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MEMBRANE-COUPLED GRAVITY WAVE SCATTERING BY A VERTICAL BARRIER WITH A GAP

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Abstract

We study the reflection of membrane-coupled gravity waves in deep water against a vertical barrier with a gap. A floating membrane is attached on both sides of the barrier. The associated mixed boundary value problem, which is not particularly well posed, is analysed. We utilize an orthogonal mode-coupling relation to reduce the problem to solving a set of dual integral equations with trigonometric kernel. We solve these by using a weakly singular integral equation. The reflection coefficient is determined explicitly, while having freedom to clamp the membrane with a spring of a certain stiffness on only one side of the vertical barrier. The physical problem is of capillary–gravity wave scattering by a vertical barrier with a gap, when the membrane density is neglected. In this case, the reflection coefficient is known up to an undetermined edge slope on either side of the barrier. The scattering quantity is computed and presented graphically against a wave parameter for different values of nondimensional parameters pertaining to the structures involved in the problem.

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1. Introduction

Scattering of free surface gravity waves in deep water by various vertical barrier configurations has been the subject of interest for many researchers. Many interesting methods of solution have been developed to tackle the mixed boundary value problems associated with them in the literature [1, 7, 15, 19, 20]. One particular type of scattering problem, that involved a vertical barrier with a gap, was first considered by Tuck [18]. He found an approximate solution by making use of matched asymptotic expansion. Porter [16] provided a complete analytical solution via a reduction procedure as well as an integral equation formulation. Later, Chakrabarti and Vijaya Bharati [4] revisited the problem with a reduction procedure of solving

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two independent Riemann–Hilbert problems. Chakrabarti et al. [3] provided a weakly singular integral equation method with an aim to get the scattering quantities in a fairly straightforward manner. More generally, gravity wave scattering by a vertical barrier with a finite number of gaps in it was tackled by Mei [15] through complex variable techniques.

Floating flexible structures received some attention for their utility as floating breakwaters in ocean-engineering applications. These floating barriers can suppress the vertical motion of water oscillation so strongly that the wave amplitude can be reduced effectively. Among them, floating membranes enjoy desirable characteristics of being transportable, relatively inexpensive, rapidly deployable, reusable and easily detachable. In this context, wave scattering by a submerged horizontal membrane was analysed by Cho and Kim [5]. They found that flexible membranes could enhance overall wave reflection better than rigid plates. A few studies on the interaction of free surface gravity waves with floating membrane structures can be found in the literature [9, 10, 21].

The governing mixed boundary value problem is mathematically the same for both the flexible membrane-covered surface and the free surface acting under surface tension, except for the edge conditions that are to be specified at intersection points of floating surface and piercing vertical barrier. These naturally arising physical edge conditions are utilized to find the velocity at the mean surface on either side of the vertical barrier, since this quantity is not the same across the barrier. In fact, study of the surface tension effect on gravity waves is not of recent interest. Evans [6] investigated surface gravity waves under surface tension in the presence of partial vertical barriers in deep water. Later, Rhodes-Robinson [17] studied capillary–gravity waves produced by incomplete wave-makers, such as surface-piercing and submerged ones. However, numerical computation has not been carried out for scattering quantities in their work. Perhaps these quantities were not in a suitable form for computation. It was presumed that the edge-slope constants can be evaluated at the edges of the barrier at the mean free surface [8]. Recently, Manam [12] has provided an analytical solution for the scattering of capillary–gravity waves by partial vertical barriers along with promising numerical results for the reflection coefficient. The solution method involves the application of an orthogonal mode-coupling relation [13] and the bounded solution of a weakly singular integral equation. The study has been implemented to membrane-coupled gravity waves under the assumption that the floating membrane can be attached with a spring of zero stiffness to the surface-piercing vertical barrier [11].

In this paper, a two-dimensional linear water wave scattering problem involving a thin vertical barrier with a gap is studied, when the free surface is either covered with a flexible floating membrane or subject to surface tension. Membrane ends can be attached with springs of certain specified stiffnesses, giving rise to suitable edge conditions at the intersection of membrane and vertical barrier. For capillary–gravity waves, the slope of the water surface on either side of the vertical barrier may not be known, but the difference between both sides can be determined. The solution procedure is an extension to that of Manam [11]. The associated mixed boundary value problem is formulated in Section 2. The method of solution is explained in Section 3 with illustrations in appendices. Numerical computations are carried out and the results are presented graphically in Section 4. Finally, some conclusions are drawn in Section 5.

