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Generating Ideals in Rings of Integer-Valued Polynomials

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Abstract. Let R be a one-dimensional locally analytically irreducible Noetherian domain with finite residue fields. In this note it is shown that if I is a finitely generated ideal of the ring $\mathrm{Int}(R)$ of integer-valued polynomials such that for each $x \in R$ the ideal $I(x) = \{f(x) \mid f \in I\}$ is strongly n-generated, $n \geq 2$, then I is n-generated, and some variations of this result.

Let R be an integral domain with quotient field K and let $\mathrm{Int}(R)$ be the ring of integer-valued polynomials on R. Thus $\mathrm{Int}(R) = \{f \in K[X] \mid f(R) \subseteq R\}$. The ring $\mathrm{Int}(R)$ has been much studied since it was considered in the 1919 articles of Ostrowski [10] and Polya [11] for the case that R is the ring of integers in an algebraic number field. For example see [2] and the references listed there. In [6] Gilmer and Smith answered a question of Brizolis [1] by showing that in the case that R is the ring Z of rational integers, each finitely generated ideal of $\mathrm{Int}(R)$ is generated by two elements. Since $\mathrm{Int}(\mathrm{Z})$ is a Prüfer domain [2, Theorem VI.1.7], the finitely generated ideals of $\mathrm{Int}(R)$ are invertible. Results showing that each invertible ideal of $\mathrm{Int}(R)$ is two-generated for larger classes of one-dimensional domains R were given in [3], [9], [12], [4], [13] and [2, Theorem VIII.4.3]. In this note we give some results on numbers of generators of possibly non-invertible finitely generated ideals of $\mathrm{Int}(R)$. In particular, if for example R is local with multiplicity $e(R)$ it follows $e(R) + 1$ is a uniform bound on the number of elements required to generate any finitely generated ideal I of the two-dimensional non-Noetherian ring $\mathrm{Int}(R)$. We say that an ideal I of a ring A is *n-generated* if it can be generated by n elements, and *strongly n-generated* if each nonzero element of I is a member of an n-element generating set for I. The ring A is said to have the *n-generator property (strong n-generator property)* if each finitely generated ideal of R is n-generated (*strongly n-generated*). It is shown that if R is a one-dimensional Noetherian locally analytically irreducible integral domain with finite residue fields, and if I is a finitely generated ideal of $\mathrm{Int}(R)$ such that for some integer $n \geq 2$, $I(x)$ is strongly n-generated for each $x \in R$, then I is n-generated. If in addition R is a semilocal, then $I(x)$ is $(n-1)$-generated for each $x \in R$ if and only if I is strongly n-generated. We give some of our results for the ring of integer-valued polynomials in several variables.

1 Preliminary Results

Let R be a Noetherian integral domain with quotient field K. If d is a positive integer we let $\mathrm{Int}(R^{(d)}) = \{f \in K[X_1, \ldots, X_d] \mid f(R^{(d)}) \subseteq R\}$. (We write $S^{(d)}$ for cartesian product to distinguish it from a product of ideals.) We write $\mathbf{X}$ for $(X_1, \ldots, X_d)$ and $\mathbf{a}$ for $(a_1, \ldots, d_d) \in R^{(d)}$. An ideal I of $\mathrm{Int}(R^{(d)})$ is said to be *unitary* if $I \cap R \neq \{0\}$. Let

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$I(\mathbf{a}) = \{f(\mathbf{a}) \mid f \in I\}$. Following [2] we say that $\mathrm{Int}(R^{(d)})$ has the *almost strong Skolem property* if for finitely generated unitary ideals I and J of $\mathrm{Int}(R^{(d)})$, $I(\mathbf{a}) = J(\mathbf{a})$ for each $\mathbf{a} \in R^{(d)} \Rightarrow I = J$. Recall that a local ring (R, m) is said to be *analytically irreducible* if its m-adic completion $(\hat{R}, \hat{m})$ is an integral domain, and a Noetherian domain R is said to be *locally analytically irreducible* if R_m is analytically irreducible for each maximal ideal m of R. The relevance of this property lies in the following theorem:

Theorem 1.1 **([2, Proposition XI.3.8])** *If R be a one-dimensional locally analytically irreducible domain with finite residue fields, then $\mathrm{Int}(R^{(d)})$ has the almost strong Skolem property.*

The following result will be needed later.

