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A Cesàro-like operator from a class of analytic function spaces to analytic Besov spaces

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ABSTRACT

Let μ be a finite positive Borel measure on $[0,1)$ and $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$. For $0<\alpha<\infty$, the generalized Cesàro-like operator $\mathcal{C}_{\mu,\alpha}$ is defined by

$$\mathcal{C}_{\mu,\alpha}(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!}a_k\right)z^n,\ z\in\mathbb{D},$$

where, for $n\geq 0$, μ_n denotes the n-th moment of the measure μ, that is, $\mu_n=\int_0^1 t^n d\mu(t)$.

For $s>1$, let X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}}\subset X\subset\mathcal{B}$. In this paper, for $1\leq p<\infty$, we characterize the measure μ for which $\mathcal{C}_{\mu,\alpha}$ is bounded (resp. compact) from X into the analytic Besov space B_p.

Keywords: Cesàro operator. Bloch space. Besov space.

MSC 2010: 47B35, 30H30, 30H20

1 Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. H^∞ denote the set of bounded analytical functions on $\mathbb{D}$.

The Bloch space $\mathcal{B}$ consists of those functions $f\in H(\mathbb{D})$ for which

$$||f||_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

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For $1 < p < \infty$, the analytic Besov space B_p consists of those functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_p} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-2} \mathrm{d}A(z)\right)^{\frac{1}{p}} < \infty,$$

where $dA(z) = \frac{dxdy}{\pi}$ is the normalized area measure on $\mathbb{D}$. When $p = 2$, then B_2 is just the classic Dirichlet space $\mathcal{D}$. If $1 < p_1 < p_2 < \infty$, then $B_{p_1} \subsetneq B_{p_2} \subsetneq \mathcal{B}$. It is known that the analytic Besov spaces are Möbius invariant and the Bloch space $\mathcal{B}$ is the largest Möbius invariant space.

The space B_1 consists of $f \in H(\mathbb{D})$ such that

$$||f||_{B_1} = |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| dA(z) < \infty.$$

The space B_1 is the smallest Möbius invariant Banach spaces of analytic function in $\mathbb{D}$ and $B_1 \subsetneq H^\infty$. See [32, Chapter 5] for the theory of these spaces.

Let $0 < p \leq \infty$, the classical Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ for which

$$||f||_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \ 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions $f \in H(\mathbb{D})$ having a non-tangential limit almost everywhere such that $\omega_p(t, f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is known (see [12]) that Λ^p_α is a subset of H^p and

$$\Lambda^p_\alpha = \left(f \in H(\mathbb{D}) : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}}\right), \ \text{as } r \to 1\right).$$

The space Λ^p_α is a Banach space with the norm $|| \cdot ||_{\Lambda^p_\alpha}$ given by

$$||f||_{\Lambda^p_\alpha} = |f(0)| + \sup_{0 \leq r < 1} (1-r)^{1-\alpha} M_p(r, f').$$

It is known (see e.g. [8]) that

$$\Lambda^p_{\frac{1}{p}} \subsetneq \mathcal{B}. \ 1 < p < \infty.$$

For $f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{D})$, the Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) = \sum_{n=0}^\infty \left(\frac{1}{n+1}\sum_{k=0}^n a_k\right) z^n, \ z \in \mathbb{D}.$$

The boundedness and compactness of the Cesàro operator $\mathcal{C}$ and its generalizations on various spaces of analytic functions such as Hardy spaces, Bergman spaces, Dirichlet spaces, the Bloch

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space, Q_p spaces, mixed norm spaces have been widely studied. See e.g. [1, 2, 10, 19, 24–26, 30]. We refer the reader to the recent survey [22] for more on Cesàro operator.

Recently, Galanopoulos, Girela and Merchán [14] introduced a Cesàro-like operator $\mathcal{C}_\mu$ on $H(\mathbb{D})$, which is a natural generalization of the classical Cesàro operator $\mathcal{C}$. They considered the following generalization: For a positive Borel measure μ on the interval $[0,1)$ they define the operator

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\mu_n \sum_{k=0}^{n} \widehat{f}(k)\right) z^n = \int_0^1 \frac{f(tz)}{1-tz} d\mu(t),\ z \in \mathbb{D}.$$

where μ_n stands for the moment of order n of μ, that is, $\mu_n = \int_0^1 t^n d\mu(t)$. They studied the operators $\mathcal{C}_\mu$ acting on distinct spaces of analytic functions (e.g. Hardy spaces, Bergman spaces, the Bloch space, etc.).

