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About the Defectivity of Certain Segre–Veronese Varieties

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Abstract. We study the regularity of the higher secant varieties of $\mathbb{P}^1 \times \mathbb{P}^n$, embedded with divisors of type $(d, 2)$ and $(d, 3)$. We produce, for the highest defective cases, a "determinantal" equation of the secant variety. As a corollary, we prove that the Veronese triple embedding of $\mathbb{P}^n$ is not Grassmann defective.

1 Introduction

The study of secant varieties has a long history, dating back to the beginning of the twentieth century; nonetheless it is still of great interest since it is connected to many problems in mathematics.

The first problem that can be associated with the study of secant varieties is the well-known Waring problem for forms.

Problem. Given a form F of degree n, how many n-th powers of linear forms do we need such that their sum gives F?

The answer is trivial and well known in the univariate case, but requires much more work in the multivariate case. Assume the variety X to be a Veronese variety $V_d^n \subseteq \mathbb{P}^{\binom{n+d}{d}-1}$: it parametrizes the powers of degree d of linear forms in the space $\mathbb{P}^{\binom{n+d}{d}-1}$ of the homogeneous forms of degree d in $n+1$ variables. The minimal h such that the secant variety X^h fills up $\mathbb{P}^{\binom{n+d}{d}-1}$ gives the solution for the Waring problem for forms.

Along the same lines, if we generalize the Waring problem, and consider (for example) multilinear forms, we are led to study secant varieties of Segre varieties. On the other hand, we may think of a multilinear form as a tensor (see [6]), and the study of secant varieties corresponds to the study of the tensor rank for tensors of a given shape. If we generalize even further and work on multihomogeneous forms, we may view the decomposable ones as points of a Segre–Veronese variety and hence study its secant varieties as a way to study the stratifications of partially symmetric tensors with respect to their "symmetric" tensor rank.

Moreover, thanks to Terracini's lemma, the study of secant varieties is also connected to specialness of linear systems and to a few conjectures on it [2, 12]

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Much work has been done, especially in recent years [6,10,15,19], but much more work is still needed, even in simple cases, especially to understand the "algebraic" structure of secant varieties, *e.g.*, their ideals and degrees.

In this paper, we focus on the case of Segre–Veronese varieties. After a preliminary section where the notations are fixed and some known results are stated, we discuss the cases of $\mathbb{P}^1 \times \mathbb{P}^n$ embedded in degree $(d, 2)$ and $(d, 3)$, describing in detail the defective cases. Then we show some cases where it is possible to give some "determinantal" equations for the secant varieties of a Segre–Veronese variety. In the end, we present some consequences of the main results about the so-called Grassman defectivity [11].

2 Preliminaries and Notations

Definition 2.1 We denote by X^{h+1} the h-th secant variety to $X \subseteq \mathbb{P}^N$, *i.e.*, the closure of the union of all h-dimensional linear spaces $\mathbb{P}^h$ that are $(h+1)$-secant to X.

Example 2.2 The first secant variety X^2 is the chordal variety, *i.e.*, the variety of secant lines.

Remark 2.3. It is possible to compute an *expected* dimension for the secant varieties $\exp\dim X^h = \min\{N, h\dim X + h - 1\}$.

Actually, the real dimension could be smaller than the expected one; therefore we can give the following

Definition 2.4 A variety $X \subseteq \mathbb{P}^N$ is said to be *h-defective* if the dimension of its h-th secant variety X^{h+1} is smaller than the expected one. Otherwise, we say that X is *h-regular*, or *nondefective*.

If X is $(h-1)$-defective, then its defect $\delta_h = \exp\dim X^h - \dim X^h$ is the difference between the expected and the effective dimension (we note here that X is $(h-1)$-defective if and only if $\delta_h \geq 1$).

We recall here some classical results that are of use in the study of defectivity of projective varieties.

Proposition 2.5 **(Terracini's Lemma)** *Let $P \in X^{h+1}$ be a general point. Assume $P \in \langle P_0, \dots, P_h \rangle$, (the linear span of $h+1$ points on X). Then*

$$T_P(X^{h+1}) = \langle T_{P_0}X, \dots, T_{P_h}X \rangle.$$

This result allows us to find the dimension of secant varieties (*i.e.*, the dimension of their tangent spaces) by studying the dual dimension of the linear system of hyperplanes that contain $T_P(X^{h+1})$, that is, contain the tangent spaces to $h+1$ general points on X. This lemma also connects the problem of defectivity of higher secant varieties with the problem of specialness of linear systems and the computation of the Hilbert function of 0-dimensional schemes made of double fat points.

Proposition 2.6 *If X is a h-regular variety, and X^{h+1} does not fill up the ambient space, then X is also $(h-1)$-regular.*

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The reason is more evident if we think in terms of linear systems. The h-regularity means that the linear system $\mathcal{L}(2^{h+1})$ of the hyperplanes that contain $h+1$ generic double fat points is non-special and has the right dimension, *i.e.,* the points impose the right (maximum) number of conditions. Therefore, a smaller number of points is forced to impose the maximum number of conditions. This result permits gathering broader information about regularity from the analysis of a small number of cases.

Proposition 2.7 **(Horace's Lemma** [17, Lemma 4.2]) *Let $X \subseteq \mathbb{P}^n$ be a scheme associated with a sheaf of ideals $\mathcal{I}_X$ and L an irreducible divisor (an invertible sheaf) of degree d. If $H^0(\mathcal{I}_{X'}(t-d)) = 0$ and $H^0(\mathcal{I}_{X_L}(t)) = 0$, then $H^0(\mathcal{I}_X(t)) = 0$, where X_L is the scheme "cut" by L on X,* i.e., *$\mathcal{I}_{X_L} = (\mathcal{I}_X + \mathcal{I}_L)$ and X' is the residual scheme given by $\mathcal{I}_{X'} = \mathcal{I}_X : \mathcal{I}_L$.*

This lemma exploits the residual sequence $0 \to \mathcal{I}_{X_L} \to \mathcal{I}_X \to \mathcal{I}_{X'} \to 0$ to split into two simpler problems the study of regularity and it has also a "differential version" (see [3, Proposition 9.1], or [7] for a more suitable statement).