Lemma 1.2 **([8, Proposition 4.2])** *If R is a zero-dimensional ring and M is a finitely generated R-module such that M_m is n-generated for each maximal ideal m of R, then M is n-generated.*

We also need the following simple lemmas which help to clarify the strong n-generator hypothesis which is often imposed on the ideals $I(x)$ in what follows. For this we note that only the ideal $\{0\}$ is 0-generated.

Lemma 1.3 *Let I be a finitely generated ideal of the integral domain A, and let $n \in \mathrm{Z}$, $n \geq 1$.*

(1) If I is strongly n-generated and S is a multiplicative subset of A, then IA_S is a strongly n-generated ideal of A_S.
(2) If A has nonzero Jacobson radical, then I is strongly n-generated if and only if I is $(n-1)$-generated.

Proof Statement (1) is clear. For (2) let J be the Jacobson radical of R. For the only if part of (2) first assume $n > 1$. If $I \neq \{0\}$ is strongly n-generated let $\{a_1, \ldots, a_n\}$ be a generating set for I with $a_1 \in JI - \{0\}$. Then $I = (a_2, \ldots, a_n)A$ by Nakayama's Lemma. The case $n = 1$ is similar. The converse implication in (2) is clear. ■

Lemma 1.4 *Let I be a finitely generated ideal of the one-dimensional Noetherian integral domain A and let $n \in \mathrm{Z}$, $n \geq 1$. Consider the following properties of I.*

(1) I is $(n-1)$-generated.
(2) I is strongly n-generated.
(3) IA_m is $(n-1)$-generated for each maximal ideal m of A.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3), and if $n \geq 3$, (1), (2) and (3) are equivalent.

Proof That (1) $\Rightarrow$ (2) is clear, and (2) $\Rightarrow$ (3) follows from Lemma 1.3.

The implication (3) $\Rightarrow$ (2) is clear if $n = 1$. Thus let $n \geq 2$ and let $I \neq \{0\}$ be such that I_m is an $(n-1)$-generated ideal of A_m for each maximal ideal m of A. Let $a_1 \in I - \{0\}$. Then I/a_1A is an ideal of A/a_1A which is locally $(n-1)$-generated. But since A/a_1A is zero-dimensional, I/a_1A is $(n-1)$-generated by Lemma 1.2. Thus a_1 is a member of an n-element generating set for I.

That (3) $\Rightarrow$ (1) if $n \geq 3$ follows from a Theorem of Forster and Swan. For example see [7, p. 108, Corollary 2.14]. ■

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The proof of the following lemma is the same as that given in the proof of [2, Proposition XI.3.10]. See [2, Proposition VII.1.11] for the one variable case.

Lemma 1.5 *Let R be a one-dimensional Noetherian domain and I a finitely generated unitary ideal of* $\mathrm{Int}(R^{(d)})$*. Then there is a nonzero ideal J of R such that if $\mathbf{a}, \mathbf{b} \in R^{(d)}$, and $a_i - b_i \in J$ for $i \in \{1, \ldots, d\}$ then $I(\mathbf{a}) = I(\mathbf{b})$.*

2 The n-Generator Property in $\mathrm{Int}(R)$

The following result includes the result [2, Proposition XI.3.10] which gives the case that R is Dedekind (and then $n = 2$).

Theorem 2.1 *Let R be a one-dimensional locally analytically irreducible domain with finite residue fields, and let I be a finitely generated unitary ideal of* $\mathrm{Int}(R^{(d)})$*. If for some integer $n \geq 2$, $I(\mathbf{x})$ is strongly n-generated for each $\mathbf{x} \in R^{(d)}$, then I can be generated by n elements. Moreover, one of the generators may be chosen to be any $r \in I \cap R - \{0\}$.*

Proof Let $r \in I \cap R - \{0\}$. It follows from [2, Proposition XI.2.9] that the ring $\mathrm{Int}(R^{(d)})/r\big(\mathrm{Int}(R)\big)$ is zero-dimensional. Thus by Lemma 1.2 it suffices to show that the ideal $I/r\big(\mathrm{Int}(R^{(d)})\big)$ is locally $(n-1)$-generated. In particular since $\mathrm{Int}(R^{(d)})_S = \mathrm{Int}\big((R_S)^{(d)}\big)$ for each multiplicative subset S of R [2, Corollary XI.1.8], we may assume R is local. Then by Lemma 1.3, $I(x)$ is $(n-1)$-generated for each $x \in R$.

