j(z)| dz \\ &\leq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y),\end{aligned}$$

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which yields a contradiction. So $r < x - y$. Note that each $f_j \in \mathrm{Lip}(\mathbb{R})$; then all f_j are absolutely continuous. By Fubini's theorem, we have that for any $l \in \{1, 2, \dots, m\}$,

$$
\begin{aligned}
(2.3)\qquad & \Big| \int_{x-r}^{x+s} |f_l(z)|dz - \int_{y}^{y+r+s} |f_l(z)|dz\Big| \\
&\le \int_{x-r}^{x+s} |f_l(z) - f_l(z + r + y - x)|dz \\
&\le \int_{x-r}^{x+s} \int_{r+y-x}^{0} |f_l'(z + t)|dtdz \\
&= \min\Big\{ \int_{r+y-x}^{0} \int_{x-r}^{x+s} |f_l'(z + t)|dzdt, (r + s) \int_{y}^{x+s} |f_l'(z)|dz\Big\} \\
&\le \min\{(x - y - r), r + s\} \int_{y}^{x+s} |f_l'(z)|dz.
\end{aligned}
$$

Then (2.1) follows from (2.3) and the following inequality

(2.4)

$$
\begin{aligned}
&\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \\
&\quad\le |\mathcal{A}_{r,s}(\vec{f})(x) - \mathcal{A}_{0,r+s}(\vec{f})(y)| \\
&\quad\le (r + s)^{\alpha-1} \sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})}\Big| \int_{x-r}^{x+s} |f_l(z)|dz - \int_{y}^{y+r+s} |f_l(z)|dz\Big|. \qquad\blacksquare
\end{aligned}
$$

Lemma 2.2 *Let $0 \le \alpha < m$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R})$.*

(i) *If each $f_j \in L^\infty(\mathbb{R})$, then there exists $M > 0$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \le M$ for all $x \in \mathbb{R}$.*

(ii) *If $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y)$ for some $x, y \in \mathbb{R}$ with $x > y$, then there exist $r, s \ge 0$ such that $r < x - y$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$. In particular, if each $f_j \in$ Lip$(\mathbb{R})$, then*

$$
(2.5)\quad \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \le \min\{r + s, x - y - r\}(r + s)^{\alpha-m} \sum_{l=1}^{m} \int_{y}^{x+s} |f_l'(z)|dz \prod_{1\le j\ne l\le m} \|f_j\|_{L^1(\mathbb{R})}.
$$

(iii) *If $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y)$ for some $x, y \in \mathbb{R}$ with $x < y$, then there exist $r, s \ge 0$ such that $s < y - x$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$. In particular, if $f_j \in$ Lip$(\mathbb{R})$, then*

$$
\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \le \min\{r + s, y - x - s\}(r + s)^{\alpha-m} \sum_{l=1}^{m} \int_{x-r}^{y} |f_l'(z)|dz \prod_{1\le j\ne l\le m} \|f_j\|_{L^1(\mathbb{R})}.
$$

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Proof (i) One can easily check that

$$\begin{aligned}\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \le \sup_{\substack{r,s\ge 0\\ 0<r+s\le 1}} \frac{1}{(r+s)^{m-\alpha}} \prod_{j=1}^m \int_{x-r}^{x+s} |f_j(z)|dz\\ + \sup_{\substack{r,s\ge 0\\ r+s>1}} \frac{1}{(r+s)^{m-\alpha}} \prod_{j=1}^m \int_{x-r}^{x+s} |f_j(z)|dz\\ \le \prod_{j=1}^m \|f_j\|_{L^\infty(\mathbb{R})} + \prod_{j=1}^m \|f_j\|_{L^1(\mathbb{R})}.\end{aligned}$$

for any $x \in \mathbb{R}$.

(ii) We will only prove (ii), as (iii) is analogous. The proof of (ii) is similar to the proof of Lemma 2.1(ii). Assume that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)$ is not attained by any average $\mathcal{A}_{r,s}(\vec{f})(x)$ with $r, s \ge 0$. Then there exists a sequence (r_j, s_j) such that $r_j, s_j \ge 0$, $r_j + s_j \to \infty$, and $\mathcal{A}_{r_j,s_j}(\vec{f})(x) \to \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)$ as $j \to \infty$. Using a similar argument to the one used for (2.2) we get