Definition 2.8 We denote by Δ_h the difference function of the dimensions of two successive secant varieties, that is, $\Delta_h(X) = \dim X^h - \dim X^{h-1}$.

Remark 2.9. Since we know what the expected dimension is, we can also compute an expected value for Δ_h, in case the expected dimension of X^h is smaller than N (the dimension of the ambient space)

$$e\Delta_h = h \dim X + h - 1 - [(h-1)\dim X + h - 2] = \dim X + 1.$$

As a consequence of Proposition 2.6, a first approach in the study of defectivity of secant varieties consists in investigating the case of the minimal h such that the variety of X^h is expected to fill up the ambient space $\mathbb{P}^N$.

If this case is non-defective, so are the lower secant varieties; otherwise, it may give a hint on the kind of defectivity.

2.1 Segre–Veronese Varieties

We focus now on the special case of Segre–Veronese varieties, *i.e.,* varieties that are obtained via embeddings of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ of type $(a_1, \dots, a_t)$.

We fix the following notation [6]. If $\mathbf{n} = (n_1, \dots, n_t)$, and $\mathbf{a} = (a_1, \dots, a_t)$ as above, we denote by $X^{\mathbf{n}}_{\mathbf{a}}$ the corresponding Segre–Veronese variety. In this case, [6] provides a useful method to study subschemes of Segre–Veronese varieties by means of schemes in ordinary projective spaces. Namely, the result below allows us to find subschemes of ordinary projective spaces with the Hilbert function that coincides (for a given degree) with that of generic double points in a Segre–Veronese variety.

In order to fix notations, we shall assume that the homogeneous coordinates in $\mathbb{P}^{n_i}$ are $\{x_{0,i}, x_{1,i}, \dots, x_{n_i,i}\}$, for all $i = 1, \dots, t$.

Theorem 2.10 **([6, Theorem 1.1])** *Let Z be a scheme of generic double fat points in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$, contained in the affine chart $\{x_{0,1} \neq 0, x_{0,2} \neq 0, \dots, x_{0,t} \neq 0\}$.*

Let $f\colon \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t} \longrightarrow \mathbb{P}^n$ be the following rational map, where $n = \sum_{i=1}^{t} n_i$:

$$(x_{0,1} : \cdots : x_{n_1,1}), \ldots, (x_{0,t} : \cdots : x_{n_t,t}) \\ \mapsto (x_{0,1}x_{0,2}\cdots x_{0,t} : x_{1,1}x_{0,2}\cdots : x_{0,t} : \cdots x_{n_t,t}x_{0,1}\cdots x_{0,t-1}).$$

Let $\Pi_i \subseteq \mathbb{P}^n$ be the linear subschemes spanned by the coordinate points $Q_{i,1}, \ldots, Q_{i,n_i}$ in $\mathbb{P}^n$ corresponding to the coordinates $z_{i,j} := x_{j,i}\prod_{h=1,h\neq i}^{t} x_{0,h}$ for $j = 1, \ldots, n$. Let Z' be the image of Z via f. Then $\dim(I_Z)_{(a_1,\ldots,a_t)} = \dim(I_W)_a$ where:

$$W = Z' \cup W_1 \cup \cdots \cup W_t, \quad W_i = (a - a_i)\Pi_i, \quad a = a_1 + \cdots + a_t.$$

3 The Segre–Veronese Variety $X^{(1,n)}_{(a,2)}$

In this section, we will focus our interest on the case $X^{(1,n)}_{(a,2)}$, *i.e.*, $\mathbb{P}^1 \times \mathbb{P}^n$ embedded with divisors of type $(a, 2)$.

In this case, the ambient space is a projective space having dimension $N = (a+1)\binom{n+2}{2} - 1$. Since $\exp\dim X^{h+1} \geq N \Leftrightarrow h \geq \frac{(a+1)(n+1)}{2}$, the first case to be considered is $h = \left\lceil \frac{(a+1)(n+1)}{2} \right\rceil$.

We show that the behaviour is different, according to a being odd or even. We start by analyzing the case $a = 2d + 1$. We have the following.

Proposition 3.1 *Let X be the Segre–Veronese variety $X^{(1,n)}_{(2d+1,2)}$. Then X is h-regular for any h.*

Proof It is enough to prove regularity for $h = (d + 1)(n + 1)$, since

$$N = (d + 1)(n + 1)(n + 2) - 1,$$

and the expected dimension of the secant variety X^h for such h is exactly N.

We work by induction on n, the case $n = 1$ being known [6]. For $n \geq 2$, we apply Horace's method and consider a scheme Z made of h generic fat double points on $\mathbb{P}^1 \times \mathbb{P}^n$. We expect to find no forms of type $(2d + 1, 2)$ in the ideal of Z. We prove that for a specialization of Z, it is impossible to find any such form, and we conclude by semicontinuity.

It is equivalent, by Theorem 2.10, to study the $(2d + 3)$-ics in $\mathbb{P}^{n+1}$ through the scheme W, which is the image of Z plus two more components, *i.e.*, a $\mathbb{P}^{n-1}$ of multiplicity $2d + 2$ and an extra double point. Assume that $n(d + 1) + 1$ of the points are general on a general hyperplane, say H. By the inductive assumption, the ideal of the scheme $W|_H$ contains no forms of degree $2d + 3$ (again by Theorem 2.10, it is equivalent to studying $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ embedded in degree $(2d + 1, 2)$), so we need to consider the system of the $(2d + 2)$-ics in $\mathbb{P}^{n+1}$ through the residual scheme W', composed of $d + 1$ generic double fat points, $n(d + 1) + 1$ generic (simple) points on H and a $2d$-tuple $\mathbb{P}^{n-1}$. This linear system has $d + 1$ fixed components, namely the hyperplanes spanned by the $\mathbb{P}^{n-1}$ and each double point; so we need to study the system of $(d + 1)$-ics through the scheme W'', made of $d + 1$ general points, $n(d + 1) + 1$ general points on H and a d-tuple $\mathbb{P}^{n-1}$.