By Lemma 1.5 there is a nonzero ideal J of R such that if $\mathbf{a}, \mathbf{b} \in R^{(d)}$, and $a_i - b_i \in J$ for each i then $I(\mathbf{a}) = I(\mathbf{b})$. Since I is finitely generated, we may choose $h_1, \ldots, h_k \in I$ such that for each $\mathbf{x} \in R^{(d)}$, $h_1(\mathbf{x}), \ldots, h_k(\mathbf{x})$ are generators of $I(\mathbf{x})$. We may assume $k > n-1$. Let $A_1, \ldots, A_e$ be the subsets of $\{h_1, \ldots, h_k\}$ having cardinality $n-1$. If $\mathbf{x} \in R^{(d)}$, then since R is local and $I(\mathbf{x})$ is n-generated, we have $I(\mathbf{x}) = A_i(\mathbf{x})R = \big(A_i(\mathbf{x}), r\big)R$ for some $i \in \{1, \ldots, e\}$.

Let $W_i = \{\mathbf{y} \in \hat{R}^{(d)} \mid I(y)\hat{R} = \big(A_i(\mathbf{y}), r\big)\hat{R}\}$. We may choose $c \in \mathrm{N}$ such that $m^c \subseteq J$ and such that if $x_i - a_i \in \hat{m}^c$ for each i then $h_j(\mathbf{x}) - h_j(\mathbf{a}) \in rR$ for each $j \in \{1, \ldots, e\}$. Then if $\mathbf{x} \in \mathbf{a} + (\hat{m}^c)^{(d)}$ we have $I(\mathbf{x}) = I(\mathbf{a})$ and $\big(A_i(\mathbf{a}), r\big)\hat{R} = \big(A_i(\mathbf{x}), r\big)\hat{R}$. It follows that W_i is an open and closed subset of $\hat{R}^{(d)}$ for each i.

Let $U_1 = W_1$ and for $i \in \{2, \ldots, e\}$ let $U_i = W_i - (U_1 \cup \cdots \cup U_{i-1})$. The subsets U_i are open and closed in $\hat{R}$. Let χ_j be the characteristic function of the set U_j for each j. Let $t > 0$ be such that $m^t \subseteq rR$. Since $\mathrm{Int}(R^{(d)})$ is dense in $C(\hat{R}^{(d)}, \hat{R})$ [2, Proposition XI.2.4], there exist $g_j \in \mathrm{Int}(R^{(d)})$ such that

$$g_j(\mathbf{x}) - \chi_j(\mathbf{x}) \in \hat{m}^t \quad \text{for} \quad \mathbf{x} \in \hat{R}^{(d)} \quad \text{and} \quad j = 1, \ldots, e.$$

Let $A_i = \{h_{i1}, \ldots, h_{in-1}\}$, and let $f_j = h_{1j}g_1 + \cdots + h_{ej}g_e$. Then for each $\mathbf{x} \in \hat{R}^{(d)}$ and $j = 1, \ldots, n-1$, we have

$$f_j(\mathbf{x}) = \sum_{i=1}^{e} h_{ij}(\mathbf{x})\big[g_i(\mathbf{x}) - \chi_i(\mathbf{x})\big] + \sum_{i=1}^{e} h_{ij}(\mathbf{x})\chi_i(\mathbf{x}).$$

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For $\mathbf{x} \in U_s$ this gives $f_j(\mathbf{x}) - h_{sj}(\mathbf{x}) \in m^t \subset rR$. If $\mathbf{x} \in U_s$ we have $I(\mathbf{x}) = \big(r, A_s(\mathbf{x})\big)R$. Since $f_j(\mathbf{x}) - h_{sj}(\mathbf{x}) \in rR$, this is $\big(r, f_1(\mathbf{x}), \dots, f_{n-1}(\mathbf{x})\big)R$. Now since $(r, f_1, \dots, f_{n-1})\operatorname{Int}(R)$ and I are unitary, $(r, f_1, \dots, f_{n-1})\operatorname{Int}(R) = I$ by Theorem 1.1. ■

In the case of integer-valued polynomials in one variable, a standard argument (given in the next proof) shows that it is not necessary to restrict to unitary ideals as was done in the previous Theorem.