$$|\mathcal{A}_{r_j,s_j}(\vec{f})(y) - \mathcal{A}_{r_j,s_j}(\vec{f})(x)| \le 2m|y-x|(r_j+s_j)^{\alpha-m} \prod_{i=1}^m \|f_i\|_{L^1(\mathbb{R})}$$

for any $j \in \mathbb{Z}$ such that $r_j + s_j > |x-y|$. It follows that

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) + \lim_{j\to\infty} 2m|y-x|(r_j+s_j)^{\alpha-m} \prod_{i=1}^m \|f_i\|_{L^1(\mathbb{R})} \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y),$$

which is a contradiction. Thus, there exist $r, s \ge 0$ with $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = \mathcal{A}_{r,s}(\vec{f})(x)$. One can easily check that $r < x - y$. Using a similar argument as in (2.4) we obtain

$$\begin{aligned}&\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(y) \le\\ &\qquad (r+s)^{\alpha-m} \sum_{l=1}^m \prod_{1\le j\ne l\le m} \|f_j\|_{L^1(\mathbb{R})} \Big| \int_{x-r}^{x+s} |f_l(z)|dz - \int_y^{y+r+s} |f_l(z)|dz \Big|,\end{aligned}$$

which together with (2.3) yields (2.5). ∎

The following two lemmas are the heart of our proofs. Its bound the q-variation of $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ in a monotone interval.

Lemma 2.3 *Let $0 \le \alpha < 1$ and $q = 1/(1-\alpha)$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in \mathrm{BV}(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$.*

(i) *Let $x_1 < x_2 < \cdots < x_N$ be a sequence of real numbers such that*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_2) \le \cdots \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N-1}) < \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N).$$

Then there exist $r, s \ge 0$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) = \mathcal{A}_{r,s}(\vec{f})(x_N)$ and

$$\begin{aligned}(2.6)\quad &\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \le\\ &4\Big(\sum_{l=1}^m \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1} \Big(\sum_{l=1}^m \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1}^{x_N+s} |f_l'(x)|dx\Big).\end{aligned}$$

In particular, if $x_N < x_{N+1}$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N+1})$, then $s < x_{N+1} - x_N$.

(ii) *Let* $x_1 < x_2 < \cdots < x_N$ *be a sequence of real numbers such that*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_2) \geq \cdots \geq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N-1}) \geq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N).$$

Then there exist $r, s \geq 0$ *such that* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) = \mathcal{A}_{r,s}(\vec{f})(x_1)$ *and*

$$\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \leq 4\Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1} \Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1-r}^{x_N} |f_l'(x)|dx\Big).$$

In particular, if $x_0 < x_1$ *and* $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_0)$*, then* $r < x_1 - x_0$.

Proof We only prove (i), as (ii) is analogous. By Lemma 2.1, there exist $r, s \geq 0$ such that $r < x_N - x_1$, $s < x_{N+1} - x_N$, $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) = \mathcal{A}_{r,s}(\vec{f})(x_N)$ and

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) \leq (r+s)^\alpha \sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1}^{x_N+s} |f'(x)|dx. \tag{2.7}$$

Let M be the smallest integer with $2 \leq M \leq N$ such that

$$\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M-1})}{x_M - x_{M-1}} = \max\Big\{\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n-1})}{x_n - x_{n-1}};\ 2 \leq n \leq N\Big\} > 0.$$

By Lemma 2.1 again there exist $u, v \geq 0$ such that $u < x_M - x_{M-1}$, $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M) = \mathcal{A}_{u,v}(\vec{f})(x_M)$ and

$$\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M-1})}{x_M - x_{M-1}} \leq (u+v)^{\alpha-1} \sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{M-1}}^{x_M+v} |f'(z)|dz. \tag{2.8}$$

Case 1. Assume that $u + v \geq r + s$. Using (2.7) and (2.8) we obtain

$$\begin{aligned}
&\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \\
&= \Big(\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M-1})}{x_M - x_{M-1}}\Big)^{q-1} (\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1)) \\
&\leq (u+v)^{(\alpha-1)(q-1)}(r+s)^\alpha \Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{M-1}}^{x_M+v} |f_l'(z)|dz\Big)^{q-1} \\
&\quad \times \Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1}^{x_N+s} |f'(x)|dx\Big) \\
&\leq \Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1} \Big(\sum_{l=1}^m \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1}^{x_N+s} |f_l'(x)|dx\Big),
\end{aligned}$$

which yields (2.6).