Let $I_\Lambda = (l_1, l_2)^d$ be the ideal of a d-tuple $\mathbb{P}^{n-1}$. We may assume, since the linear forms l_1, l_2 are generic, that $I_\Lambda = (x_0, x_1)^d$ for a suitable set of homogeneous coordinates in $\mathbb{P}^{n+1}$. The dimension of the vector space $(I_\Lambda)_{d+1}$ is $(d+2)+(n-1)(d+1) = n(d+1)+1$. Then if the points on H impose independent conditions on forms in $(I_\Lambda)_{d+1}$, the linear system of $(d+1)$-ics in $\mathbb{P}^{n+1}$ through W'' is empty, and we are done.

If this were not the case, then the linear system had to be made of the fixed component H plus the d-ics through the d-tuple $\mathbb{P}^{n-1}$ and the general points. But then each of the hyperplanes generated by the $\mathbb{P}^{n-1}$ and one of the points should be a fixed component again. This is impossible since there are $d+1$ of them. ∎

On the other hand, in the even case there is always defectivity. We start by recalling some known results about defectivity of maximal higher secant varieties, *i.e.*, those for $h = \left\lceil \frac{(a+1)(n+1)}{2} \right\rceil$ [6].

Proposition 3.2 ([**6**]) *Let X be the Segre–Veronese variety $X^{(1,n)}_{(2d,2)}$. Then X^h is defective for $h = \left\lceil \frac{(2d+1)(n+1)}{2} \right\rceil$.*

The proof consists of finding two divisors of type $(d,1)$, each one passing through the h (simple) points, so that the product of the associated bihomogeneous forms is a form of bidegree $(2d,2)$ which was not expected to exist in the ideal of the subscheme made of the h double points.

Another result by Bocci [4] gives other examples of defective secant varieties (as a consequence of a slightly different question, regarding special linear systems) and states that there is defectivity also for $h = \left\lfloor \frac{(2d+1)(n+1)}{2} \right\rfloor$. In the following proposition, the existence of a *special effect* variety forces defectivity.

Proposition 3.3 ([**4, Proposition 6.3**]) *Let $Y \subseteq \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ be a divisor of type (e_1, e_2), with $e_i \neq 0$ for at least one i. Then Y is a 2-special effect variety for $\mathcal{L}_{(d_1,d_2)}(2^h)$, with $d_1 \cdot d_2 \neq 0$ in the following cases:*

$\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$	(d_1, d_2)	(e_1, e_2)	h
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2, 2e_2)$	$(1, e_2)$	$2e_2 + 1$
$\mathbb{P}^1 \times \mathbb{P}^1$	$(2e_1, 2)$	$(e_1, 1)$	$2e_1 + 1$
$\mathbb{P}^1 \times \mathbb{P}^{n_2}$	$(2e_1, 2)$	$(e_1, 1)$	$\left\lfloor \frac{(2e_1+1)(n_2+1)}{2} \right\rfloor \leq h \leq e_1 n_2 + e_1 + n_2$
$\mathbb{P}^2 \times \mathbb{P}^{n_2}$	$(2, 2)$	$(1, 1)$	$\left\lfloor \frac{3n_2^2+9n_2+5}{n_2+3} \right\rfloor \leq h \leq 3n_2 + 2$
$\mathbb{P}^3 \times \mathbb{P}^3$	$(2, 2)$	$(1, 1)$	15
$\mathbb{P}^3 \times \mathbb{P}^4$	$(2, 2)$	$(1, 1)$	19

According to the previous proposition, for $e_1 = d$, $n_2 = n$, we are left to investigate the cases $h < \left\lfloor \frac{(2d+1)(n+1)}{2} \right\rfloor$. In such situations, the secant variety is not expected to fill the ambient space, so it is useful to compute its expected codimension. The dimension of the ambient space is

$$N = \frac{(2d+1)(n+1)(n+2)}{2} - 1, \ \exp\dim X^h = h(n+1)+h-1 = (h-1)(n+2)+(n+1).$$

So

$$\begin{aligned}N - \exp\dim X^h &= \frac{(2d+1)(n+1)(n+2)}{2} - 1 - [(h-1)(n+2) + (n+1)]\\ &= \frac{(n+1)(n+2)}{2} + d(n+1)(n+2) - 1 - (h-1)(n+2) - (n+1)\\ &= \frac{n(n+1)}{2} - 1 + (n+2)[d(n+1) - (h-1)]\\ &= \binom{n+1}{2} - 1 + (n+2)[d(n+1) - (h-1)].\end{aligned}$$

3.1 A Different Approach

The main result in this section can be seen as a special case of a statement about 2-tuple embeddings of varieties of minimal degree, as we explicitly show here.

Theorem 3.4 *Let X be Segre–Veronese variety $X^{(1,n)}_{(2d,2)}$. Then X^h is defective if and only if $d(n+1)+1 \le h \le (d+1)(n+1)-1$.*

We recall that a nondegenerate projective variety $Y \subseteq \mathbb{P}^n$ is said to be of *minimal degree* if $n = \dim Y + \deg Y - 1$.

The classification of such varieties is well known [16]. They can be either projective spaces, quadric hypersurfaces, rational normal scrolls, the Veronese surface in $\mathbb{P}^5$ or a cone over the Veronese surface.

Remark 3.5. The variety $S \subseteq \mathbb{P}^{(d+1)(n+1)-1}$, given by $S = X^{(1,n)}_{(d,1)}$, is a rational normal scroll of dimension $n+1$ and degree $d(n+1)$.