Theorem 2.2 *Let R be a one-dimensional locally analytically irreducible domain with finite residue fields, and let I be a finitely generated ideal of $\operatorname{Int}(R)$. If $I(x)$ is strongly n-generated for each $x \in R$, $n \geq 2$, then I is n-generated. Moreover, one of the generators may be chosen to be any $g \in I$ such that $gK[X] = IK[X]$.*

Proof If I is not unitary, choose a finite subset A of I such that $I = A\big(\operatorname{Int}(R)\big)$. If $g \in I$ is such that $IK[X] = gK[X]$, then $A = gA_1$ for some finite subset A_1 of $K[X]$, and $A_1K[X] = K[X]$. Let $a \in R - \{0\}$ be such that $aA_1 \subseteq R[X]$. Then $aA_1\big(\operatorname{Int}(R)\big) = I_1$ is unitary, $gI_1 = aI$ and $I_1(x)$ is strongly n-generated since $I(x)$ is. Further, if I_1 is n-generated, I is also. Thus it suffices to consider the case that I is unitary, and $g \in I \cap R$. The result now follows from Theorem 2.1. ■

Corollary 2.3 *Let R be a one-dimensional locally analytically irreducible Noetherian domain with finite residue fields. If R has the strong n-generator property, $n \geq 2$, then $\operatorname{Int}(R)$ has the n-generator property.*

Recall that a one-dimensional local Noetherian domain (R, m) has the n-generator property for $n = e(R)$, the multiplicity of R [14, Theorem 3.1.1]. Thus we have the following:

Corollary 2.4 *Let R be a one-dimensional locally analytically irreducible Noetherian domain with finite residue fields, and $I \subseteq \operatorname{Int}(R^{(d)})$ a finitely generated ideal such that $e(R_m) \leq n$ for each maximal ideal m of R containing $I \cap R$. If either I is unitary or $d = 1$, then I can be generated by $n + 1$ elements.*

We end this section by noting that Theorem 2.2 gives, via a result of Gilmer [5], an alternate proof of the following well-known result. See [2, Chapter VI] for an exposition of when $\operatorname{Int}(R)$ is Prüfer.

Theorem 2.5 *If R is a Dedekind domain with finite residue fields, then $\operatorname{Int}(R)$ is Prüfer.*

Proof Since R is Dedekind, each ideal of R is strongly 2-generated, and thus by Theorem 2.2 each finitely generated ideal of $\operatorname{Int}(R)$ is 2-generated. But by [5, Corollary 3], if for some integer n each finitely generated ideal of an integral domain D is n-generated, the integral closure D' of D is Prüfer. But $\operatorname{Int}(R)$ is easily seen to be integrally closed since R is [2, Proposition VI]. Thus $\operatorname{Int}(R)$ is Prüfer. ■

3 The Strong n-Generator Property in $\operatorname{Int}(R)$

We now consider what can be said when the hypothesis of the strong n-generator property on the ideals $I(x)$ is weakened to the n-generator property. Since the n-generator property

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trivially implies the strong $(n+1)$-generator property, then for R as in Theorem 2.2, if I is a finitely generated ideal of $\operatorname{Int}(R)$ such that $I(x)$ is n-generated for each $x \in R$, then by Theorem 2.2 I is $(n+1)$-generated. The following result shows that in the case that R is semilocal, $\operatorname{Int}(R)$ has the *strong* $(n+1)$-generator property. Further, there is a converse.

Theorem 3.1 *Let R be a semilocal one-dimensional domain which is locally analytically irreducible and has finite residue fields, and let I be a finitely generated ideal of* $\operatorname{Int}(R)$*. Then I is strongly $(n+1)$-generated if and only if $I(x)$ is n-generated for each $x \in R$.*

Proof $(\Rightarrow)$ Let $x \in R$. For any $a \in I(x) - \{0\}$ let $f_1 \in I$ be such $a = f_1(x)$. Since I is strongly $(n+1)$-generated there exist $f_2, \dots, f_{n+1} \in I$ such that $I = (f_1, f_2, \dots, f_{n+1})$. Then $I(x) = \big(a, f_2(x), \dots, f_{n+1}(x)\big)R$. Thus $I(x)$ is strongly $(n+1)$-generated. Since R is semilocal, it follows from part (2) of Lemma 1.3 that $I(x)$ is n-generated.

