Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–13, *2024*
DOI:10.1017/prm.2024.98

Liouville theorems for the sub-linear Lane–Emden equation on the half space

Huxiao Luo
Department of Mathematics, Zhejiang Normal University, 688 Yingbin Avenue, Jinhua, Zhejiang 321004, P. R. China and
School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai, 200240, P. R. China (luohuxiao@zjnu.edu.cn)

(Received 9 March 2024; revised 1 August 2024; accepted 2 August 2024)

In this article, we study the following Dirichlet problem to the sub-linear Lane–Emden equation

$$\begin{cases} -\Delta u = u^p, \quad u(x) \geq 0, \quad x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, \quad x \in \partial\mathbb{R}^n_+, \end{cases}$$

where $n \geq 3$, $0 < p \leq 1$. By establishing an equivalent integral equation, we give a lower bound of the Kelvin transformation $\bar{u}$. Then, by constructing a new comparison function, we apply the maximum principle based on comparisons and the method of moving planes to obtain that u only depends on x_n. Based on this, we prove the non-existence of non-negative solutions.

Keywords: comparison principles; Lane–Emden equation; Liouville theorems; method of moving planes

2010: Mathematics subject classification: 35A01; 35B09; 35B50; 35B53

1. Introduction

There are some typical nonlinear Liouville theorems about the Lane–Emden equation

$$-\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

which go back to J. Serrin in the 1970s. In 1981, Gidas and Spruck [11] proved that equation (1.1) has no nontrivialpt non-negative classical solution if $n \geq 2$ and $p < \frac{n+2}{(n-2)_+}$.

However, a full answer to the existence of classical solutions for Lane–Emden equation (1.1) is currently not available for proper subdomains of $\mathbb{R}^n$. This is so even for the Dirichlet problem

$$\begin{cases} -\Delta u = u^p, \quad u(x) \geq 0, \ x \in \mathbb{R}^n_+, \\ u(x) \equiv 0, \quad x \in \partial\mathbb{R}^n_+. \end{cases} \tag{P}$$

where $\mathbb{R}^n_+$ is the half space

$$\mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | x_n > 0\},$$

despite its long history and the large number of works on that problem, see e.g. [5–7, 9, 10].

Since the half space is the simplest unbounded domain with an unbounded boundary and performing a blow-up close to the boundary for some elliptic equations in a smooth domain leads to the Lane–Emden equation in a half space, studying the elliptic equations in half space is very meaningful.

Combining moving planes argument with Kelvin transform, Gidas and Spruck [10] reduces the Dirichlet problem (P) to the one-dimensional case and then proved that (P) has no solutions in half space $\mathbb{R}^n_+$, provided $1 < p \leq \frac{n+2}{n-2}$.

The question of existence of bounded solutions of the Dirichlet problem for (P) in half space was fully answered by Chen et al. [5]. By selecting a good auxiliary function involving derivatives of u and using convexity considerations, the authors proved that (P) has no bounded solutions for any $1 < p < +\infty$.

We notice that the condition $p > 1$ is indispensable in these Liouville-type theorems for the Lane–Emden equation [5–7, 9, 10] in half space and fractional Lane–Emden equation [3] in half space. To our knowledge, the non-existence of non-negative solutions of (P) in $\mathbb{R}^n_+$ is completely open in the sublinear range $0 < p \leq 1$. Here, we study this range and prove that

THEOREM 1.1 *Assume that $n \geq 3$ and $0 < p \leq 1$. If $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ is a non-negative solution of (P), then $u \equiv 0$.*

REMARK 1.2. The author believe that using a process similar to proving Theorem 1.1, the Liouville result also holds for $n = 2$. This requires changing the fundamental solution of $-\Delta$ to $\ln \frac{1}{|x|}$, the comparison function ϕ defined in (3.10) to $x_2^r + \ln|x|$ for $r \in (0, 1)$, and making some corresponding adjustments.

Very recently, Montoro, Muglia, and Sciunzi [13, 14] provided a classification result for positive solutions to (P) for $n \geq 1$ in singular case: $p < -1$ and non-existence of positive solutions to (P) for $n \geq 1$ in the case: $-1 \leq p < 0$. Theorem 1.1 together with [10] and the results in [13, 14] provide a complete description of the solution to the Lane–Emden problem in $\mathbb{R}^n_+$ (P).