In fact, it is not difficult to write down explicitly the $2 \times d(n+1)$ matrix of linear forms whose maximal minors define S: if we assume that the embedding is given by $(x_0 : x_1), (y_0 : \ldots : y_n) \mapsto (x_0^d y_0 : x_0^{d-1} x_1 y_0 : \ldots : x_1^d y_0 : x_0^d y_1 : \ldots : x_1^d y_n)$ and we denote by $z_{(d,0)}, z_{(d-1,0)}, \ldots, z_{(0,n)}$ the coordinates in $\mathbb{P}^{(d+1)(n+1)-1}$, the matrix is

$$\left(\begin{array}{cccc|cc|cccc} z_{(0,0)} & z_{(1,0)} & \cdots & z_{(d-1,0)} & \cdots & \cdots & z_{(0,n)} & z_{(1,n)} & \cdots & z_{(d-1,n)} \\ z_{(1,0)} & z_{(2,0)} & \cdots & z_{(d,0)} & \cdots & \cdots & z_{(1,n)} & z_{(2,n)} & \cdots & z_{(d,n)} \end{array}\right)$$

Definition 3.6 ([1, Definition 2.4]) A projective variety X of dimension m is said to be *extremal of type* (m, i) if and only if X is i-defective and X^{m+i-1} is not a cone.

Theorem 3.7 ([1, Theorem 3.10]) *Let X be a variety of dimension $m \ge 2$. Then X is extremal of type (m, i) if and only if it is the 2-tuple embedding of a variety of minimal degree (of degree i and dimension m).*

Proof of Theorem 3.4 As a straightforward consequence of the above statement, the 2-tuple embedding X of $S = X^{(1,n)}_{(d,1)}$ in $\mathbb{P}^N$ is extremal of type $(n+1, d(n+1))$, so it has defective higher secant varieties.

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By [1, Observation 2.5] we also have that for such an extremal variety of degree $d(n+1)$ and dimension $n+1$,

$$\Delta_h(X) = \begin{cases} n+2 & \text{for } 1 \le h \le d(n+1), \\ (d+1)(n+1) - h + 1 & \text{for } d(n+1) \le h \le d(n+1) + (n+1) + 1. \end{cases}$$

By the definition of Δ_h, we have that the dimension of the last nondefective secant variety is $d(n+1)(\dim X + 1) - 1$.

It follows that the smallest defective secant variety is $X^{d(n+1)+1}$ with defect 1, and this forces all higher (non-filling) secant varieties to be defective, with increasing defects from 1 up to $\binom{n+2}{2}$.

Assume, in fact, that $h \ge d(n+1)+1$, *i.e.*, $h = d(n+1)+t$, for a positive integer t.

$$\begin{aligned} \text{(3.1)} \qquad \dim X^h &= \Delta_h + \dim X^{h-1} \\ &= \Delta_{d(n+1)+t} + \Delta_{d(n+1)+t-1} + \dim X^{d(n+1)+t-2} + \cdots \\ &\qquad + \Delta_{d(n+1)+t} + \Delta_{d(n+1)+t-1} + \cdots + \Delta_{d(n+1)+1} + \dim X^{d(n+1)} \\ &= \sum_1^t \Delta_{d(n+1)+i} + d(n+1)(n+2) - 1 \\ &= \sum_1^t ((d+1)(n+1) - i + 1) + d(n+1)(n+2) - 1 \\ &= t((d+1)(n+1)+1) - \binom{t+1}{2} + d(n+1)(n+2) - 1. \end{aligned}$$

Then for any $h \ge d(n+1)+1$,

$$\begin{aligned} \dim X^h &= [h - d(n+1)][(d+1)(n+1)+1] + d(n+1)(n+2) \\ &\qquad - \binom{h - d(n+1) + 1}{2} - 1 \\ &= h[(d+1)(n+1)+1] - \binom{d(n+1)}{2} - \binom{h+1}{2} - 1 \end{aligned}$$

Therefore, the defect $\delta_h = \exp\dim X^h - \dim X^h$ for $d(n+1)+1 \le h \le (d+1)(n+1)$ is the following:

$$\begin{aligned} \delta_h =& h(n+2) - 1 - h[(d+1)(n+1)+1] + \binom{d(n+1)}{2} + \binom{h+1}{2} + 1 = \\ =& \binom{d(n+1)}{2} + \binom{h+1}{2} - hd(n+1), \end{aligned}$$

where we assume $h \ge d(n+1)+1$, that is, the defect is a positive integer. This proves Theorem 3.4. ∎

Remark 3.8. We notice that Theorem 3.4 can actually be proved along the same lines as Proposition 3.1, *i.e.,* by means of the Horace method and Theorem 2.10.

Remark 3.9. If we were in the nondefective case, we would expect that the last significant (nontrivial, nonfilling) secant variety is the one for $h = \left\lfloor \frac{(2d+1)(n+1)}{2} \right\rfloor = d(n+1) + \left\lfloor \frac{n+1}{2} \right\rfloor$; we get instead that for $h = (d+1)(n+1)$, X^h has dimension $\dim X^h = (d+1)(n+1)[(d+1)(n+1)+1] - \binom{d(n+1)}{2} - \binom{(d+1)(n+1)+1}{2} - 1 = \frac{(2d+1)(n+1)(n+2)}{2} - 1 = N$, so it coincides with the ambient space, while the highest (significant) secant variety, for $h = (d+1)(n+1) - 1$, has dimension $N - 1$ and is defective with defect $\binom{n+1}{2}$.

3.2 The Equations for Some Secant Varieties

Following an idea of Schreyer, in some cases we are able to describe the equations of the secant varieties. Consider the following diagram:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{P}^n \arrow[r, "(\nu_a, \nu_2)"] \arrow[dr, "\Phi_{(a,2)}"'] & \mathbb{P}^d \times \mathbb{P}^{\binom{n+2}{2}-1} \arrow[d, "\sigma"] \\
& \mathbb{P}^N
\end{tikzcd}$$

where the Segre–Veronese embedding Φ is realized as composition of the horizontal map (the product of the two Veronese embeddings (ν_a, ν_2)) and the vertical map σ (the Segre embedding).