The Cesàro-like operator $\mathcal{C}_\mu$ defined above has attracted the interest of many mathematicians. For instance, Jin and Tang [18] studied the boundedness (resp. compactness) of $\mathcal{C}_\mu$ from one Dirichlet-type space $\mathcal{D}_\alpha$ into another one $\mathcal{D}_\beta$. Bao, Sun and Wulan [4] studied the range of $\mathcal{C}_\mu$ acting on H^∞. Blasco [6] investigated the operators $\mathcal{C}_\mu$ induced by complex Borel measures on $[0,1)$, and extended the results of [14] to this more general case. Galanopoulos et al. [15] studied the behaviour of the operators $\mathcal{C}_\mu$ on the Dirichlet space and on the analytic Besov spaces B_p. Recently, Bao et al. [3] have complectly characterized the measure μ such that $\mathcal{C}_\mu$ is bounded (resp. compact) on the Dirichlet space. The reader is referred to [5, 7, 13, 28, 29, 31] for more on Cesàro-like operators on spaces of analytic functions.

In [4], Bao et al. introduced a more general Cesàro-like operator. Suppose that $0 < \alpha < \infty$ and μ is a finite positive Borel measure on $[0,1)$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, they defined

$$\mathcal{C}_{\mu,\alpha}(f)(z) = \sum_{n=0}^{\infty}\left(\mu_n \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!} a_k\right) z^n,\ z \in \mathbb{D}.$$

A simple calculation with power series gives the integral form of $\mathcal{C}_{\mu,\alpha}$ as follows.

$$\mathcal{C}_{\mu,\alpha}(f)(z) = \int_0^1 \frac{f(tz)}{(1-tz)^\alpha} d\mu(t).$$

For $1 < s < \infty$, let X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. There are many well known spaces located between the mean Lipschitz space $\Lambda^s_{\frac{1}{s}}$ and the Bloch space $\mathcal{B}$. In [4], the authors investigated the range of $\mathcal{C}_{\mu,\alpha}$ acting on H^∞. They proved that if $\max\{1, \frac{1}{\alpha}\} < s < \infty$, then $\mathcal{C}_{\mu,\alpha}(H^\infty) \subset X$ if and only if μ is an α-Carleson measure. Zhou [31] considered the same problem for the measure μ supported on $\mathbb{D}$. Guo, Tang and Zhang [17] investigated the boundedness (resp. compactness) of $\mathcal{C}_{\mu,\alpha}$ acting from X to weighted Bergman spaces A^p_β. Galanopoulos, Siskakis and Zhao [16] characterized the measure μ such that $\mathcal{C}_{\mu,\alpha}$ is bounded from weighted Bergman space A^p_β to A^q_β when $1 \le p \le q < \infty$ and $\beta > -1$. Sun et al. [27] studied the boundedness (resp. compactness) of the operator $\mathcal{C}_{\mu,1}$ acting from B_p to X. It remains open to characterize the boundedness and the compactness of $\mathcal{C}_{\mu,\alpha}$ from B_p to B_p when $p > 1$ and $p \neq 2$. The Besov spaces B_p and Bloch space $\mathcal{B}$ are Möbius invariant and the Bloch space $\mathcal{B}$ can be regarded as the limit case of B_p as $p \to +\infty$. The purpose of this paper is describe the measure μ such that the operator $\mathcal{C}_{\mu,\alpha}$ is bounded (resp. compact) from X to B_p for $1 \le p < \infty$.

Our main results are included in the following.

Theorem 1.1. *Suppose $0 < \alpha < \infty$, $1 < s < \infty$ and $\max\{1, \frac{1}{\alpha}\} \le p < \infty$. Let μ be a finite positive Borel measure on $[0,1)$ and X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. Then the following statements are equivalent.*
(1) The operator $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_p.
(2) The operator $\mathcal{C}_{\mu,\alpha}$ is compact from X to B_p.
(3) The measure μ satisfies

$$\sum_{n=0}^{\infty}(n+1)^{p\alpha-1}\mu_n^p \log^p(n+2) < \infty.$$

For $p = 1$, we have the following corollary.