Case 2. Assume that $u+v<r+s$ and $x_N-x_M\le v$. In this case note that $x_M+v\le X_N+s$ (which is clear if $M=N$, and if $M<N$, we note that $r<x_N-x_M$). By Lemma 2.1 again we have $u<x_M-x_1$ and

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1)\le(u+v)^\alpha\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_M+v}|f'(x)|dx,$$

which, combining (2.8) with the similar argument as in Case 1, implies

$$\sum_{n=1}^{M-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\le\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_M+v}|f_l'(x)|dx\Big)^q.\tag{2.9}$$

On the other hand, we get from (2.8) that

$$\begin{aligned}\sum_{n=M}^{N-1}&\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\\&\le\Big(\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_M)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M-1})}{x_M-x_{M-1}}\Big)^q\sum_{n=M}^{N-1}|x_{n+1}-x_n|\\&\le v(u+v)^{(\alpha-1)q}\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{M-1}}^{x_M+v}|f_l'(z)|dz\Big)^q\\&\le\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{M-1}}^{x_M+v}|f_l'(z)|dz\Big)^q.\end{aligned}\tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\sum_{n=1}^{N-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\le2\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_N+s}|f_l'(z)|dz\Big)^q,$$

which gives (2.6).

Case 3. Assume that $u+v<r+s$ and $x_N-x_M>v$. Let i_1 be the unique integer such that $x_{i_1}\le x_M+v<x_{i_1+1}$ and $M\le i_1\le N-1$. By (2.9) and the fact that $x_M+v<x_{i_1+1}$, we get

$$\sum_{n=1}^{M-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\le\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_{i_1+1}}|f_l'(x)|dx\Big)^q.\tag{2.11}$$

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By a similar argument to the one used to obtain (2.9) we have

$$\sum_{n=M}^{i_1-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\leq \Big(\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_{i_1+1}}|f_l'(z)|dz\Big)^q. \tag{2.12}$$

Combining (2.11) with (2.12) yields that

$$\sum_{n=1}^{i_1-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\leq 2\Big(\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_{i_1+1}}|f_l'(z)|dz\Big)^q. \tag{2.13}$$

Let M_1 be the smallest integer with $i_1<M_1\leq N$ such that

$$\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1})-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1-1})}{x_{M_1}-x_{M_1-1}}=\max\Big\{\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n-1})}{x_n-x_{n-1}};\ i_1\leq n\leq N\Big\}>0.$$

Then by Lemma 2.1 again there exist $u_{M_1},v_{M_1}\geq 0$ such that $u_{M_1}<x_{M_1}-x_{M_1-1}$, $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1})=\mathcal{A}_{u_{M_1},v_{M_1}}(\vec{f})(x_{M_1})$, and

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{i_1})\leq (r+s)^\alpha\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_1}}^{x_N+s}|f'(x)|dx; \tag{2.14}$$

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1})-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{i_1})\leq (u_{M_1}+v_{M_1})^\alpha\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_1}}^{x_{M_1}+v_{M_1}}|f'(x)|dx; \tag{2.15}$$

$$\frac{\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1})-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_1-1})}{x_{M_1}-x_{M_1-1}}\leq (u_{M_1}+v_{M_1})^{\alpha-1}\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{M_1-1}}^{x_{M_1}+v_{M_1}}|f'(z)|dz. \tag{2.16}$$

If $u_{M_1}+v_{M_1}\geq r+s$ or $u_{M_1}+v_{M_1}<r+s$ and $x_N-x_{M_1}\leq v_{M_1}$, then by (2.14)–(2.16) and similar arguments as in Cases 1 and 2, we get

$$\sum_{n=i_1}^{N-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\leq 2\Big(\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1}\times\Big(\sum_{l=1}^m\prod_{1\leq j\neq l\leq m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_1}}^{x_N+s}|f_l'(x)|dx\Big).$$

When $u_{M_1}+v_{M_1}<r+s$ and $x_N-x_{M_1}>v_{M_1}$, let i_2 be the unique integer such that $x_{i_2}\leq x_{M_1}+v_{M_1}<x_{i_2+1}$ and $M_1\leq i_2\leq N-1$. Then by a similar argument as in (2.13)

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we have

$$\sum_{n=i_1}^{i_2-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}$$
$$\le 2\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_1}}^{x_{i_2+1}}|f_l'(z)|dz\Big)^q$$
$$\le 2\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1}\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_1}}^{x_{i_2+1}}|f_l'(z)|dz\Big).$$

