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REGULAR QUOTIENTS OF METRIC SPACES

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1. **Introduction.** In this note we consider the class of topological spaces whose every Hausdorff quotient is regular (or equivalently, normal). Sufficient conditions in order that a space belongs to this class have appeared in papers by Morita [3], and more recently by MacDonald and Willard [2]. In both cases, the authors impose conditions implying paracompactness. We shall derive additional sufficient conditions here which show that paracompactness is not necessary in general. A characterization of those metrizable spaces whose every Hausdorff quotient is regular is obtained as a particular case of our main result:

THEOREM. *Let X be a first countable paracompact space. Then the following are equivalent:*

(a) *every Hausdorff quotient of X is paracompact;*

(b) *every Hausdorff quotient of X is regular;*

(c) acc *X*(*=set of non-isolated points of X*) *is compact, or else X is locally compact and* acc *X is Lindelöf.*

COROLLARY. *If X is a metric space, then conditions* (a), (b) *and* (c) *above are equivalent.*

All spaces considered in this paper are assumed to be Hausdorff topological spaces. Our terminology is standard, except that *acc X* is used to denote the set of non-isolated (or accumulation) points of *X*.

2. **Preliminary lemmas.** We first establish some lemmas which we need in proving the theorem.

LEMMA 1. (Morita [3]). *If X is locally compact and Lindelöf, then every Hausdorff quotient of X is paracompact.*

That *X* need not be Lindelöf is apparent from the following slightly stronger result.

LEMMA 2. *If X is locally compact and acc X is Lindelöf, then every Hausdorff quotient of X is paracompact.*

Proof. Let f be a quotient map of X onto a Hausdorff space Y and let $\{U_i\}_{i=1}^{\infty}$ be a countable open cover of *acc X* such that $\overline{U_i}$ is compact for each i, where $\overline{U_i}$ denotes the closure of U_i in X. Then $U=\bigcup_{i=1}^{\infty} U_i=\bigcup_{i=1}^{\infty} \overline{U_i}$ is an open, hence

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locally compact, subspace of X containing *acc* X. Since $\bigcup_{i=1}^{\infty} \overline{U_i}$ is also Lindelöf, U is paracompact. Noting that X is the disjoint union of the clopen set U and its discrete complement, we see that f restricted to U remains a quotient map, and $f(U)$ is paracompact by Lemma 1. That Y is paracompact now follows easily from the fact that $Y-f(U)$ is a closed set consisting of isolated points.

LEMMA 3 [CH]. *If X is paracompact and* acc X *contains a dense first countable subspace, then every Hausdorff quotient of X is regular only if* acc X *is Lindelöf.*

Proof. Under the assumptions that X is paracompact, acc X contains a dense first countable subspace and acc X is not Lindelöf, MacDonald and Willard [2; Theorem 4.3] have constructed (on the continuum hypothesis) a non-normal regular quotient of X. This in turn will yield a non-regular Hausdorff quotient of X, by identifying to a point one of two disjoint closed subsets which can't be separated.

The following lemma improves Theorem 2.2 of [2], employing the same method of proof.

LEMMA 4. *If X is first countable and every Hausdorff quotient of X is regular, then either*

(a) acc X *is countably compact, or*

(b) *X is locally countably compact.*

Proof. If neither (a) nor (b) holds, we can find in acc X a point y having no countably compact neighborhood and a sequence $(x_n)_{n=1}^{\infty}$ in acc $X-\{y\}$ having no cluster point. Let U be a closed neighborhood of y containing no x_n, and choose disjoint sequences (x_{1i}), (x_{2i}), ... in $(X-U)$ such that $x_{ni} \to x_n$ for each $n \in N$. Let $\{U_n\}_{n=1}^{\infty}$ be a nested neighborhood base at y with $U_1 = U$ and such that for each $n \in N$, $U_n - U_{n+1}$ contains a sequence $(y_{ni})_{i=1}^{\infty}$ having no cluster point. Now identify n_{ni} with y_{ni} for all n, $i \in N$. The resulting (Hausdorff) quotient is non-regular, since the closed set $\{x_n : n \in N\}$ cannot be separated from the point y. But this contradicts our assumption, so that either (a) or (b) must hold.

Proof of the theorem. (a)$\Rightarrow$(b): This is obvious.

(b)$\Rightarrow$(c). This implication follows from Lemmas 3 and 4 combined with the fact that a countably compact, paracompact space is compact.

(c)$\Rightarrow$(a). If acc X is compact, then every Hausdorff quotient of X is regular [2; Theorem 2.1] and, because acc X in Lindelöf, every regular quotient of X is paracompact. If acc X is Lindelöf and X is locally compact, then every Hausdorff quotient of X is paracompact by Lemma 2. This completes the proof of the theorem.

3. **Examples and a related result.** The independence of the two parts of condition (c) of the theorem is easily checked. A convergent sequence with each non-limit point expanded to an infinite discrete set satisfies the first part but not the second, while the real line works the other way.

EXAMPLE. The previous theorem suggests that whenever every Hausdorff quotient of a Hausdorff space X is regular, then every Hausdorff quotient of X is paracompact. However, the space of ordinals less than the first uncountable ordinal (with the order topology) together with the following proposition forms a counterexample.

Recall that a space is *sequential* iff a subset is closed whenever it contains all of its sequential limit points.

PROPOSITION. *If X is a normal, countably compact, sequential space, then every Hausdorff quotient of X is normal and hence regular.*

Proof. Let f be a quotient map of X onto a Hausdorff space Y. Since normality is preserved by closed maps, it suffices to show that f is closed. If F is a closed subset of X, then F is countably compact and hence so is $f(F)$. That $f(F)$ is closed in Y follows from the facts that every quotient of a sequential space is sequential [1], and a countably compact subset of a sequential space is closed. Thus f is a closed map, as required.

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