2. Mathematical formulation

A two-dimensional mixed boundary value problem that models membrane-coupled gravity wave scattering by a rigid barrier with a gap is formulated under the assumptions of linearized water wave theory and small-amplitude membrane response. The fluid surface is covered with a floating membrane, which may be considered as a one-dimensional string. A two-dimensional Cartesian coordinate system is used in which the positive y-axis is pointed vertically downward and the region $y > 0$, $x \in \boldsymbol{R}$ is occupied by the fluid. The surface-piercing structure is located at $x = 0,\ y \in (0, a) \cup (b, \infty)$ in the fluid region. An irrotational time-harmonic motion is assumed in the incompressible inviscid fluid under the action of gravity. Hence, there exists a velocity potential $\Phi(x, y, t) = \mathrm{Re}(\phi(x, y)e^{-i\omega t})$, where $\omega\,(> 0)$ denotes angular frequency and t denotes time. Also, surface deflection is of the form $\zeta(x, t) = \mathrm{Re}(\eta(x)e^{-i\omega t})$. The time-dependent factor $e^{-i\omega t}$ is naturally suppressed throughout the analysis. Then, the spatial velocity potential $\phi(x, y)$ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \quad |x| > 0,\ y > 0. \tag{2.1}$$

A linear boundary condition is derived on the membrane-covered region [11], by balancing the hydrodynamic pressure at the interface and the effective pressure on the membrane, as

$$M\frac{\partial^3\phi}{\partial y^3} + \frac{\partial\phi}{\partial y} + K\phi = 0 \quad \text{on } y = 0,\ x \in \boldsymbol{R}, \tag{2.2}$$

where $M = S/(\rho g - m_s\omega^2)$ with S, ρ, g representing membrane tension, density of water, acceleration due to gravity, respectively, and $K = \rho\omega^2/(\rho g - m_s\omega^2)$. When the mass of the membrane is zero, that is, $m_s = 0$, (2.2) represents the boundary condition for water waves under surface tension S. In this case, the boundary value problem governs scattering of capillary–gravity waves by the vertical barrier with a gap in it.

On the rigid vertical structure located at $x = 0,\ y \in (0, a) \cup (b, \infty)$ with $a < b$, $\phi(x, y)$ satisfies the Neumann boundary condition

$$\frac{\partial\phi}{\partial x} = 0. \tag{2.3}$$

Since the fluid flow is continuous across the gap $x = 0,\ y \in (a, b)$, $\phi(x, y)$ satisfies

$$\phi(0^-, y) = \phi(0^+, y), \tag{2.4}$$

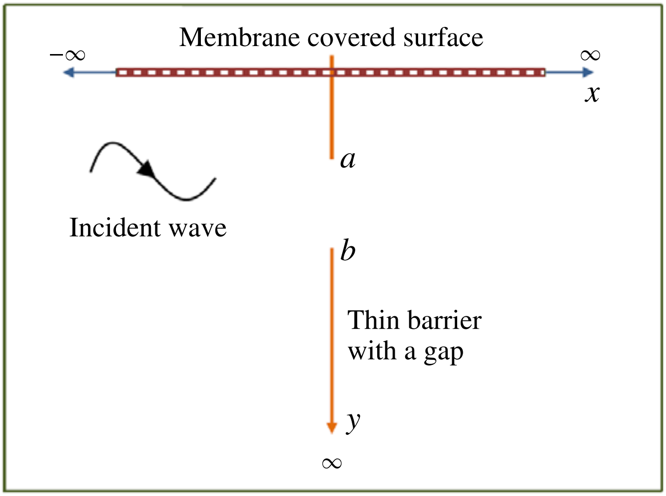

FIGURE 1. Schematic diagram for the physical problem.

in the usual notation, and

$$\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \to 0 \quad \text{as } y \to \infty, \tag{2.5}$$

which represents no motion at larger depth.