Reasoning as above, we can obtain $\{(i_k, M_k, u_{M_k}, v_{M_k})\}_{k=1}^L$ such that

(i) $u_{M_k}, v_{M_k}\ge 0$, $u_{M_k}<x_{M_k}-x_{M_k-1}$, $i_k<M_k$, $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{M_k})=\mathcal{A}_{u_{M_k},v_{M_k}}(\vec{f})(x_{M_k})$ for $k=1,2,\dots,L$;

(ii) $M_k\le i_{k+1}\le N-1$, $i_k+1\le i_{k+1}$ and

$$\sum_{n=i_k}^{i_{k+1}-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\le \tag{2.17}$$
$$2\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1}\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_k}}^{x_{i_{k+1}+1}}|f_l'(z)|dz\Big).$$

for $k=1,2,\dots,L-1$;

(iii) $u_{M_L}+v_{M_L}\ge r+s$ or $u_{M_L}+v_{M_L}<r+s$ and $x_N-x_{M_L}\le v_{M_L}$. In both cases we have

$$\sum_{n=i_L}^{N-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}\le \tag{2.18}$$
$$\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1}\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_{i_L}}^{x_N+s}|f_l'(x)|dx\Big).$$

It follows from (2.13), (2.17), and (2.18) that

$$\sum_{n=1}^{N-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}$$
$$=\sum_{n=1}^{i_1-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}+\sum_{k=1}^{L-1}\sum_{n=i_k}^{i_{k+1}-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}$$
$$+\sum_{n=i_L}^{N-1}\frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n)-\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n-x_{n+1}|^{q-1}}$$
$$\le 4\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1}$$
$$\times\Big(\sum_{l=1}^m\prod_{1\le j\ne l\le m}\|f_j\|_{L^\infty(\mathbb{R})}\int_{x_1}^{x_N+s}|f_l'(x)|dx\Big).$$

This yields (2.6) and finishes the proof. ■

Applying Lemma 2.2 and the similar arguments as in the proof of Lemma 2.3, we can obtain the following lemma. The details are omitted.

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Lemma 2.4 *Let $m-1 \le \alpha < m$, $q = 1/(m-\alpha)$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$.*

(i) *Let $x_1 < x_2 < \cdots < x_N$ be a sequence of real numbers such that*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_2) \le \cdots \le \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N-1}) < \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N).$$

Then there exist $r, s \ge 0$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) = \mathcal{A}_{r,s}(\vec{f})(x_N)$ and

$$\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \le 4\Big(\sum_{l=1}^{m} \prod_{1 \le j \ne l \le m} \|f_j\|_{L^1(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1} \Big(\sum_{l=1}^{m} \prod_{1 \le j \ne l \le m} \|f_j\|_{L^1(\mathbb{R})} \int_{x_1}^{x_N+s} |f_l'(x)| dx\Big).$$

In particular, if $x_N < x_{N+1}$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N+1})$, then $s < x_{N+1} - x_N$.

(ii) *Let $x_1 < x_2 < \cdots < x_N$ be a sequence of real numbers such that*

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_2) \ge \cdots \ge \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{N-1}) \ge \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N).$$

Then there exist $r, s \ge 0$ such that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) = \mathcal{A}_{r,s}(\vec{f})(x_1)$ and

$$\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \le 4\Big(\sum_{l=1}^{m} \prod_{1 \le j \ne l \le m} \|f_j\|_{L^1(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q-1} \Big(\sum_{l=1}^{m} \prod_{1 \le j \ne l \le m} \|f_j\|_{L^1(\mathbb{R})} \int_{x_1-r}^{x_N} |f_l'(x)| dx\Big).$$

In particular, if $x_0 < x_1$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_1) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_0)$, then $r < x_1 - x_0$.

The following proposition is a classical result of F. Riesz (see [18, Chapter IX §4, Theorem 7]), which plays a key role in the proof of Theorem 1.3.

Proposition 2.5 *Let $f: \mathbb{R} \to \mathbb{R}$ be a given function and $1 < q < \infty$. Then $\mathrm{Var}_q(f) < \infty$ if and only if f is absolutely continuous and its derivative f' belongs to $L^q(\mathbb{R})$. Moreover, in this case, we have that $\|f'\|_{L^q(\mathbb{R})} = \mathrm{Var}_q(f)$.*

We end this section by presenting the q-variation of $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ with each $f_j \in \mathrm{Lip}(\mathbb{R})$.