Corollary 1.2. *Suppose $1 \le \alpha < \infty$ and $1 < s < \infty$. Let μ be a finite positive Borel measure on $[0,1)$ and X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. Then the following statements are equivalent.*
(1) The operator $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_1.
(2) The operator $\mathcal{C}_{\mu,\alpha}$ is compact from X to B_1.
(3) The measure μ satisfies

$$\sum_{n=0}^{\infty}(n+1)^{\alpha-1}\mu_n \log(n+2) < \infty.$$

(4) The measure μ satisfies

$$\int_0^1 \frac{\log \frac{e}{1-t}}{(1-t)^{\alpha}} d\mu(t) < \infty.$$

When $p = 2$, the space B_2 is the classic Dirichlet space $\mathcal{D}$, so we have the following corollary.

Corollary 1.3. *Suppose $1 \le \alpha < \infty$ and $1 < s < \infty$. Let μ be a finite positive Borel measure on $[0,1)$ and X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. Then the following statements are equivalent.*
(1) The operator $\mathcal{C}_{\mu,\alpha}$ is bounded from X to $\mathcal{D}$.
(2) The operator $\mathcal{C}_{\mu,\alpha}$ is compact from X to $\mathcal{D}$.
(3) The measure μ satisfies

$$\sum_{n=0}^{\infty}(n+1)^{2\alpha-1}\mu_n^2 \log^2(n+2) < \infty.$$

Throughout the paper, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation "$P \lesssim Q$" if there exists a constant $C = C(\cdot)$ such that "$P \le CQ$", and "$P \gtrsim Q$" is understood in an analogous manner. In particular, if "$P \lesssim Q$" and "$P \gtrsim Q$" , then we will write "$P \asymp Q$".

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2 Preliminaries

To prove our main results, we need some preliminary results which will be repeatedly used throughout the rest of the paper. We begin with a characterization of the functions $f \in H(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing which belong to B^p. For a proof, see e.g., [11, Theorem 3.10].

Lemma 2.1. *Let $1 < p < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. Suppose that the sequence $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of non-negative real numbers. Then $f \in B_p$ if and only if*

$$\sum_{n=1}^{\infty} n^{p-1} a_n^p < \infty.$$

The following lemma contains a characterization of L^p-integrability of power series with non-negative coefficients. For a proof, see [21, Theorem 1].

Lemma 2.2. *Let $0 < \beta, p < \infty$, $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers. Then*

$$\int_0^1 (1-r)^{p\beta-1} \left(\sum_{n=0}^{\infty} \lambda_n r^n \right)^p dr \asymp \sum_{n=0}^{\infty} 2^{-np\beta} \left(\sum_{k \in I_n} \lambda_k \right)^p$$

where $I_0 = \{0\}$, $I_n = [2^{n-1}, 2^n) \cap \mathbb{N}$ for $n \in \mathbb{N}$.

The following lemma is a consequence of Theorem 2.31 on page 192 of the classical monograph [33].

Lemma 2.3. *(a) The Taylor coefficients a_n of the function*

$$f(z) = \frac{1}{(1-z)^\beta} \log^\gamma \frac{2}{1-z}, \quad \beta > 0, \gamma \in \mathbb{R}, \ z \in \mathbb{D}$$

have the property $a_n \asymp n^{\beta-1} (\log(n+1))^\gamma$.

(b) The Taylor coefficients a_n of the function

$$f(z) = \log^\gamma \frac{2}{1-z}, \quad \gamma > 0, z \in \mathbb{D}$$

have the property $a_n \asymp n^{-1} (\log(n+1))^{\gamma-1}$.

We also need the following estimates (see, e.g. Proposition 1.4.10 in [23]).

Lemma 2.4. *Let α be any real number and $z \in \mathbb{D}$. Then*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^\alpha} \asymp \begin{cases} 1 & \text{if } \alpha < 1, \\ \log \frac{2}{1-|z|^2} & \text{if } \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}} & \text{if } \alpha > 1, \end{cases}$$

The following lemma is useful in dealing with the compactness. The proof is similar to that of Proposition 3.11 in [9]. The details are omitted.