The main innovation of this article:

To use the comparison principle (see § 2), the low bound of the Lipschitz coefficient

$$c(x) = -\frac{p}{|x|^{n+2-p(n-2)}} \bar{u}^{p-1}$$

is required. Due to $p < 1$, we need some low bound of $\bar{u}$ for the sublinear problem in half space. Different with the whole space problem, for $|x|$ large, $C\frac{1}{|x|^{n-2}}$ is only the upper bound of the Kelvin transformation $\bar{u}$ but not the low bound for the Dirichlet problem on half space.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

To overcome this difficulty, by establishing an equivalent integral equation, we obtain a lower bound $\bar{u}(x) \geq C\frac{x_n}{|x|^n}$ for $|x|$ large. Based on the suitable lower bound of $\bar{u}$, we find a new comparison function $\phi(x) = \frac{1}{|x|^q} + x_n^r$ and then use the maximum principle based on comparisons to find that u only depends on x_n. Then, based on this and the equivalent integral equation, we prove the non-existence of non-negative solutions.

2. Preliminaries

PROPOSITION 2.1. (Strong Maximum Principle, [12]). *Consider a domain $\Omega \subset \mathbb{R}^n$ and define*

$$L = -\Delta + \sum_i b_i \partial_i + c,$$

where b_i and c are bounded on Ω. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ and $u \geq 0$ in Ω. If u vanishes at some point in Ω, then $u \equiv 0$ in Ω. In particular, if there exists a point on $\partial\Omega$, where $u > 0$, then $u > 0$ in Ω.

PROPOSITION 2.2. (Hopf's Lemma, § 9.5 Lemma 1 in [8]). *Let B be a ball in $\mathbb{R}^n$ and consider the elliptic operator*

$$L = -\Delta + c,$$

where c is bounded in B. Assume further that $u \in C^2(B) \cap C^1(\overline{B})$ satisfies $Lu \geq 0$ in B. If there exists $x \in \partial B$ such that

$$0 = u(x^0) < u(x) \quad \forall x \in B,$$

then one has $\frac{\partial u}{\partial \nu}(x^0) < 0$ for any outward pointing directional derivative ν and, in particular, $\nabla u(x^0) \neq 0$.

PROPOSITION 2.3. (Comparison Principle). *Assume that Ω is a domain. Let ϕ be a positive function on $\bar{\Omega}$ satisfying*

$$-\Delta\phi + \lambda(x)\phi \geq 0.$$

Assume that $w \in C^2(\Omega) \cap C(\bar{\Omega})$ solves

$$\begin{cases} -\Delta w + c(x)w \geq 0, & x \in \Omega, \\ w \geq 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

If

$$c(x) > \lambda(x), \quad \forall x \in \Omega,$$

and

$$\liminf_{|x|\to\infty, x\in\Omega} \frac{w(x)}{\phi(x)} \geq 0, \tag{2.2}$$

then $w \geq 0$ in Ω.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Proof. The Comparison Principle can be found in [4, Theorem 4.1]. For the convenience of readers, we provide its proof.

Suppose that there is a point $x \in \Omega$ such that $w(x) < 0$. Let $\tilde{w}(x) = \frac{w(x)}{\phi(x)}$. By $\phi(x) > 0$ and condition (2.2), $\tilde{w}$ has a minimum point $x^o \in \Omega$ such that $\tilde{w}(x^o) < 0$. By straight calculation,

$$-\Delta\tilde{w}(x) = 2\nabla\tilde{w} \cdot \frac{\nabla\phi}{\phi} + \left(-\Delta w(x) + \frac{\Delta\phi}{\phi} w\right)\frac{1}{\phi}. \tag{2.3}$$

On the one hand, since x^o is the minimum point of $\tilde{w}$, then

$$-\Delta\tilde{w}(x^o) \leq 0 \quad \text{and} \quad \nabla\tilde{w}(x^o) = 0.$$

Then by (2.3), we get

$$-\Delta w(x^o) + \frac{\Delta\phi}{\phi} w(x^o) \leq 0. \tag{2.4}$$

On the other hand, by $w(x^o) < 0$ and the assumption of the proposition,

$$-\Delta w(x^o) + \frac{\Delta\phi}{\phi} w(x^o) \geq -\Delta w(x^o) + \lambda(x^o)w(x^o) > -\Delta w(x^o) + c(x^o)w(x^o) \geq 0.$$

This contradicts (2.4). □

Remark 2.4. From the proof of proposition 2.3 (Comparison Principle), one can see that condition (2.1) is required only at the points where $\tilde{w}$ attains its minimum.