The behaviour of $\phi(x, y)$ at the extremities in the horizontal direction is given by

$$\phi(x, y) \to \begin{cases} e^{i\lambda x - \lambda y} + Re^{-i\lambda x - \lambda y} & \text{as } x \to -\infty, \\ Te^{i\lambda x - \lambda y} & \text{as } x \to \infty, \end{cases} \tag{2.6}$$

representing progressive waves. The unknown complex constants R and T are the reflected and transmitted amplitudes of the incident wave $e^{i\lambda x - \lambda y}$, where $\lambda > 0$. The constant λ satisfies the dispersion equation $Mx^3 + x - K = 0$. This equation also has two complex conjugate roots $\lambda_1, \bar{\lambda}_1$ with negative real parts.

The edge conditions, as required for the energy to be finite in the neighbourhood of all edges associated with the flow [14], are given by

$$\frac{\partial \phi}{\partial x}(0, y) \sim O(|y - t|^{-1/2}) \quad \text{as } y \to t, \tag{2.7}$$

where $t = a^+$ and b^-, the edge points of thin vertical structure under consideration. A schematic diagram for the physical problem is displayed in Figure 1.

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3. Method of solution

The unknown velocity potential $\phi(x, y)$ in the regions $x < 0$ and $x > 0$ may be expanded as [13]

$$\phi(x,y) = \begin{cases} e^{i\lambda x-\lambda y} + Re^{-i\lambda x-\lambda y} + \dfrac{2}{\pi}\displaystyle\int_0^\infty A(\xi)[\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]e^{\xi x}\,d\xi, \\ \hfill x<0, \\ Te^{i\lambda x-\lambda y} + \dfrac{2}{\pi}\displaystyle\int_0^\infty B(\xi)[\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]e^{-\xi x}\,d\xi, \quad x>0, \end{cases}$$

where $A(\xi)$ and $B(\xi)$ are unknown functions along with the constants R and T. The above potential function $\phi(x, y)$ automatically satisfies the partial differential equation (2.1) and the conditions (2.2), (2.5) and (2.6) for an appropriate choice of the functions $A(\xi)$ and $B(\xi)$.

Since the horizontal velocity component is continuous across the positive y-axis except at the intersection point of the membrane and the rigid barrier, the difference in the velocities on both sides of the interface $x = 0$ is obtained as

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty \xi[A(\xi)+B(\xi)][\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]\,d\xi \\ &\quad = (\phi_x(0^-,y)-\phi_x(0^+,y)) + i\lambda(T+R-1)e^{-\lambda y} \quad \text{for } y\ge 0. \end{aligned}$$

The functions $[\xi(1-M\xi^2)\cos\xi y - K\sin\xi y]$ and $e^{-\lambda y}$ satisfy the mode-coupling relation [13]

$$\langle f, g\rangle = \int_0^\infty f(y)g(y)\,dy + \frac{M}{K}f'(0)g'(0),$$

where $'$ denotes differentiation. Hence,

$$T = 1 - R + \frac{2iM(\mu^+-\mu^-)}{1+3M\lambda^2}; \quad B(\xi) = -A(\xi) + \frac{M(\mu^+-\mu^-)}{\Delta(\xi)},$$

where $\Delta(\xi) = \xi^2(1-M\xi^2)^2 + K^2$ and $\mu^\pm = \phi_{xy}(0^\pm, 0)$ are the unknown edge-slope constants. It will be clear a little later that one of these edge-slope constants needs to be prescribed mathematically for the unique solution. Physically, they are calibrated slopes at the attached points of the floating membrane to the vertical barrier. For water waves under surface tension, these edge-slope constants can be found experimentally, as explained in Hocking [8].