Proposition 2.6 (i) *Let $0 \le \alpha < 1$ and $q_1 = 1/(1-\alpha)$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in \mathrm{Lip}(\mathbb{R})$. Then*

$$\mathrm{Var}_{q_1}(\widetilde{\mathfrak{M}}_\alpha(\vec{f})) \le 8^{1/q_1} \sum_{l=1}^{m} \mathrm{Var}(f_l) \prod_{1 \le j \ne l \le m} \|f_j\|_{L^\infty(\mathbb{R})}. \tag{2.19}$$

(ii) *Let $m-1 \le \alpha < m$ and $q_2 = 1/(m-\alpha)$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R})$. Then*

$$\mathrm{Var}_{q_2}(\widetilde{\mathfrak{M}}_\alpha(\vec{f})) \le 8^{1/q_2} \sum_{l=1}^{m} \mathrm{Var}(f_l) \prod_{1 \le j \ne l \le m} \|f_j\|_{L^1(\mathbb{R})}. \tag{2.20}$$

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Proof First we prove (i). We can assume that all $f_j \in BV(\mathbb{R})$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ is not constant (since (2.19) is obvious in other cases). Let $x_1 < x_2 < \cdots < x_N$. For a generic function $g : \mathbb{R} \to \mathbb{R}$ and a partition $\mathcal{P} = \{x_1 < x_2 < \cdots < x_N\}$, we say that an interval $[x_n, x_l]$ is *a string of local maxima* of g if

$$g(x_{n-1}) < g(x_n) = \cdots = g(x_l) > g(x_{l+1}),$$

provided $n \neq 1$ and $l \neq N$. When $n \neq l$, we say that the leftmost point x_n is *a left local maximal* of g. Respectively, the rightmost point x_l is *a right local maximum* of g. When $n = l$, we say that x_n is *a left local maximum* and *a right local maximum* of g. We define *string of local minima*, *left local minimum* and *right local minimum* analogously. Fix a partition $\mathcal{P} = \{x_1 < x_2 < \cdots < x_N\}$. Without loss of generality we can assume that $\{[x_{i_l^-}, x_{i_l^+}]\}_{l=1}^L$ and $\{[x_{j_l^-}, x_{j_l^+}]\}_{l=1}^L$ are the ordered strings of local minima and local maxima of $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ (relative to the partition $\mathcal{P}$), *i.e.*,

$$1 < i_1^- \le i_1^+ < j_1^- \le j_1^+ < i_2^- \le i_2^+ < j_2^- \le j_2^+ < \cdots < i_L^- \le i_L^+ < j_L^- \le j_L^+ < N.$$

Obviously, $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{j_L^+}) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{i_L^-})$. By Lemma 2.3 there exist $r,\ r_L \ge 0$ such that $r_L < x_{j_L^+} - x_{i_L^-}$ and

(2.21)
$$\sum_{n=1}^{i_1^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \le$$
$$4\Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1-r}^{x_{i_1^-}} |f_l'(z)|dz\Big);$$

(2.22)
$$\sum_{n=j_L^+}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q-1}} \le$$
$$4\Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{j_L^+}-r_L}^{x_N} |f_l'(z)|dz\Big).$$

Since $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{j_k^-}) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{i_{k+1}^+})$ for any $1 \le k \le L-1$ and $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{j_L^-}) > \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_N)$. By Lemma 2.3 again there exists a sequence $\{s_k\}_{k=1}^L$ such that $s_k < x_{i_{k+1}^+} - x_{j_k^-}$,

$$\sum_{n=i_k^+}^{j_k^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \le \tag{2.23}$$
$$4\Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^m \prod_{1\le j\neq l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{i_k^+}}^{x_{j_k^-}+s_k} |f_l'(z)|dz\Big)$$

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for all $1 \le k \le L-1$ and $s_L < x_N - x_{j_L^-}$,

$$(2.24)\quad \sum_{n=i_L^+}^{j_L^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \le 4\Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{i_L^+}}^{x_{j_L^-}+s_L} |f_l'(z)|dz\Big).$$

It follows from (2.23) and (2.24) that

$$(2.25)\quad \sum_{k=1}^{L}\sum_{n=i_k^+}^{j_k^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q-1}} \le 4\Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_{i_1^-}}^{x_N} |f_l'(z)|dz\Big).$$