Lemma 2.5. *Let $p \geq 1$, $s > 1$, X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. Suppose that T is a bounded operator from X to B_p. Then T is compact if and only if for any bounded sequence $\{f_k\}$ in X which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\lim_{k\to\infty} ||T(f_k)||_{B_p} = 0$.*

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3 Proofs of the main results

We now present the proofs of Theorem 1.1.
Proof of the implication $(1) \Rightarrow (3)$

Since the definition of the space B_1 is slightly different from B_p when $p > 1$, we split the proof into $p = 1$ and $p > 1$.

Case 1: $p = 1$**.**

Assume that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_1. Let $g(z) = \log \frac{1}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k}$. It is easy to check that $g \in \Lambda^s_{\frac{1}{s}} \subset X$. This implies that $\mathcal{C}_{\mu,\alpha}(g) \in B_1$. For $z \in \mathbb{D}$, by the definition of $\mathcal{C}_{\mu,\alpha}$ we get

$$\mathcal{C}_{\mu,\alpha}(g)''(z) = \sum_{n=0}^{\infty} \left((n+2)(n+1)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) z^n.$$

For $0 < r < 1$, by Hardy's inequality we have that

$$M_1(r, \mathcal{C}_{\mu,\alpha}(g)'') \gtrsim \sum_{n=0}^{\infty} \left((n+2)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) r^n.$$

It follows that

$$\begin{aligned} 1 &\gtrsim ||g||_X \gtrsim ||\mathcal{C}_{\mu,\alpha}(g)||_{B_1} = \int_{\mathbb{D}} |\mathcal{C}_{\mu,\alpha}(g)''(z)| dA(z) \\ &= 2\int_0^1 M_1(r, \mathcal{C}_{\mu,\alpha}(g)'') r dr \\ &\gtrsim \int_0^1 \sum_{n=0}^{\infty} \left((n+2)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) r^{n+1} dr \\ &\gtrsim \sum_{n=0}^{\infty} \mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k}. \end{aligned}$$

Using the Stirling's formula we get

$$\sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \asymp \sum_{k=1}^{n+2} \frac{(n+3-k)^{\alpha-1}}{k}.$$

For $n \geq 1$, simple estimations lead us to the following

$$\begin{aligned} \sum_{k=1}^{n+2} \frac{(n+3-k)^{\alpha-1}}{k} &= \left(\sum_{k=1}^{[\frac{n+2}{2}]} + \sum_{k=[\frac{n+2}{2}]+1}^{n+2} \right) \frac{(n+3-k)^{\alpha-1}}{k} \\ &\asymp (n+1)^{\alpha-1} \sum_{k=1}^{[\frac{n+2}{2}]} \frac{1}{k} + \frac{1}{n+1} \sum_{k=[\frac{n+2}{2}]+1}^{n+2} (n+3-k)^{\alpha-1} \\ &\asymp (n+1)^{\alpha-1} \log(n+2) + (n+1)^{\alpha-1} \\ &\asymp (n+1)^{\alpha-1} \log(n+2). \end{aligned}$$

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Therefore,

$$
\begin{aligned}
1 &\gtrsim \sum_{n=0}^{\infty} \mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \\
&\gtrsim \sum_{n=0}^{\infty} (n+1)^{\alpha-1} \mu_n \log(n+2).
\end{aligned}
$$

Case 2: $p > 1$**.**

Let q be the conjugate index of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $(B_q)^* \cong B_p$ (see [32, Theorem 5.24]) under the paring

$$
\langle F, G \rangle = \int_{\mathbb{D}} F'(z) \overline{G'(z)} dA(z), \quad F \in B_p, G \in B_q.
$$

This means that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_p if and only if

$$
|\langle \mathcal{C}_{\mu,\alpha}(F), G \rangle| \lesssim ||F||_X ||G||_{B_q} \text{ for all } F \in X, G \in B_q.
$$

Now, suppose that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_p. Take $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n \in B_q$ and the sequence of its Taylor coefficients is a decreasing sequence of the non-negative real numbers. Let $f(z) = \log \frac{1}{1-z} = \sum_{n=1} \frac{z^n}{n} \in X$, we have that

$$
|\langle \mathcal{C}_{\mu,\alpha}(f), g \rangle| \lesssim ||f||_X ||g||_{B_q}.
$$

A simple calculation shows that

$$
|\langle \mathcal{C}_{\mu,\alpha}(f), g \rangle| = \sum_{n=1}^{\infty} n \mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k} \right) \widehat{g}(n).
$$