Let x_0, x_1 and $y_0, \dots, y_n$ be the coordinates, respectively, in $\mathbb{P}^1$ and $\mathbb{P}^n$. Then the coordinates in $\mathbb{P}^N$, where $N = (d+1)\binom{n+2}{2} - 1$, can be defined as

$$z_{i,\alpha} = \Phi(x_0^i x_1^{a-i} y_0^{a_0} \cdots y_n^{a_n}),$$

where we assume that $\alpha = (a_0, \dots, a_n)$ is a multiindex with $|\alpha| = 2$.

Assume that $a = 2d$. We may then consider the following two matrices:

$$A = \begin{pmatrix} x_0^{2d} & x_0^{2d-1}x_1 & \dots & x_0^d x_1^d \\ x_0^{2d-1}x_1 & x_0^{2d-2}x_1^2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ x_0^d x_1^d & x_0^{d-1}x_1^{d+1} & \dots & x_1^{2d} \end{pmatrix} \quad B = \begin{pmatrix} y_0^2 & y_0 y_1 & \dots & y_0 y_n \\ y_0 y_1 & y_1^2 & \dots & y_1 y_n \\ \vdots & \vdots & \ddots & \vdots \\ y_0 y_n & y_1 y_n & \dots & y_n^2 \end{pmatrix}.$$

On the Veronese varieties $\nu_d(\mathbb{P}^1)$ and $\nu_2(\mathbb{P}^n)$, the matrices A and B respectively have rank 1.

If we take the tensor product $A \otimes B$ of A and B, we get a square matrix of dimension $(d+1)(n+1)$. The secant variety X^h to the $(a, 2)$-Segre–Veronese variety is contained in the rank h locus of $\Phi(A \otimes B)$, since any point on the secant is a linear combination of h points where the rank of $\Phi(A \otimes B)$ is 1.

In particular, $X^{(d+1)(n+1)-1}$ is contained in the rank $(d+1)(n+1) - 1$ locus of $\Phi(A \otimes B)$, *i.e.,* in the hypersurface defined by the determinant of $\Phi(A \otimes B)$. But we already know (by the previous section) that $X^{(d+1)(n+1)-1}$ is a hypersurface, so we conclude that the ideal of $X^{(d+1)(n+1)-1}$ is generated by the determinant of $\Phi(A \otimes B)$.

4 The Embedding $(d, 3)$

Lemma 4.1 *The Segre–Veronese variety $X^{(1,n)}_{(1,3)}$ is regular for any $n \neq 2$.*

Proof First we recall that the case $n = 1$ is known to be h-regular for any h [6], while the case $n = 2$ is 5-defective [5]. Then we proceed by induction on n, the basic case given by $n = 3$. In this case, we need to show that $\mathbb{P}^1 \times \mathbb{P}^3$ embedded in degree $(1, 3)$ is h-regular for any h. It is enough to show regularity for $h = 8$, since for this integer the expected dimension of $X^8 \subseteq \mathbb{P}^{39}$ is 39.

We assume, by Theorem 2.10, that we are considering the hypersurfaces of degree 4 in $\mathbb{P}^4$, passing through eight double fat points and a triple point, and containing a plane $\simeq \mathbb{P}^2$. We show that we cannot find such hypersurfaces. In fact, by running out an example using CoCoA [13], we find (as expected) that the affine dimension of the space of the quartic hypersurfaces containing that scheme is 0, *i.e.,* it is empty.

Assume then $n \geq 4$ and consider a 0-dimensional scheme W made of $h = \lceil \frac{2(n+1)(n+3)}{3} \rceil$ double points. We specialize on a divisor $H \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$ the subscheme W_H made of $h' = \lceil \frac{2n(n+2)}{3} \rceil$ double points. By inductive hypothesis on n, there are no forms of type $(1, 3)$ passing through W_H; we then need to study on $\mathbb{P}^1 \times \mathbb{P}^n$ the residual scheme W', in bidegree $(1, 2)$. However, thanks to [6, Proposition 2.3], we know that it is regular for any h, since it is of the form $X^{(r,k)}_{(1,r+1)}$. Then the total number of conditions imposed by W' is $(h - h')(n + 2) + h' = h(n + 2) - h'(n + 1)$, which is always bigger than $2\binom{n+2}{2} - 1$, so there are no divisors of type $(1, 2)$ containing W'.

The specialization is not excessive, since if H were a fixed component, a simple computation shows that for $n \geq 4$ the linear system of divisors containing the residual scheme W'' is empty. ∎

Theorem 4.2 *The Segre–Veronese variety $X^{(1,n)}_{(d,3)}$ is regular, for any n and d, except for the case $d = 1, n = 2$.*

Proof We proceed by double induction on d and n. The starting cases are the following. For $n = 1$ and any d, $\mathbb{P}^1 \times \mathbb{P}^1$, $(d, 3)$ is known to be h-regular for any h (see [6, Theorem 2.1]). For the case $d = 1, n > 2$, $\mathbb{P}^1 \times \mathbb{P}^n$, $(1, 3)$ is also h-regular for any h by Lemma 4.1. Then we can assume that $n \geq 3$, $d \geq 2$ and go on by induction.

The ambient space has dimension $N = (d + 1)\binom{n+3}{3} - 1$. We expect that the minimal value of h such that the secant variety X^h fills up $\mathbb{P}^N$ is $h = \lceil \frac{(d+1)(n+1)(n+3)}{6} \rceil$.

Again we use Theorem 2.10 and Horace's method. Namely, instead of working on forms of bidegree $(d, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^n$, we work on $\mathbb{P}^{n+1}$ with forms of degree $d + 3$. So it is enough (by Terracini's lemma and Theorem 2.10) to show that there are no hypersurfaces of degree $d+3$ in $\mathbb{P}^{n+1}$ containing a scheme W made of h double points, a triple point, and a d-tuple $\mathbb{P}^{n-1}$.

We shall distinguish between two cases, according to the congruence class mod 6 of d and n. These are summarized in the following table. In particular, for the pairs (d, n) corresponding to an "S" we shall proceed with a standard Horace method, while for the cases corresponding to a "D" a differential Horace method will be needed.