The idea in the following arguments is similar to that in the proof of [3, Theorem 4.1].

Proposition 2.5. *Assume that* $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ *is a positive solution of problem* (P). *Then,* u *is also a solution of integral equation*

$$u(x) = \int_{\mathbb{R}^n_+} G(x,y)u^p(y)dy. \tag{2.5}$$

Here, $G(x,y)$ *is the Green function of* $-\Delta$ *on half space* $\mathbb{R}^n_+$*:*

$$G(x,y) = c_n\left[\frac{1}{|x-y|^{n-2}} - \frac{1}{(|x-y|^2 + 4x_n y_n)^{\frac{n-2}{2}}}\right], \quad c_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}.$$

Proof. Let u be a positive solution of (P). First, we show that

$$\int_{\mathbb{R}^n_+} G(x,y)u^p(y)dy < +\infty. \tag{2.6}$$

Set

$$v_R(x) = \int_{B_R(P_R)} G_R(x,y)u^p(y)dy,$$

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

where $G_R(x,y)$ is the Green's function on $B_R(P_R)$, and $P_R = (0,\cdots,0,R)$,

$$G_R(x,y) = c_n \left[\frac{1}{|x-y|^{n-2}} - \frac{1}{\left(|x-y|^2 + \left(R - \frac{|x-P_R|^2}{R}\right)\left(R - \frac{|y-P_R|^2}{R}\right)\right)^{\frac{n-2}{2}}} \right].$$

Obviously, the Green's function G_R on $B_R(P_R)$ converges pointwise and monotonically to the Green's function G on $\mathbb{R}^n_+$. From the assumption on u, one can see that, for each $R>0$, $v_R(x)$ is well-defined and is continuous. Moreover,

$$\begin{cases} -\Delta v_R(x) = u^p(x), & x \in B_R(P_R), \\ v_R(x) = 0, & x \notin B_R(P_R). \end{cases}$$

Let $w_R(x) = u(x) - v_R(x)$, then

$$\begin{cases} -\Delta w_R(x) = 0, & x \in B_R(P_R), \\ w_R(x) \geq 0, & x \notin B_R(P_R). \end{cases}$$

Now, by the Maximum Principle [12], we derive

$$w_R(x) \geq 0, \quad \forall x \in B_R(P_R).$$

Then, letting $R \to \infty$, we arrive at

$$u(x) \geq \int_{\mathbb{R}^n_+} G(x,y)u^p(y)dy := v(x),$$

and thus, (2.6) holds. Here, $v(x)$ satisfies

$$\begin{cases} -\Delta v(x) = u^p(x), & x \in \mathbb{R}^n_+, \\ v(x) = 0, & x \notin \mathbb{R}^n_+. \end{cases}$$

Setting $w = u - v$, we have

$$\begin{cases} -\Delta w(x) = 0, \quad w(x) \geq 0, & x \in \mathbb{R}^n_+, \\ w(x) = 0, & x \notin \mathbb{R}^n_+. \end{cases} \tag{2.7}$$

Based on Boundary Harnack Inequality [1, 2], the uniqueness of harmonic functions on half spaces is well known: either

$$w(x) \equiv 0, \quad \forall x \in \mathbb{R}^n$$

or there is a constant c, such that

$$w(x) \geq cx_n.$$

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

We will derive a contradiction in the latter case. In fact, in this case, we have

$$u(x) = w(x) + v(x) \geq w(x) \geq cx_n. \tag{2.8}$$

Denote $x = (x', x_n)$, $y = (y', y_n) \in \mathbb{R}^{n-1} \times (0, +\infty)$. It follows from (2.8) that, for each fixed x and for sufficiently large R,

$$\begin{aligned} u(x) \geq v(x) &= \int_{\mathbb{R}^n_+} G(x,y) u^p(y) dy \geq c \int_{\mathbb{R}^n_+} G(x,y) y_n^p dy \\ &\geq c \int_R^{+\infty} y_n^p dy_n \int_{\mathbb{R}^{n-1}} G(x,y) dy'. \end{aligned}$$