Similarly, we obtain

$$(2.26)\quad \sum_{k=1}^{L-1}\sum_{n=j_k^+}^{i_{k+1}^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \le 4\Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1-1} \Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \int_{x_1}^{x_{i_L^-}} |f_l'(z)|dz\Big).$$

It follows from (2.21), (2.22), (2.25), and (2.26) that

$$\begin{aligned}
&\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \\
&= \sum_{n=1}^{i_1^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \\
&\quad + \sum_{k=1}^{L}\sum_{n=i_k^+}^{j_k^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \\
&\quad + \sum_{k=1}^{L-1}\sum_{n=j_k^+}^{i_{k+1}^- -1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \\
&\quad + \sum_{n=j_L^+}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^{q_1}}{|x_n - x_{n+1}|^{q_1-1}} \\
&\le 8\Big(\sum_{l=1}^{m} \prod_{1\le j\ne l\le m} \|f_j\|_{L^\infty(\mathbb{R})} \|f_l'\|_{L^1(\mathbb{R})}\Big)^{q_1}.
\end{aligned}$$

Combining this with the fact that $\|f_l'\|_{L^1(\mathbb{R})} = \mathrm{Var}(f_l)$ for any $1 \le l \le m$ yields (2.19). Using Lemma 2.4 and similar arguments as in leading to (2.19) we get (2.20). ■

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3 Proofs of Main Results

This section is devoted to proving Theorems 1.3 and 1.4. In what follows, let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in \mathrm{BV}(\mathbb{R})$. Without loss of generality we assume that all $f_j \geq 0$, since $\mathrm{Var}(|f|) \leq \mathrm{Var}(f)$. Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ be a nonnegative smooth function such that $\mathrm{supp}(\varphi) = [-1,1]$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\epsilon > 0$, we define $\varphi_\epsilon(x) = \frac{1}{\epsilon}\varphi(\frac{x}{\epsilon})$. For $\epsilon > 0$, let $\vec{f}_\epsilon = (f_{1,\epsilon}, \dots, f_{m,\epsilon})$ with each $f_{j,\epsilon} = \varphi_\epsilon * f_j$. Note that for any $\epsilon > 0$, all functions $f_{j,\epsilon}$ are Lipschitz continuous, $\mathrm{Var}(f_{j,\epsilon}) \leq \mathrm{Var}(f)$ and $\|f_{j,\epsilon}\|_{L^\infty(\mathbb{R})} \leq \|f_j\|_{L^\infty(\mathbb{R})}$.

We now proceed with the proofs of Theorems 1.3 and 1.4.

Proof of Theorem 1.3 We first claim that

$$\lim_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x) = \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \tag{3.1}$$

for all $x \in \mathbb{R}$ and $0 \leq \alpha < m$. Note that $\lim_{\epsilon\to 0} f_{l,\epsilon}(x) = f(x)$ a.e. $x \in \mathbb{R}$ for all $1 \leq l \leq m$. Fix $x \in \mathbb{R}$. We assume that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) > 0$, since $\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) = 0$ implies $f_{l,\epsilon} \equiv 0$ for all $1 \leq l \leq m$ and then (3.1) is obvious. By Fatou's lemma, for any $r, s \geq 0$ with $r + s > 0$,

$$\mathcal{A}_{r,s}(\vec{f})(x) \leq \liminf_{\epsilon\to 0} \mathcal{A}_{r,s}(\vec{f}_\epsilon)(x) \leq \liminf_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x),$$

which leads to

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \leq \liminf_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x). \tag{3.2}$$

Thus, to prove (3.1), it suffices to show that

$$\limsup_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x) \leq \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x). \tag{3.3}$$

We now prove (3.3) by contradiction. Assume that there exists $\eta > 0$ such that

$$\limsup_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x) > (1+3\eta)\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x).$$

It follows that there exists a sequence $\{\epsilon_k\}_{k\geq 1}$ such that $0 \leq \epsilon_k < 1$ and $\epsilon_k \to 0$ as $k \to \infty$ and

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f}_{\epsilon_k})(x) > (1+2\eta)\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x). \tag{3.4}$$

This yields that there exists a subsequence $\{\epsilon_{\iota,k}\}_{k\geq 1}$ of $\{\epsilon_k\}_{k\geq 1}$ such that $0 \leq \epsilon_{\iota,k} < 1$ and $\epsilon_{\iota,k} \to 0$ as $k \to \infty$ and