$d \setminus n$	0	1	2	3	4	5
0	S	S	D	D	S	D
1	S	S	D	S	S	D
2	S	D	S	D	S	D
3	S	S	D	S	S	D
4	S	D	S	D	S	D
5	S	S	S	S	S	S

In the standard cases (S), we specialize on a generic hyperplane $H \simeq \mathbb{P}^n$, h' double points, and the triple point, with $h' = \left\lceil \frac{(d+1)n(n+2)}{6} \right\rceil$. The scheme W_H is made of h' double points, a triple point, and a d-tuple $\mathbb{P}^{n-2}$. By inductive assumption on n, there are no $(d+3)$-ics through W_H, so we need to study the residual scheme W', made of $h - h'$ general double points, a d-tuple $\mathbb{P}^{n-1}$, and h' simple points plus another double point on H. Again by Theorem 2.10, this corresponds to studying the variety $\mathbb{P}^1 \times \mathbb{P}^n$ embedded in degree $(d, 2)$.

By Theorems 3.4 and 3.1, we know that these Segre–Veronese varieties are k-regular for any k, when d is odd, and for $k \le \frac{d(n+1)}{2}$ when d is even.

By an explicit computation (Remark 4.4), we get that $k = h - h' \le \frac{d(n+1)}{2}$ for any possible d, n except for the case $d = 2$, n even that will be treated separately below.

When $h - h' \le \frac{d}{2}(n+1)$, the dimension of the linear system of $(d+2)$-ics in $\mathbb{P}^{n+1}$ through $h - h'$ double points, *i.e.,* the codimension of the secant variety $X^{h-h'}$ in $\mathbb{P}^{(d+1)\binom{n+2}{2}-1}$ is $(d+1)\binom{n+2}{2} - 1 - (h-h')(n+2) + 1$; if we also impose the conditions given by the residual scheme made of $h' = \left\lceil \frac{(d+1)n(n+2)}{6} \right\rceil$ simple points on H, we find (Remark 4.5) that the affine dimension $N(h, h')$ of such a system is smaller than zero, *i.e.,* the system is always empty.

In the cases corresponding to the letter D in the table, we proceed with a slightly different specialization of W. Let $h'' = \left\lfloor \frac{(d+1)n(n+2)}{6} \right\rfloor$ and

$$h^* = (d+1)\binom{n+2}{3} - 1 - h''(n+1).$$

Then specialize on a general hyperplane H the triple point, h'' double fat points, and h^* simple points. This is equivalent to considering $X = X^{(1,n-1)}_{(d,3)}$ and the regularity of $X^{h''}$. By inductive assumption on n, there are no forms of degree $d+3$ of $\mathbb{P}^n$ in the ideal of the scheme W_H made of h'' double points, a d-tuple $\mathbb{P}^{n-2}$, a triple point, and h^* simple points. The residual scheme W' is made of the d-tuple $\mathbb{P}^{n-1}$, $h - h'' - h^*$ double generic fat points, h'' simple points, and h^* differential residues plus a double point on H. So we need to verify that the linear system of forms of degree $d+2$ in $\mathbb{P}^{n+1}$ containing the residual scheme is empty. Since for such a number of points the secant variety to $X^{(1,n)}_{(d,2)}$ is regular (by Proposition 3.1 and Theorem 3.4), by a direct computation (explicitly done in Remark 4.6) we get that the dimension $N(h, h'', h^*)$ of the linear system above is

$$(d+1)\binom{n+2}{2} - 1 - (h - h'' - h^*)(n+2) - h'' - h^*(n+1) \le -1,$$

i.e., the system is empty.

In both cases (the standard Horace method or the differential one), the specialization is not excessive. To show this, let us assume we are studying (by Theorem 2.10) the system of hypersurfaces of degree $d+3$ in $\mathbb{P}^{n+1}$ that contain the scheme W made of h double fat points, a triple point, and a d-tuple $\mathbb{P}^{n-1}$, say Π. If the hyperplane H were a fixed component, it would be easy to verify that the linear system of $(d+1)$-ics containing the residual scheme is empty (this is explicitly done in Remarks 4.5 and 4.6), by considering the fixed components originated by the hyperplanes spanned by Π and one of the remaining double points.

The only case left is $d=2$, n even. Assume then $n=2m$ and $d=2$. We may write down explicitly $h=\left\lceil \frac{(2m+1)(2m+3)}{2}\right\rceil = 2m^2+4m+2$ in terms of m.

We need to study $X^{(1,2m)}_{(2,3)}$. So, again by Theorem 2.10, we may work on forms of degree 5 in $\mathbb{P}^{2m+1}$. We study the quintics through a scheme W, made of h double fat points plus a triple point and a double $\mathbb{P}^{2m-1}$. We consider the specialized scheme W where $h'=2m^2+2m$ double points and the triple point are on a generic hyperplane H. The subscheme W_H cut by H corresponds exactly to h' double points on a $\mathbb{P}^1\times\mathbb{P}^{2m-1}$. Since $X^{(1,2m-1)}_{(2,3)}$ is regular by inductive assumption, we can just consider the quartics containing the residual scheme W' in $\mathbb{P}^{2m+1}$, made of $h-h'=2m+2$ double general points, a double $\mathbb{P}^{2m-1}$ and $2m^2+2m$ simple points plus a double point on H. This corresponds to studying $X^{(1,2m)}_{(2,2)}$. Therefore, by Theorem 3.4 the $(2m+1)$-th secant variety is defective and by the formula (3.1), its defect is 1. Hence, the complete linear system of quartics through $2m+2$ double points and a double $\mathbb{P}^{2m-1}$ in $\mathbb{P}^{2m+1}$ has dimension

$$3(m+1)(2m+1)-1-[(2m+2)^2-2]-1=(m+1)(2m-1)=2m^2+m-1.$$

Then by imposing the conditions of the remaining points, we can conclude that in this case also, the system is actually empty if the points impose independent conditions. So it remains to check that the specialization is not excessive. But if H were a fixed component, we would need to study the system of cubics in $\mathbb{P}^{n+1}$ through $2m+2$ double points, a simple point, and a triple $\mathbb{P}^n$, which is empty. ■

Remark 4.3. In the proof of the theorem above, we needed to do several computations involving the integers h, h', h'', and h^*, approximating expressions like $\frac{(d+1)(n+1)(n+3)}{6}$. In order to determine the exact value of such integers, this task requires treating many different cases separately, depending on congruence classes modulo 6 of n and d. The computations were done partially by hand and partially by Maple [18]. The following remarks describe the required computations.