Notice that

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} G(x,y) dy' \qquad (2.9) \\ &= \int_{\mathbb{R}^{n-1}} \left[\frac{1}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n-2}{2}}} - \frac{1}{(|x'-y'|^2 + |x_n + y_n|^2)^{\frac{n-2}{2}}} \right] dy' \\ &= \int_{\mathbb{R}^{n-1}} \left[\frac{1}{(|x'-y'|^2)^{\frac{n-2}{2}}} - \frac{1}{(|x'-y'|^2 + |x_n + y_n|^2)^{\frac{n-2}{2}}} \right] dy' \\ &\quad - \int_{\mathbb{R}^{n-1}} \left[\frac{1}{(|x'-y'|^2)^{\frac{n-2}{2}}} - \frac{1}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n-2}{2}}} \right] dy' \\ &= C\left(|x_n + y_n| - |y_n - x_n|\right), \qquad (2.9) \end{aligned}$$

where

$$C = \int_{\mathbb{R}^{n-1}} \left(\frac{1}{|\xi|^{n-2}} - \frac{1}{(|\xi|^2+1)^{\frac{n-2}{2}}} \right) d\xi \in (0, +\infty).$$

Therefore, for $x_n < R$, we get

$$u(x) \geq cx_n \int_R^{+\infty} y_n^p dy_n = +\infty,$$

which is a contradiction. Therefore, we must have $w \equiv 0$, i.e., (2.5) holds. □

3. Non-existence of positive solutions in the half space $\mathbb{R}^n_+$

We will employ the method of moving planes to prove the radial symmetry of u. However, without any decay conditions on u, we are not able to carry the method

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

of moving planes on u directly. To overcome this difficulty, we employ the Kelvin transformation of u centred at $x^o \in \partial\mathbb{R}^n_+$,

$$\bar{u}_{x^o}(x) = \frac{1}{|x-x^o|^{n-2}} u\left(\frac{x-x^o}{|x-x^o|^2} + x^o\right), \quad \forall x \in \mathbb{R}^n \setminus \{x^o\}.$$

Specifically, let $\bar{u}$ be the Kelvin transformation of u centred at the original point

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right), \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Clearly,

$$\bar{u}(x) = O\left(\frac{1}{|x|^{n-2}}\right), \quad |x| \to \infty. \tag{3.1}$$

Let u be the solution of (P). By direct calculation, the Kelvin transform $\bar{u}$ satisfies the equation

$$\begin{cases} -\Delta \bar{u} = \frac{\bar{u}^p}{|x|^\tau}, & x \in \mathbb{R}^n_+, \\ \bar{u}(x) \equiv 0, & x \in \partial\mathbb{R}^n_+, \end{cases} \tag{3.2}$$

where $\tau = n + 2 - p(n-2)$.

For any real number λ, let

$$T_\lambda = \{x \in \mathbb{R}^n_+ | x_1 = \lambda\}$$

be the plane perpendicular to the x_1-axis. Let Σ_λ be the region to the left of the plane T_λ

$$\Sigma_\lambda = \{x \in \mathbb{R}^n_+ | x_1 < \lambda\} \ \forall \lambda \in \mathbb{R}.$$

Denote

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n).$$

Set

$$w(x) = w_\lambda(x) = \bar{u}(x^\lambda) - \bar{u}(x).$$

According to equation (3.2),

$$-\Delta w_\lambda(x) = \frac{\bar{u}^p(x^\lambda)}{|x^\lambda|^\tau} - \frac{\bar{u}^p(x)}{|x|^\tau} = \frac{1}{|x|^\tau}(\bar{u}^p(x^\lambda) - \bar{u}^p(x)) + \bar{u}^p(x^\lambda)\left(\frac{1}{|x^\lambda|^\tau} - \frac{1}{|x|^\tau}\right). \tag{3.3}$$