This implies that

$$
|\langle \mathcal{C}_{\mu,\alpha}(f), g \rangle| = \sum_{n=1}^{\infty} n^{\frac{1}{q}} \mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k} \right) \widehat{g}(n) n^{\frac{q-1}{q}} < \infty.
$$

By Lemma 2.1, the sequence $\{\widehat{g}(n) n^{\frac{q-1}{q}}\}_{n=1}^{\infty} \in l^q$. The well known duality $(l^q)^* = l^p$ yields that

$$
\left\{ n^{\frac{1}{q}} \mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k} \right) \right\}_{n=1}^{\infty} \in l^p.
$$

Using the estimate $\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k} \asymp (n+1)^{\alpha-1} \log(n+2)$ we deduce that

$$
\sum_{n=0}^{\infty} (n+1)^{p\alpha-1} \mu_n^p \log^p(n+2) < \infty.
$$

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□

Proof of the implication $(3) \Rightarrow (2)$ Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in X which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Without loss of generality, we may assume that $f_k(0) = 0$ and $\sup_{k\ge 1} ||f||_X \le 1$. It suffices to prove that $\lim_{k\to\infty} ||\mathcal{C}_{\mu,\alpha}(f_k)||_{B_p} = 0$ by using Lemma 2.5. As before, we divide the proof into $p = 1$ and $p > 1$.

Case 1: $p > 1$**.**

Assume that $\sum_{n=1}^{\infty}(n+1)^{p\alpha-1}\mu_n^p \log^p(n+1) < \infty$, then

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)^{p\alpha-1}\mu_n^p \log^p(n+1) &= \sum_{n=1}^{\infty}\left(\sum_{k=2^{n-1}}^{2^n-1}(k+1)^{p\alpha-1}\mu_k^p \log^p(k+1)\right)\\ &\gtrsim \sum_{n=1}^{\infty} 2^{np\alpha}\mu_{2^n}^p \log^p(2^n+1)\\ &\gtrsim \sum_{n=1}^{\infty} 2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k \log(k+1)\right)^p.\end{aligned}$$

This shows that

$$\sum_{n=1}^{\infty} 2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k \log(k+1)\right)^p < \infty.$$

By Lemma 2.2 we have that

$$\begin{aligned}&\int_0^1 (1-r)^{p-2}\left(\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_n \log(n+1) r^n\right)^p dr\\ \asymp &\sum_{n=0}^{\infty} 2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k \log(k+1)\right)^p < \infty.\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists a $0 < r_0 < 1$ such that

$$\int_{r_0}^1 (1-r)^{p-2}\left(\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_k \log(k+1) r^n\right)^p dr < \varepsilon. \tag{1}$$

It is clear that

$$\begin{aligned}||\mathcal{C}_{\mu,\alpha}(f_k)||_{B_p}^p &= \left(\int_{|z|\le r_0} + \int_{r_0<|z|<1}\right)|\mathcal{C}_{\mu,\alpha}(f_k)'(z)|^p(1-|z|)^{p-2}dA(z)\\ &:= J_{1,k} + J_{2,k}.\end{aligned}$$

By the integral representation of $\mathcal{C}_{\mu,\alpha}$ we get

$$\mathcal{C}_{\mu}(f_k)'(z) = \int_0^1 \frac{t f_k'(tz)}{(1-tz)^{\alpha}}d\mu(t) + \int_0^1 \frac{\alpha t f_k(tz)}{(1-tz)^{\alpha+1}}d\mu(t). \tag{2}$$

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Cauchy integral theorem implies that the sequence $\{f_k'\}_{k=1}^{\infty}$ is also converge to 0 uniformly on every compact subset of $\mathbb{D}$. Thus, for $|z| \le r_0$ we have that

$$\begin{aligned}|\mathcal{C}_{\mu,\alpha}(f_k)'(z)| &\lesssim \int_0^1 \frac{|f_k'(tz)|}{|1-tz|^{\alpha}} + \frac{|f_k(tz)|}{|1-tz|^{\alpha+1}} d\mu(t)\\ &\lesssim \sup_{|w|<r_0} \left(|f_k(w)| + |f_k'(w)|\right) \int_0^1 \frac{1}{(1-tr_0)^{\alpha+1}} d\mu(t)\\ &\lesssim \sup_{|w|<r_0} \left(|f_k(w)| + |f_k'(w)|\right).\end{aligned}$$

It follows that

$$J_{1,k} \to 0, \ (k \to \infty).$$

Next, we estimate $J_{2,k}$.