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f}_{\epsilon_{\iota,k}})(x) = \sup_{\substack{r,s\geq 0\\ 0<r+s\leq 1}} \mathcal{A}_{r,s}(\vec{f}_{\epsilon_{\iota,k}})(x) \quad \forall\, k \geq 1 \tag{3.5}$$

or

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f}_{\epsilon_{\iota,k}})(x) = \sup_{\substack{r,s\geq 0\\ r+s\geq 1}} \mathcal{A}_{r,s}(\vec{f}_{\epsilon_{\iota,k}})(x) \quad \forall\, k \geq 1. \tag{3.6}$$

Assume that (3.5) holds. Fix $k \geq 1$; we can write

$$
\begin{aligned}
&|\widetilde{\mathfrak{M}}_\alpha(\vec{f_{\epsilon_{\iota,k}}})(x) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)| \\
&\quad \leq \sup_{\substack{r,s\geq 0\\ 0<r+s\leq 1}} |\mathcal{A}_{r,s}(\vec{f_{\epsilon_{\iota,k}}})(x) - \mathcal{A}_{r,s}(\vec{f})(x)| \\
&\quad \leq \sup_{\substack{r,s\geq 0\\ 0<r+s\leq 1}} \sum_{l=1}^{m} \frac{1}{(r+s)^{m-\alpha}} \prod_{\mu=1}^{l-1} \int_{x-r}^{x+s} |f_\mu(y)|dy \prod_{\nu=l+1}^{m} \int_{x-r}^{x+s} |f_{\nu,\epsilon_{\iota,k}}(y)|dy \\
&\qquad \times \int_{x-r}^{x+s} |f_{l,\epsilon_{\iota,k}}(y) - f_l(y)|dy \\
&\quad \leq \sum_{l=1}^{m} \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})} \sup_{\substack{r,s\geq 0\\ 0<r+s\leq 1}} \frac{1}{r+s} \int_{x-r}^{x+s} |f_{l,\epsilon_{\iota,k}}(y) - f_l(y)|dy.
\end{aligned}
$$

Fix $l \in \{1,2,\ldots,m\}$ and a pair (r,s) satisfying $r,s \geq 0$ and $0 < r+s \leq 1$. Since $|f_{l,\epsilon_{\iota,k}}(y) - f_l(y)|\chi_{[x-r,x+s]}(y) \leq 2|f_l(y)|\chi_{[x-1,x+1]}(y)$ a.e. $y \in \mathbb{R}$, by Fatou's lemma,

$$
\lim_{k\to\infty} \sup_{\substack{r,s\geq 0\\ 0<r+s\leq 1}} \frac{1}{r+s} \int_{x-r}^{x+s} |f_{l,\epsilon_{\iota,k}}(y) - f_l(y)|dy = 0,
$$

which implies $\lim_{\epsilon\to 0} \widetilde{\mathfrak{M}}_\alpha(\vec{f_{\epsilon_{\iota,k}}})(x) = \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x)$ and yields a contradiction. Assume that (3.6) holds. Then by (3.4), fix $k \in \mathbb{Z}$; there are $r_k,\ s_k \geq 0$ such that $r_k + s_k \geq 1$ and

$$
\mathcal{A}_{r_k,s_k}(\vec{f_{\epsilon_{\iota,k}}})(x) > (1+\eta)\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x). \tag{3.7}
$$

For any fixed $1 \leq l \leq m$ and $k \in \mathbb{Z}$, by Fubini's theorem,

$$
\begin{aligned}
\int_{x-r_k}^{x+s_k} |f_{l,\epsilon_{\iota,k}}(z)|dz &\leq \int_{x-r_k}^{x+s_k} \int_{-\epsilon_{\iota,k}}^{\epsilon_{\iota,k}} |f_l(z-t)|\varphi_{\epsilon_{\iota,k}}(t)dtdz \\
&= \int_{-\epsilon_{\iota,k}}^{\epsilon_{\iota,k}} \int_{x-r_k}^{x+s_k} |f_l(z-t)|dz\varphi_{\epsilon_{\iota,k}}(t)dt \\
&\leq \int_{-\epsilon_{\iota,k}}^{\epsilon_{\iota,k}} \int_{x-r_k-\epsilon_{\iota,k}}^{x+s_k+\epsilon_{\iota,k}} |f_l(z)|dz\varphi_{\epsilon_{\iota,k}}(t)dt \\
&= \int_{x-r_k-\epsilon_{\iota,k}}^{x+s_k+\epsilon_{\iota,k}} |f_l(z)|dz.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathcal{A}_{r_k,s_k}(\vec{f_{\epsilon_{\iota,k}}})(x) &\leq \frac{1}{(r_k+s_k)^{m-\alpha}} \prod_{j=1}^{m} \int_{x-r_k-\epsilon_{\iota,k}}^{x+s_k+\epsilon_{\iota,k}} f_j(u)du \\
&\leq \Big(\frac{r_k+s_k+2\epsilon_{\iota,k}}{r_k+s_k}\Big)^{m-\alpha} \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x),
\end{aligned}
$$