For $x \in \Sigma_\lambda$ and $\lambda \le 0$, we have $|x^\lambda| \le |x|$ and

$$-\Delta w_\lambda(x) \ge \frac{p}{|x|^\tau} \xi_\lambda^{p-1} w_\lambda(x), \tag{3.4}$$

where $\xi_\lambda(x)$ is between $\bar{u}(x^\lambda)$ and $\bar{u}(x)$. For $0 < p \le 1$, if $\bar{u}(x^\lambda) \le \bar{u}(x)$, then

$$\frac{p}{|x|^\tau} \xi_\lambda^{p-1} w_\lambda(x) \ge \frac{p}{|x|^\tau} \bar{u}(x^\lambda)^{p-1} w_\lambda(x).$$

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Define

$$c(x) = -\frac{p}{|x|^\tau}\bar{u}(x^\lambda)^{p-1}.$$

Assume $\lambda \le 0$, $x \in \Sigma_\lambda \setminus \{0^\lambda\}$ such that $w_\lambda(x) \le 0$. Then,

$$-\Delta w_\lambda(x) + c(x)w_\lambda(x) \ge 0. \tag{3.5}$$

Lemma 3.1. *For $|x|$ large,*

$$\bar{u}(x) \ge C\frac{x_n}{|x|^n} \tag{3.6}$$

and

$$0 > c(x) > -\frac{Cx_n^{p-1}}{|x|^{2(p+1)}}, \tag{3.7}$$

where the constant $C > 0$ is independent of x and λ.

Proof. For $y \in B_1(2e_n)$, we have $u^p(y) \ge C$. Denote $x^* = (x', -x_n)$, for $|x| < 1$, by (2.5),

$$\begin{aligned} u(x) =& c_n \int_{\mathbb{R}^n_+} \left[\frac{1}{|x-y|^{n-2}} - \frac{1}{(|x-y|^2 + 4x_n y_n)^{\frac{n-2}{2}}}\right] u^q(y)dy \\ \ge& C\int_{B_1(2e_n)} \left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|x^*-y|^{n-2}}\right] dy. \end{aligned} \tag{3.8}$$

Applying the mean value theorem to (3.8), we have for $|x| < 1$,

$$u(x) \ge Cx_n. \tag{3.9}$$

Then, for $|x|$ large,

$$\bar{u}(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) \ge C\frac{x_n}{|x|^n}$$

and

$$0 < -c(x) < \frac{Cx_n^{p-1}}{|x|^{\tau+(p-1)n}} = \frac{Cx_n^{p-1}}{|x|^{n+2-p(n-2)+(p-1)n}} = \frac{Cx_n^{p-1}}{|x|^{2(p+1)}},$$

which imply (3.6) and (3.7). □

Lemma 3.2. (Decay at Infinity). *If w solves*

$$\begin{cases} -\Delta w + c(x)w \ge 0, & in\ B_R^c, \\ w \ge 0, & on\ \partial B_R^c, \end{cases}$$

then $w \ge 0$ in B_R^c.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Proof. In order to get '*Decay at infinity*' by proposition 2.3 (Comparison Principle), based on the low bound of $c(x)$ (3.7), we construct a new comparison function. Let

$$\tilde{w}_\lambda(x) = \tilde{w}(x) = \frac{w(x)}{\phi},$$

where

$$\phi(x) = x_n^r + \frac{1}{|x|^q}, \text{ with } 0 < r < 1 \text{ and } 0 < q < n-2. \tag{3.10}$$

For $|x| > R$, by calculation and using $|x| \geq x_n > 0$,

$$\begin{aligned} -\Delta\phi(x) &= -\Delta(x_n^r) - \Delta\left(\frac{1}{|x|^q}\right) = \frac{r(1-r)}{x_n^{2-r}} + \frac{q(n-2-q)}{|x|^{2+q}} \\ &\geq \frac{r(1-r)}{x_n^{2-r}} \\ &\geq C\frac{1}{|x|^{2(p+1)}}\frac{1}{x_n^{1-p-r}} + C\frac{1}{|x|^{2(p+1)+q}}\frac{1}{x_n^{1-p}} \\ &= C\frac{1}{|x|^{2(p+1)}}\frac{1}{x_n^{1-p}}\phi(x). \end{aligned} \tag{3.11}$$

$w(x) = O(\frac{1}{|x|^{n-2}})$ ensure that

$$\lim_{|x|\to\infty} \frac{w(x)}{\phi} = \lim_{|x|\to\infty} w(x)\frac{|x|^q}{x_n^r|x|^q + 1} = 0. \tag{3.12}$$

Then, by proposition 2.3 (Comparison Principle), we get the conclusion. □

Remark 3.3. If we choose $\phi(x) = x_n^r$, although (3.11) also holds, $\lim\limits_{|x|\to\infty} \frac{w(x)}{\phi} = 0$ is false. Here, $\lim\limits_{|x|\to\infty} \frac{w(x)}{\phi} = 0$ ensure that $\tilde{w}$ can attain its minimum, and thus, proposition 2.3 (Comparison Principle) works.