Application of the conditions (2.3) and (2.4), after integration, provides a pair of integral equations

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\,d\xi \\ &= \begin{cases} B_1e^{\lambda y} + B_2e^{\lambda_1 y} + B_3e^{\bar\lambda_1 y} + \dfrac{i\lambda(1-R)}{Q(\lambda)}e^{-\lambda y} \equiv f_1(y) & \text{if } y\in(0,a), \\ C_1e^{\lambda_1 y} + C_2e^{\bar\lambda_1 y} + C_3e^{\lambda y} + \frac{i\lambda(1-R)}{Q(\lambda)}e^{-\lambda y} \equiv f_2(y) & \text{if } y\in(b,\infty), \end{cases} \end{aligned} \tag{3.1}$$

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$$\begin{aligned}
&\frac{2}{\pi}\int_0^\infty A(\xi)\sin\xi y\,d\xi\\
&\quad= D_1 e^{\lambda y} + D_2 e^{\lambda_1 y} + D_3 e^{\bar{\lambda}_1 y} + \left[R - \frac{iM(\mu^+ - \mu^-)}{1+3M\lambda^2}\right]\frac{e^{-\lambda y}}{Q(\lambda)}\\
&\qquad+\frac{M}{\pi}(\mu^+ - \mu^-)\int_0^\infty \frac{\sin\xi y}{\Delta(\xi)}\,d\xi \equiv h(y) \quad \text{if } y\in(a,b),
\end{aligned} \tag{3.2}$$

where $B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, D_3$ are arbitrary constants, and $Q(\lambda) = \lambda(1 + M\lambda^2) + K$. Clearly, the constant C_3 in (3.1) must be zero. Otherwise, the function $f_2(y)$ is unbounded as $y \to \infty$. Note that the function $f_1(y)$ must satisfy $f_1(0) = f_1''(0) = 0$. These conditions translate into

$$B_1 + B_2 + B_3 - \frac{i\lambda}{Q(\lambda)}R = -\frac{i\lambda}{Q(\lambda)} \tag{3.3}$$

and

$$\lambda^2 B_1 + \lambda_1^2 B_2 + \bar{\lambda}_1^2 B_3 - \frac{i\lambda^3}{Q(\lambda)}R = -\frac{i\lambda^3}{Q(\lambda)}. \tag{3.4}$$

Suppose that

$$\frac{2}{\pi}\int_0^\infty \xi A(\xi)\sin\xi y\,d\xi = g_1(y) \quad \text{for } y\in(a,b),$$

where $g_1(y)$ is an unknown function. Then, by the inverse Fourier sine transform, one can get

$$A(\xi) = \frac{1}{\xi}\int_0^\infty P(y)\sin\xi y\,dy, \tag{3.5}$$

where

$$P(y) = \begin{cases} f_1(y) & \text{for } y\in(0,a),\\ g_1(y) & \text{for } y\in(a,b),\\ f_2(y) & \text{for } y\in(b,\infty).\end{cases}$$

It is routine to obtain a weakly singular integral equation, as in Manam [11], for the unknown function $g_1(y)$, and it is given by

$$\frac{1}{\pi}\int_a^b g_1(u)\log\left|\frac{u+t}{u-t}\right|du = h_1(t) \quad \text{for } t\in(a,b), \tag{3.6}$$

where

$$h_1(t) = h(t) - \frac{1}{\pi}\int_0^a f_1(u)\log\left|\frac{u+t}{u-t}\right|du - \frac{1}{\pi}\int_b^\infty f_2(u)\log\left|\frac{u+t}{u-t}\right|du.$$

On differentiating the relations (3.1) and (3.2) twice, again by the application of the inverse Fourier sine transform, one can obtain

$$A(\xi) = -\frac{1}{\xi^3}\int_0^\infty \frac{d^2P}{dy^2}\sin\xi y\,dy, \tag{3.7}$$

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where

$$\frac{d^2P}{dy^2}=\begin{cases}\dfrac{d^2f_1}{dy^2} & \text{for } y\in(0,a),\\ g_2 & \text{for } y\in(a,b),\\ \dfrac{d^2f_2}{dy^2} & \text{for } y\in(b,\infty).\end{cases}$$

By substituting (3.7) into the twice-differentiated form of (3.2), we find that the function $g_2\equiv d^2g_1/dy^2$ satisfies the weakly singular integral equation

$$\frac{1}{\pi}\int_a^b g_2(u)\log\left|\frac{u+t}{u-t}\right|du=h_2(t)\quad\text{for } t\in(a,b),\tag{3.8}$$

where

$$h_2(t)=\frac{d^2h}{dt^2}-\frac{1}{\pi}\int_0^a\frac{d^2f_1}{du^2}\log\left|\frac{u+t}{u-t}\right|du-\frac{1}{\pi}\int_b^\infty\frac{d^2f_2}{du^2}\log\left|\frac{u+t}{u-t}\right|du.$$