which together with (3.7) implies

$$
\Big(\frac{r_k+s_k+2\epsilon_{\iota,k}}{r_k+s_k}\Big)^{m-\alpha} > 1+\eta.
$$

This implies $r_k + s_k < 2((1+\eta)^{\frac{1}{m-\alpha}} - 1)^{-1}$. We can choose two subsequences $\{r_{\iota,k}\}$ of $\{r_k\}$ and $\{s_{\iota,k}\}$ of $\{s_k\}$ such that $r_{\iota,k} \to r_0$ and $s_{\iota,k} \to s_0$ as $k \to \infty$ and $r_0 + s_0 \geq 1$.

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From (3.7) we get

$$\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x) \geq \mathcal{A}_{r_0,s_0}(\vec{f})(x) = \lim_{k\to\infty} \mathcal{A}_{r_{\iota,k},s_{\iota,k}}(\vec{f_{\epsilon_{\iota,k}}})(x) \geq (1+\eta)\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x),$$

which is a contradiction, and (3.3) holds. Equation (3.1) follows from (3.2) and (3.3).

We first prove Theorem 1.3(i). Let $0 \leq \alpha < 1$ and $q = 1/(1-\alpha)$. Fix a partition $\mathcal{P} = \{x_1 < x_2 < \cdots < x_N\}$. We get from Proposition 2.6(i) that

$$\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \leq 8\Big(\sum_{l=1}^m \mathrm{Var}(f_{l,\epsilon}) \prod_{1\leq j\neq l\leq m} \|f_{j,\epsilon}\|_{L^\infty(\mathbb{R})}\Big)^q.$$

Combining this inequality with (3.1) implies

$$\Big(\sum_{n=1}^{N-1} \frac{|\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}}\Big)^{1/q} \leq 8^{1/q} \sum_{l=1}^m \mathrm{Var}(f_l) \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})}.$$

This yields that

$$\mathrm{Var}_q(\widetilde{\mathfrak{M}}_\alpha(\vec{f})) \leq 8^{1/q} \sum_{l=1}^m \mathrm{Var}(f_l) \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})}$$

since the original partition $\mathcal{P}$ was arbitrary. By Proposition 2.5 and the fact that $q > 1$, we know that $\widetilde{\mathfrak{M}}_\alpha(\vec{f})$ is absolutely continuous with

$$\|(\widetilde{\mathfrak{M}}_\alpha(\vec{f}))'\|_{L^q(\mathbb{R})} = \mathrm{Var}_q(\widetilde{\mathfrak{M}}_\alpha(\vec{f})).$$

Theorem 1.3(i) follows from this. Similarly, we can get Theorem 1.3(ii) by Propositions 2.5–2.6 and (3.1). ∎

Proof of Theorem 1.4 We first prove (i). Fix a partition $\mathcal{P} = \{x_1 < x_2 < \cdots < x_N\}$. We get from (i) of Proposition 2.6 that

$$\sum_{n=1}^{N-1} |\widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f}_\epsilon)(x_{n+1})| \leq 8\sum_{l=1}^m \mathrm{Var}(f_{l,\epsilon}) \prod_{1\leq j\neq l\leq m} \|f_{j,\epsilon}\|_{L^\infty(\mathbb{R})},$$

which together with (3.1) implies that

$$\sum_{n=1}^{N-1} |\widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_n) - \widetilde{\mathfrak{M}}_\alpha(\vec{f})(x_{n+1})| \leq 8\sum_{l=1}^m \mathrm{Var}(f_l) \prod_{1\leq j\neq l\leq m} \|f_j\|_{L^\infty(\mathbb{R})}.$$

This yields Theorem 1.4(i), since the original partition $\mathcal{P}$ was arbitrary. Similarly, we can get Theorem 1.4(ii) by Proposition 2.6(ii) and (3.1). ∎

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