Lemma 3.4. *For $0 < p \leq 1$, assume that $u \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ is a solution of (P). Then, u(x) only depends on x_n variable, i.e., $u(x) = u(x_n)$.*

Proof. We employ the method of moving planes along any direction in $\mathbb{R}^{n-1}$ called the x_1 direction. Next, we will move the plane T_λ along the x_1 direction until $\lambda = 0$ to show that the positive solution is axially symmetric about the x_n-axis. We will go through the following two steps.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Step 1. We start from $-\infty$ to the right. In this step, we want to show that, for λ sufficiently negative,

$$\tilde{w}_\lambda(x) \geq 0, \text{ i.e., } w_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{3.13}$$

Otherwise, there exists some convergent sequence $\{x^k\}_{k=1}^{\infty} \subset \Sigma_\lambda$ such that

$$\tilde{w}_\lambda(x^k) \to \inf_{\Sigma_\lambda} \tilde{w}_\lambda(x) < 0, \quad \text{as } k \to \infty. \tag{3.14}$$

Note that

$$\tilde{w}_\lambda(0^\lambda) = w_\lambda(0^\lambda)/\phi(0^\lambda) \geq 0.$$

Thus, x^k will not converge to the singular point $0^\lambda \in \Sigma_\lambda$. Thus, combining (3.1) implies that

$$x^k \to \hat{x} \in \Sigma_\lambda, \quad \text{as } k \to \infty.$$

Then, by the continuity of w_λ, we obtain that

$$\tilde{w}_\lambda(\hat{x}) = \inf_{\Sigma_\lambda} \tilde{w}_\lambda(x) < 0. \tag{3.15}$$

Therefore, '*Decay at infinity*' implies that there exist $R > 0$ independent of λ such that

$$|\hat{x}| < R. \tag{3.16}$$

This is impossible since $\hat{x} \in \Sigma_\lambda$ and λ is sufficiently negative. Thus, (3.13) holds.

Step 2. Now, we move the plane T_λ towards the right, i.e., increasing the value of λ as long as the inequality (3.13) holds. Define

$$\lambda_0 = \sup\{\lambda | w_\mu(x) \geq 0, x \in \Sigma_\mu, \mu \leq \lambda, \lambda < 0\}.$$

In the step, we will show that

$$\lambda_0 = 0. \tag{3.17}$$

Suppose $\lambda_0 < 0$, by the strong maximum principle (proposition 2.1), we either have $w_{\lambda_0} \equiv 0$ or

$$w_{\lambda_0} > 0 \text{ in } \Sigma_{\lambda_0}. \tag{3.18}$$

We can derive that the plane T_{λ_0} can be moved further to the right. To be more rigorous, there exists some small $\epsilon_0 > 0$, such that, for $\epsilon \in (0, \epsilon_0)$,

$$w_{\lambda_0+\epsilon}(x) \geq 0, \quad x \in \Sigma_{\lambda_0+\epsilon}. \tag{3.19}$$

We delay proving (3.19). This inequality (3.19) contradicts with the definition of λ_0. Hence, (3.17) is valid.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Now, we prove (3.19). Suppose that (3.19) is violated for any $\epsilon > 0$. Then, there exists a sequence of numbers ϵ_i tending to 0, and for each i, the corresponding negative minimum x^i of $w_{\lambda_0+\epsilon_i}$. Let

$$\tilde{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi},$$

where $\phi(x)$ defined in (3.10). By straight calculation,

$$-\Delta\tilde{w}_\lambda(x) = 2\nabla\tilde{w}_\lambda \cdot \frac{\nabla\phi}{\phi} + \left(-\Delta w_\lambda(x) + \frac{\Delta\phi}{\phi}w_\lambda\right)\frac{1}{\phi}. \tag{3.20}$$