$(\Leftarrow)$ As in the proof of Theorem 2.2 we may assume I is unitary. Let $g \in I - \{0\}$. To show that g is one of $n+1$ generators let $b \in J(I \cap R) - \{0\}$ where J is the Jacobson radical of R. By Theorem 2.2 there exist $f_1, \dots, f_n \in I$ such that $I = (b, f_1, \dots, f_n)\operatorname{Int}(R)$. For each $d \in R$ the polynomials $h_i = f_i + bd$ also have the property that $I = (b, h_1, \dots, h_n)\operatorname{Int}(R)$. Since R is not a field, R is infinite, and thus we may choose d so that $(g, h_1, \dots, h_n)K[X] = K[X]$. (In fact if $f_1 \neq 0$ we can choose d so that $(g, h_1)K[X] = K[X]$.)

To show $I = (g, h_1, \dots, h_n)\operatorname{Int}(R)$ let $ug + \sum_{i=1}^n v_i h_i = 1$, $u, v_i \in K[X]$. Then for some $c \in R$ we have $cu, cv_i \in R[X]$, and then $(cu)g + \sum_{i=1}^n (cv_i)h_i = c \in I$. Then $I = (b, h_1, \dots, h_n)\operatorname{Int}(R) \subseteq (c, b, h_1, \dots, h_n)\operatorname{Int}(R) \subseteq (g, b, h_1, \dots, h_n)\operatorname{Int}(R) \subseteq I$. Thus $I = (g, b, h_1, \dots, h_n)\operatorname{Int}(R)$. But for each $x \in R$, $I(x) = \big(g(x), b, h_1(x), \dots, h_n(x)\big)R \subseteq JI(x) + \big(g(x), h_1(x), \dots, h_n(x)\big)R$. Thus we have $I(x) = \big(g(x), h_1(x), \dots, h_n(x)\big)R$ by Nakayama's Lemma. Since R is locally analytically irreducible, $\operatorname{Int}(R)$ has the almost strong Skolem property by Theorem 1.1. Thus since I and $(g, h_1, \dots, h_n)\operatorname{Int}(R)$ are unitary, $I = (g, h_1, \dots, h_n)\operatorname{Int}(R)$. ■

Corollary 3.2 **([2, Proposition VIII.3.9])** *Let R be a one-dimensional local domain which is analytically irreducible and has finite residue fields, and let I be a finitely generated unitary ideal of* $\operatorname{Int}(R)$*. Then I is invertible if and only if $I(x)$ is principal for each $x \in R$.*

Proof If $I(x)$ is principal for each $x \in R$, then I is strongly 2-generated by Theorem 3.1, and thus locally principal by Lemma 1.3. The converse is clear. ■

We now have the following counterpart to Corollaries 2.3 and 2.4.

Corollary 3.3 *Let R be a one-dimensional locally analytically irreducible semilocal Noetherian domain with finite residue fields and let $n \geq 2$. The following are equivalent:*

(1) $e(R_m) \leq n - 1$ *for each maximal ideal m of R;*
(2) *R has the $(n-1)$-generator property;*
(3) $\operatorname{Int}(R)$ *has the strong n-generator property.*

If instead of the strong n-generator hypothesis on the ideals $I(x)$ we have an n-generator hypothesis on the localization I_M for each maximal ideal M of $\operatorname{Int}(R)$, as occurs when I is invertible, it is easier to bound the generators of I. To illustrate we conclude with a generalization of [2, Theorem VIII.4.3] which is the case $n = 1$ of the following result. Although the proof is essentially the same, we include it for the convenience of the reader.

Theorem 3.4 *Let R be a one-dimensional Noetherian domain and I a finitely generated ideal of* $\mathrm{Int}(R)$ *such that the ideal* I_M *of* $\mathrm{Int}(R)_M$ *is n-generated for each maximal ideal M of* $\mathrm{Int}(R)$. *Then I is generated by* $n+1$*-elements, one of which can be chosen to be any element* $g \in I$ *such that* $gK[X] = IK[X]$.

Proof We can reduce to the case where I is unitary and $g \in I \cap R$ as in the proof of Theorem 2.2. Then $\mathrm{Int}(R)/g\big(\mathrm{Int}(R)\big)$ is zero-dimensional by [2, Theorem V.2.2], and the ideal $I/(g)$ of $\mathrm{Int}(R)/g\big(\mathrm{Int}(R)\big)$ is locally n-generated. Since $\mathrm{Int}(R)/g\big(\mathrm{Int}(R)\big)$ is zero-dimensional, $I/(g)$ is n-generated by Lemma 1.2. Thus I is $(n+1)$-generated. ■

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