Remark 4.4. In order to assume that $X^{(1,n)}_{(d,2)}$, appearing as a residue in the proof, is regular, we had to determine the pairs (d,n) such that $k=h-h'\leq\frac{d}{2}(n+1)$, where $h=\left\lceil\frac{(d+1)(n+1)(n+3)}{6}\right\rceil$ and $h'=\left\lceil\frac{(d+1)n(n+2)}{6}\right\rceil$.

Since (by Proposition 3.1) we were interested only in the case d even (d odd is never defective), we can assume that $d+1$ is odd and we can compute the values of h and h' by considering the congruence class of d modulo 3 and of n modulo 3 or 2). Thus we check that $h-h'\leq\frac{d}{2}(n+1)$ for all pairs (d,n), with $d\geq 3$, $n\geq 1$, and $d=2$, n odd.

Remark 4.5. We can compute the affine dimension $N(h,h')$ of the linear system of $(d+2)$-ics in $\mathbb{P}^{n+1}$ that contain the residual scheme W_H made of $h-h'$ generic double points, a d-tuple $\mathbb{P}^{n-1}$, and h' simple (generic) points in the cases denoted by "S" in the table. We do it by "counting" the conditions imposed by the residual scheme, under the assumption that these conditions are independent. Namely, we start from the affine dimension of the space of divisors of type $(d,2)$, that is $(d+1)\binom{n+2}{2}$, and subtract the conditions corresponding to the double or simple points in $\mathbb{P}^{n+1}$ (respectively, $n+2$ and 1). We get

$$N(h,h') = (d+1)\binom{n+2}{2} - (h-h')(n+2) - h' \leq 0$$

for any (d,n), except in the following cases (for which we use the differential Horace method):

- $d \geq 3$ odd, $n \equiv 2(\mathrm{mod} 3)$;
- $d \equiv 3(\mathrm{mod} 6)$ odd, $n \equiv 2(\mathrm{mod} 3)$;
- $d \geq 2$ even, $n \equiv 3(\mathrm{mod} 6)$;
- $d \equiv 2(\mathrm{mod} 6)$, $n \equiv 1(\mathrm{mod} 6)$, $n \geq 7$;
- $d \equiv 4(\mathrm{mod} 6)$, $n \equiv 1(\mathrm{mod} 3)$, $n \geq 7$;
- $d \equiv 0(\mathrm{mod} 3)$, $d \geq 1$, $n \equiv 5(\mathrm{mod} 6)$;
- $d \equiv 2(\mathrm{mod} 6)$, $n \equiv 5(\mathrm{mod} 6)$;
- $d \equiv 4(\mathrm{mod} 6)$, $n \equiv 5(\mathrm{mod} 6)$;
- $d \equiv 0(\mathrm{mod} 6)$, $d \geq 1$, $n \equiv 2(\mathrm{mod} 6)$.

We also check that the system of the hypersurfaces of degree $d+1$ in $\mathbb{P}^{n+1}$ containing the residual scheme made of $h-h'$ double fat points and the d-tuple $\mathbb{P}^{n-1}$, is empty, since $h-h' \geq d+1$.

Remark 4.6. In each of the 9 cases above, we computed the affine dimension $N(h,h'',h^*)$ of the space of divisors of type $(d,2)$ in $\mathbb{P}^1 \times \mathbb{P}^n$ that contain the residual scheme made of $h-h''-h^*$ double fat points, h^* differential double residue points, and h'' simple points. We proceeded exactly as in the standard case, up to the assumption that a differential residue in $\mathbb{P}^{n+1}$ imposes $n+1$ conditions. We conclude that

$$N(h,h'',h^*) = (d+1)\binom{n+2}{2} - (h-h''-h^*)(n+2) - h'' - h^*(n+1), \leq 0$$

that is, the linear system is empty. We also check, in each case, that the system of the hypersurfaces of degree $d+1$ in $\mathbb{P}^{n+1}$ containing $h-h''-h^*$ double fat points and a d-tuple $\mathbb{P}^{n-1}$, is empty since $h-h''-h^* \geq d+1$.

Remark 4.7. Essentially along the same lines as Section 3.2, we may consider the diagram

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^n \arrow[r, "(\nu_d,\nu_3)"] \arrow[dr, "\Phi_{(d,3)}"'] & \mathbb{P}^d \times \mathbb{P}^{\binom{n+3}{3}-1} \arrow[d, "\sigma"] \\ & \mathbb{P}^N \end{tikzcd}$$

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and study the tensor product of the two catalecticant matrices A, B, associated (respectively) to the Veronese embeddings ν_d, ν_3.

If we assume that x_0, x_1 and $y_0, \dots, y_n$ are coordinates, respectively, in $\mathbb{P}^1$ and $\mathbb{P}^n$, then the coordinates in $\mathbb{P}^N$, where $N = (d+1)\binom{n+3}{3} - 1$, can be defined as before by $z_{i,\alpha} = \Phi(x_0^i x_1^{a-i} y_0^{a_0} \cdots y_n^{a_n})$, where we assume that $\alpha = (a_0, \dots a_n)$ and $|\alpha| = 3$.