Notice that for any $\lambda \in \mathbb{R}$, the function $\tilde{w}_\lambda(x)$ tends to 0 as $|x| \to \infty$ since $\bar{u} \in O(\frac{1}{|x|^{n-2}})$. It follows that the function $\tilde{w}_{\lambda_0+\epsilon_i}$ attains its negative minimum at some point $x^i \in \Sigma_{\lambda_0+\epsilon_i}$ for each $i \in \mathbb{N}$. By Step 1, there exists $R > 0$ (independent of λ) such that

$$|x^i| \leq R, \ \forall i \in \mathbb{N}^+.$$

Then, there is a subsequence of $\{x^i\}$ (still denoted by $\{x^i\}$), which converges to some point $x^0 \in \mathbb{R}^n_+$. Now, we have

$$0 \leq \tilde{w}_{\lambda_0}(x^0) = \lim_{i\to\infty}\tilde{w}_{\lambda_0+\epsilon_i}(x^i) \leq 0, \quad \nabla\tilde{w}_{\lambda_0}(x^0) = \lim_{i\to\infty}\nabla\tilde{w}_{\lambda_0+\epsilon_i}(x^i) = 0.$$

That is, $\tilde{w}_{\lambda_0}(x^0) = 0$ and $\nabla\tilde{w}_{\lambda_0}(x^0) = 0$. Now, we compute

$$w_{\lambda_0}(x^0) = \tilde{w}_{\lambda_0}(x^0)\phi(x^0) = 0, \quad \nabla w_{\lambda_0}(x^0) = \phi(x^0)\nabla\tilde{w}_{\lambda_0}(x^0) + \tilde{w}_{\lambda_0}(x^0)\nabla\phi(x^0) = 0.$$

Recalling (3.18), we see that $x^0 \in \partial\Sigma_{\lambda_0}$. However, by Hopf's Lemma (proposition 2.2), we have the outward normal derivative $\frac{\partial w_{\lambda_0}}{\partial\nu}(x^o) < 0$, which yields a contradiction. Thus, (3.19) holds.

We have already pointed out earlier that (3.19) implies (3.17), then by (3.17)

$$w_0(x) \geq 0, \quad x \in \Sigma_0. \tag{3.21}$$

Similarly, we can move the plane from near $+\infty$ to the left limiting position, and we have

$$w_0(x) \leq 0, \quad x \in \Sigma_0. \tag{3.22}$$

Combining (3.21) with (3.22), we can conclude

$$w_0(x) = 0, \quad x \in \mathbb{R}^n_+. \tag{3.23}$$

Since the direction of the x_1-axis is arbitrary, we derive that the solution $\bar{u}(x)$ of (3.2) is axially symmetric about the x_n-axis.

Now, for any $x^0 \in \partial\mathbb{R}^n_+$, let $\bar{u}$ be the Kelvin transformation of u centred at x^0,

$$\bar{u}_{x^0}(x) = \frac{1}{|x-x^0|^{n-2}} u\left(\frac{x-x^0}{|x-x^0|^2} + x^0\right), \quad \forall x \in \mathbb{R}^n \setminus \{x^0\}.$$

Using an entirely similar argument, one can verify that $\bar{u}$ is axially symmetric about the line parallel to the x_n axis and passing through x^0. For the arbitrariness of x^0, we can conclude that $\bar{u}$ is rotationally symmetric with respect to the line parallel to the x_n axis. Choosing any two points x^1 and x^2 in $\mathbb{R}^n_+$ we have

$$x^1_n = x^2_n.$$

Let z^0 be the projection of the midpoint $x^0 = \frac{x^1+x^2}{2}$, where $z^0 \in \partial\mathbb{R}^n_+$. By the proof of above, we know $\bar{u}$ is axially symmetric with respect to $\overline{x^0 z^0}$. Setting

$$y^1 = \frac{x^1 - z^0}{|x^1 - z^0|^2} + z^0, \quad y^2 = \frac{x^2 - z^0}{|x^2 - z^0|^2} + z^0,$$

it is easy to see $\bar{u}(y^1) = \bar{u}(y^2)$. Hence $u(x^1) = u(x^2)$. This implies that the positive solution of (P) only depends on x_n variable, i.e., $u(x) = u(x_n)$. This completes the proof of lemma 3.4. □