Since $X \subset \mathcal{B}$, we have

$$|f_k(z)| \lesssim \log \frac{e}{1-|z|} \ \text{ and } \ |f_k'(z)| \lesssim \frac{1}{1-|z|} \ \text{ for all } \ k \ge 1, z \in \mathbb{D}. \tag{3}$$

By (2) and (3), Minkowski inequity, Lemma 2.4 we get

$$\begin{aligned}M_p(r, \mathcal{C}_{\mu,\alpha}(f_k)') &= \left\{ \int_0^{2\pi} \left| \int_0^1 \frac{t f_k'(tre^{i\theta})}{(1-tre^{i\theta})^{\alpha}} + \frac{t f_k(tre^{i\theta})}{(1-tre^{i\theta})^{\alpha+1}} d\mu(t) \right|^p d\theta \right\}^{\frac{1}{p}}\\ &\lesssim \left\{ \int_0^{2\pi} \left(\int_0^1 \frac{1}{(1-tr)|1-tre^{i\theta}|^{\alpha}} d\mu(t) \right)^p d\theta \right\}^{\frac{1}{p}}\\ &\quad + \left\{ \int_0^{2\pi} \left(\int_0^1 \frac{\log \frac{e}{1-tr}}{|1-tre^{i\theta}|^{\alpha+1}} d\mu(t) \right)^p d\theta \right\}^{\frac{1}{p}}\\ &\lesssim \int_0^1 \frac{1}{1-tr} \left(\int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^{p\alpha}} \right)^{\frac{1}{p}} d\mu(t)\\ &\quad + \int_0^1 \log \frac{e}{1-tr} \left(\int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^{p(\alpha+1)}} \right)^{\frac{1}{p}} d\mu(t)\\ &\lesssim \int_0^1 H(t,r) d\mu(t),\end{aligned}$$

where

$$H(t,r) = \begin{cases} \dfrac{\log \frac{e}{1-tr}}{(1-tr)^{\alpha+1-\frac{1}{p}}}, & \text{if } \ p > \dfrac{1}{\alpha},\\ \dfrac{\log \frac{e}{1-tr}}{1-tr}, & \text{if } \ p = \dfrac{1}{\alpha}.\end{cases}$$

Lemma 2.3 yields that

$$M_p(r, \mathcal{C}_{\mu,\alpha}(f_k)') \lesssim \int_0^1 H(t,r) d\mu(t) \asymp \sum_{n=0}^{\infty} (n+1)^{\alpha-\frac{1}{p}} \mu_n \log(n+1) r^n.$$

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This together with (1) implies

$$
\begin{aligned}
J_{2,k} &= \int_{r_0<|z|<1} |\mathcal{C}_{\mu,\alpha}(f_k)'(z)|^p(1-|z|)^{p-2}dA(z) \\
&\lesssim \int_{r_0}^1 (1-r)^{p-2} M_p^p(r, \mathcal{C}_{\mu,\alpha}(f_k)')dr \\
&\lesssim \int_{r_0}^1 (1-r)^{p-2} \left(\sum_{n=0}^{\infty} (n+1)^{\alpha-\frac{1}{p}} \mu_n \log(n+1) r^n \right)^p dr \\
&\lesssim \varepsilon.
\end{aligned}
$$

Consequently,

$$\lim_{k\to\infty} ||\mathcal{C}_{\mu,\alpha}(f_k)||_{B_p} = 0.$$

Case 2: $p = 1$.

When $p = 1$, Lemma 2.3 shows that the condition $\sum_{n=0}^{\infty}(n+1)^{\alpha-1}\mu_n \log(n+2) < \infty$ is equivalent to $\int_0^1 \frac{\log \frac{e}{1-t}}{(1-t)^\alpha} d\mu(t) < \infty$. Hence, for any $\varepsilon > 0$ there exists a $0 < t_0 < 1$ such that

$$\int_{t_0}^1 \frac{\log \frac{e}{1-t}}{(1-t)^\alpha} d\mu(t) < \varepsilon. \tag{4}$$