Assume $k = [d/2]$ $(\leq d/2)$ and let

$$A = \begin{pmatrix} x_0^d & x_0^{d-1}x_1 & \dots & x_0^k x_1^{d-k} \\ x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ x_0^{d-k}x_1^k & x_0^{d-k-1}x_1^{k+1} & \dots & x_1^d \end{pmatrix} \quad B = \begin{pmatrix} y_0^3 & y_0^2 y_1 & \dots & y_0 y_n^2 \\ y_0^2 y_1 & y_1^3 & \dots & y_1 y_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ y_0^2 y_n & y_0 y_1 y_n & \dots & y_n^3 \end{pmatrix}.$$

We already noticed that on the Veronese varieties $\nu_d(\mathbb{P}^1)$, $\nu_3(\mathbb{P}^n)$, the matrices A and B respectively have rank 1. The tensor product $C = A \otimes B$ of A and B, is a matrix of dimension $(k+1)(n+1) \times (d-k+1)\binom{n+2}{2}$. So for $X = X_{(d,3)}^{(1,n)}$, the secant variety X^h is contained in the rank h locus of $\Phi(C)$, since any point on the secant is a linear combination of h points where the rank of $\Phi(C)$ is 1.

In particular, for $h = (k+1)(n+1) - 1$ all the maximal minors of $\Phi(C)$ vanish, defining equations for X^h. Unlike the case in Section 3.2, however, we cannot conclude that those are all the equations required to define the secant variety.

5 A Few Consequences of Grassmann Defectivity

The study of the defectivity of projective varieties can be enlarged to the Grassmann defectivity, by looking at secant variety from a different point of view.

Definition 5.1 Let $X \subseteq \mathbb{P}^N$ be a nondegenerate projective variety. The (h,k)-Grassmann secant variety of X, denoted by $\mathrm{Sec}_{(h,k)}$ is the Zariski closure of the set.

$$\{\Lambda \in \mathbb{G}(k,N) \mid \Lambda \text{ lies in the span of } h+1 \text{ points of } X\}.$$

As in the case of standard secant varieties, there is an expected dimension for $\mathrm{Sec}_{(h,k)}$:

$$\exp\dim \mathrm{Sec}_{(h,k)}(X) = \min\{(h+1)\dim X + (k+1)(h-k), (k+1)(N-k)\}.$$

A variety is said to be *Grassmann* (h,k)*-defective* when its dimension is strictly smaller than the expected one and the difference $\delta_{h,k}$ between the two dimensions is called (h,k)*-defect*.

There are a few results [8, 9] classifying some Grassmann defective varieties (of lower dimension) and other statements that link ordinary defectivity and Grassmann defectivity [6, 11]. In particular, it is possible to check the Grassmann defectivity of a variety V by knowing if the Segre embedding of a product of V and a projective space is defective.

Proposition 5.2 (**[14, Proposition 1.3]**) *Let $V \subseteq \mathbb{P}^r$ be an irreducible nondegenerate projective variety of dimension n. Let $\sigma\colon \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$ be the Segre embedding of $\mathbb{P}^k \times V$. Then V is (k,h)-defective with defect $\delta_{k,h}(V) = \delta$ if and only if $\sigma(\mathbb{P}^k \times V)$ is h-defective with defect $\delta_h(\sigma(\mathbb{P}^k \times V)) = \delta$.*

As a straightforward consequence of Lemma 4.1, we have the following.

Theorem 5.3 *If $n \neq 2$, then the 3-Veronese variety $V_{3,n} = \nu_3(\mathbb{P}^n)$ is not $(1,h)$-defective for any h.*

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References

[1] B. Ådlandsvik, *Varieties with an extremal number of degenerate higher secant varieties.* J. Reine Angew. Math. **392**(1988), 16–26.
[2] J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables.* J. Algebraic Geom. **4**(1995), no. 2, 201–222.
[3] ———, *An asymptotic vanishing theorem for generic unions of multiple points.* Invent. Math. **140**(2000), no. 2, 303–325.
[4] C. Bocci, *Special effect varieties in higher dimension.* Collect. Math. **56**(2005) no. 3, 299–326.
[5] E. Carlini and M. V. Catalisano, *Existence results for rational normal curves.* J. Lond. Math. Soc. (2) **76**(2007), no. 1, 73–86.
[6] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, *Higher secant varieties of Segre-Veronese varieties.* In: Projective Varieties with Unexpected Properties, Walter de Gruyter, Berlin, 2005, pp. 81–107.
[7] ———, *Higher secant varieties of the Segre varieties* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$. J. Pure Appl. Algebra **201**(2005), no. 1-3, 367–380.
[8] L. Chiantini, *Lectures on the structures of projective embeddings.* Rend. Sem. Mat. Univ. Politec. Torino **62**(2004), no. 4, 335–388.
[9] L. Chiantini, and C. Ciliberto, *The classification of* $(1,k)$*-defective surfaces.* Geom. Dedicata **111**(2005), 107–123.
[10] ———, *Weakly defective varieties.* Trans. Amer. Math. Soc. **354**(2001), no. 1, 151–178.
[11] L. Chiantini, and M. Coppens, *Grassmannians of secant varieties.* Forum Math. **13**(2001), no. 5, 615–628.
[12] C. Ciliberto and R. Miranda, *The Segre and Harbourne-Hirschowitz Conjectures.* In: Applications of Algebraic Geometry to Coding Theory, Physics and Computation. NATO Sci. Ser. II Math. Phys. Chem. 36, Kluwer, Dordrecht, 2001, 37–51.
[13] CoCoATeam, *CoCoA: A System for Doing Computations in Commutative Algebra.* http://cocoa.dima.unige.it
[14] C. Dionisi and C. Fontanari, *Grassman defectivity à la Terracini,* Matematiche, **56**(2001), no. 2, 245–255.
[15] C. Fontanari, *On Waring's problem for partially symmetric tensors. Variations on a theme of Mella.* Ann. Univ. Ferrara Sez. VII Sci. Mat. **52**(2006), no. 1, 37–43.
[16] J. Harris, *A bound on the geometric genus of projective varieties.* Ann. Scuola Norm. Sup. Pisa Cl. Sci.**8**(1981), no. 1, 35–68.
[17] A. Hirschowitz, *La méthode d'Horace pour l'interpolation à plusieurs variables.* Manuscripta Math. **50**(1985), 337–388.
[18] *Maple.*http://www.maplesoft.com.
[19] M. Mella, *Singularities of linear systems and the Waring problem.* Trans. Amer. Math. Soc. **358**(2006), no. 12, 5523–5538.

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