Proposition 3.5. *If $u = u(x_n) > 0$, then*

$$u(x_n) = \int_{\mathbb{R}^n_+} G(x,y) u^p(y) dy = +\infty.$$

Proof. Let $R > 0$ be any fixed number. For $x_n > R$, by (2.9), we have

$$\begin{aligned} +\infty > u(x_n) &\geq C \int_0^R u^q(y_n) \mathrm{d}y_n \int_{\mathbb{R}^{n-1}} G(x,y) dy' \\ &\geq C \int_0^R u^q(y_n) \left(|x_n + y_n| - |x_n - y_n|\right) dy_n \\ &= 2C \int_0^R u^q(y_n) 2y_n dy_n. \end{aligned}$$

This implies that

$$+\infty > u(x_n) \geq C_1, \quad \forall x_n > R. \tag{3.24}$$

For $x_n \in (0, R)$, using (2.9) again, we obtain

$$\begin{aligned} +\infty > u(x_n) &\geq C \int_R^\infty u^q(y_n) dy_n \int_{\mathbb{R}^{n-1}} G(x,y) dy' \\ &\geq C \int_R^\infty u^q(y_n) \left(|x_n + y_n| - |y_n - x_n|\right) dy_n \\ &\geq 2C x_n \int_R^\infty u^q(y_n) dy_n. \end{aligned} \tag{3.25}$$

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press

Then by (3.24), for $x_n \in (0, R)$ we get

$$+\infty > u(x_n) \geq Cx_n \int_R^\infty C_1^q dy_n = +\infty. \tag{3.26}$$

□

Proof of Theorem 1.1.. Combining lemma 3.4, proposition 2.5, and proposition 3.5, we complete the proof of theorem 1.1. □

Acknowledgements

The author would like to thank Professor Congming Li for many precious suggestions.

This work was partially supported by National Natural Science Foundation of China (NSFC) (11901532) and Shanghai Jiao Tong University Scientific and Technological Innovation Funds.

References

[1] A. Ancona. Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien. *Ann. Inst. Fourier* **28** (1978), 169–213.

[2] R. F. Bass and K. Burdzy. A probabilistic proof of the boundary Harnack principle. In *Seminar on stochastic processes*, pp. 1–16 (Birkhäuser, Boston, 1990).

[3] W. Chen, Y. Fang and R. Yang. Liouville theorems involving the fractional Laplacian on a half space. *Adv. Math.* **274** (2015), 167–198.

[4] W. Chen, and C. Li. Maximum principles and the method of moving planes. In *Textbook in Chinese* 1–65 (2009).

[5] Z. Chen, C.-S. Lin and W. Zou. Monotonicity and nonexistence results to cooperative systems in the half space. *J. Funct. Anal.* **266** (2014), 1088–1105.

[6] L. Dupaigne, A. Farina and T. Petitt. Liouville-type theorems for the Lane–Emden equation in the half-space and cones. *J. Funct. Anal.* **284** (2023), 109906.

[7] L. Dupaigne, B. Sirakov and P. Souplet. A Liouville-type theorem for the Lane–Emden equation in a half-space. *Int. Math. Res. Not.* **12** (2022), 9024–9043.

[8] L. C. Evans. *Partial differential equations*, 2nd ed. (American Mathematical Society, Rhode Island, 2010).

[9] Y. Fang and W. Chen. A Liouville type theorem for poly-harmonic Dirichlet problems in a half space. *Adv. Math.* **229** (2012), 2835–2867.

[10] B. Gidas and J. Spruck. A priori bounds for positive solutions of nonlinear elliptic equations. *Comm. Partial Differ. Equ.* **6** (1981), 883–901.

[11] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1981), 525–598.

[12] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order,* Classics in Mathematics (Springer, Berlin, 2001). Reprint of the 1998 edition.

[13] L. Montoro, L. Muglia and B. Sciunzi. Classification of solutions to $-\Delta u = u^{-\gamma}$ in the half-space. *Math. Ann.* (3) **389** (2024), 3163–3179

[14] L. Montoro, L. Muglia and B. Sciunzi. The classification of all weak solutions to $-\Delta u = u^{-\gamma}$ in the half-space. arXiv:2404.03343v1.

https://doi.org/10.1017/prm.2024.98 Published online by Cambridge University Press