By the integral representation of $\mathcal{C}_{\mu,\alpha}$ we have

$$\mathcal{C}_{\mu,\alpha}(f)''(z) = \int_0^1 \left(\frac{t^2 f''(tz)}{(1-tz)^\alpha} + \frac{2\alpha t^2 f'(tz)}{(1-tz)^{\alpha+1}} + \frac{\alpha(\alpha+1)t^2 f(tz)}{(1-tz)^{\alpha+2}} \right) d\mu(t). \tag{5}$$

For $0 < r < 1$, we have

$$
\begin{aligned}
M_1(r, \mathcal{C}_{\mu,\alpha}(f_k)'') \lesssim & \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) \int_0^{t_0} \frac{d\mu(t)}{(1-t_0 r)^{\alpha+2}} \\
& + \int_0^{2\pi} \int_{t_0}^1 \frac{|f_k''(tz)|}{|1-tre^{i\theta}|^\alpha} + \frac{|f_k'(tz)|}{|1-tre^{i\theta}|^{\alpha+1}} + \frac{|f_k(tz)|}{|1-tre^{i\theta}|^{\alpha+2}} d\mu(t) d\theta.
\end{aligned}
$$

Since $\{f_k\} \subset X \subset \mathcal{B}$, we see that

$$|f_k''(z)| \lesssim \frac{1}{(1-|z|)^2} \text{ for all } k \ge 1. \tag{6}$$

The assumption of p means that $\alpha \ge 1$. By Fubini's theorem, (3), (6) and Lemma 2.4 we have

$$
\begin{aligned}
& \int_0^{2\pi} \int_{t_0}^1 \frac{|f_k''(tre^{i\theta})|}{|1-tre^{i\theta}|^\alpha} + \frac{|f_k'(tre^{i\theta})|}{|1-tre^{i\theta}|^{\alpha+1}} + \frac{|f_k(tre^{i\theta})|}{|1-tre^{i\theta}|^{\alpha+2}} d\mu(t) d\theta \\
\lesssim & \int_{t_0}^1 \int_0^{2\pi} \left(\frac{1}{(1-tr)^2 |1-tre^{i\theta}|^\alpha} + \frac{1}{(1-tr)|1-tre^{i\theta}|^{\alpha+1}} + \frac{\log \frac{e}{1-tr}}{|1-tre^{i\theta}|^{\alpha+2}} \right) d\theta d\mu(t) \\
\lesssim & \int_{t_0}^1 \frac{\log \frac{e}{1-tr}}{(1-tr)^{\alpha+1}} d\mu(t).
\end{aligned}
$$

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Hence,

$$
\begin{aligned}
&\int_{t_0<|z|<1} |\mathcal{C}_{\mu,\alpha}(f_k)''(z)|dA(z) \\
&= 2\int_{t_0}^{1} M_1(r, \mathcal{C}_{\mu,\alpha}(f_k)'')rdr \\
&\lesssim \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^{1}\int_{t_0}^{1} \frac{\log\frac{e}{1-tr}}{(1-tr)^{\alpha+1}} d\mu(t)dr \\
&\lesssim \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^{1} \log\frac{e}{1-t}\int_0^1 \frac{dr}{(1-tr)^{\alpha+1}} d\mu(t) \\
&\lesssim \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^{1} \frac{\log\frac{e}{1-t}}{(1-t)^{\alpha}} d\mu(t) \\
&\lesssim \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \varepsilon.
\end{aligned}
$$

The uniform convergence of $\{f_k\}$ on compact subsets of $\mathbb{D}$ implies that

$$
\int_{|z|\le t_0} |\mathcal{C}_{\mu,\alpha}(f_k)''(z)|dA(z) \lesssim \sup_{|w|\le t_0} (|f_k''(w)| + |f_k'(w)| + |f_k(w)|) \to 0, \text{ as } k\to 0.
$$

Since $f_k(0)=0$, so we have that $|\mathcal{C}_{\mu,\alpha}(f_k)(0)| + |\mathcal{C}_{\mu,\alpha}(f_k)'(0)| \asymp |f_k'(0)| \to 0$ as $k\to\infty$.

Therefore, we deduce that

$$
\lim_{k\to\infty} ||\mathcal{C}_{\mu,\alpha}(f_k)||_{B_1} = 0.
$$

Thus, the operator $\mathcal{C}_{\mu,\alpha}$ is compact from X to B_1. □

Conflicts of Interest

The authors declare that there is no conflict of interest.

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