The edge condition (2.7) physically describes that the horizontal velocity component $\partial\phi/\partial x$ has an integrable singularity at the sharp barrier edges, as required for the wave energy to be finite. We utilize this to show that the functions $g_1(y)$ and d^2g_1/dy^2 are bounded [11] at the end points $y=a$ and $y=b$ with the behaviour

$$\frac{d^{(2-2i)}g_1}{dy^2}\sim O(|y-t|^{\frac{1}{2}+2i})\quad\text{as } y\to a^+,\ b^-\ \text{for } i=0,1.$$

Hence, the bounded solutions [2] of the integral equations (3.6) and (3.8) are given by

$$g_1(u)=\frac{2}{\pi}\sqrt{(u^2-a^2)(b^2-u^2)}\int_a^b\frac{th_1'(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}(u^2-t^2)}\,dt,\quad u\in(a,b),$$

provided that

$$\int_a^b\frac{th_1'(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt=0,\tag{3.9}$$

$$\frac{(a\pi-J_1)}{J_2}\int_a^b\frac{h_1(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}h_1(t)\,dt$$
$$+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}th_1'(t)\,dt=0\tag{3.10}$$

and

$$g_2(u)=\frac{2}{\pi}\sqrt{(u^2-a^2)(b^2-u^2)}\int_a^b\frac{th_2'(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}(u^2-t^2)}\,dt,\quad u\in(a,b),$$

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provided that

$$\int_a^b \frac{t h_2'(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt = 0, \tag{3.11}$$

$$\frac{(a\pi - J_1)}{J_2}\int_a^b \frac{h_2(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt + \int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}h_2(t)\,dt + \int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}t h_2'(t)\,dt = 0, \tag{3.12}$$

where

$$J_1 = \int_a^b \sqrt{\frac{t^2-a^2}{b^2-t^2}}\log\left|\frac{a+t}{a-t}\right|dt \quad \text{and} \quad J_2 = \int_a^b \frac{1}{\sqrt{(t^2-a^2)(b^2-t^2)}}\log\left|\frac{a+t}{a-t}\right|dt.$$

It may be noted that the function $A(\xi)$ is expressed in two integral representations (3.5) and (3.7). After equating these and integrating by parts, we obtain the conditions

$$f_1(a) = g_1(a), \quad \text{that is, } e^{\lambda a}B_1 + e^{\lambda_1 a}B_2 + e^{\bar{\lambda}_1 a}B_3 - \frac{i\lambda}{Q(\lambda)}e^{-\lambda a}R = -\frac{i\lambda}{Q(\lambda)}e^{-\lambda a}, \tag{3.13}$$

$$f_2(b) = g_1(b), \quad \text{that is, } e^{\lambda_1 b}C_1 + e^{\bar{\lambda}_1 b}C_2 - \frac{i\lambda}{Q(\lambda)}e^{-\lambda b}R = -\frac{i\lambda}{Q(\lambda)}e^{-\lambda b}, \tag{3.14}$$

$$f_1'(a) = g_1'(a) \tag{3.15}$$

and

$$f_2'(b) = g_1'(b). \tag{3.16}$$

Using integrals in Appendix A, the conditions in (3.9) and (3.11) are expressed as linear equations

$$\begin{aligned} &I_1(\lambda)B_1 + I_1(\lambda_1)B_2 + I_1(\bar{\lambda}_1)B_3 - I_2(\lambda_1)C_1 - I_2(\bar{\lambda}_1)C_2 + \lambda I_3(\lambda)D_1 + \lambda_1 I_3(\lambda_1)D_2 \\ &\quad + \bar{\lambda}_1 I_3(\bar{\lambda}_1)D_3 + [-iI_1(-\lambda) + iI_2(-\lambda) - I_3(-\lambda)]\frac{\lambda}{Q(\lambda)}R \\ &\quad + \left(\frac{M}{\pi}\int_a^b\int_0^\infty \frac{t\xi\cos\xi t}{\sqrt{(t^2-a^2)(b^2-t^2)}\Delta(\xi)}\,d\xi\,dt + \frac{iM\lambda}{(1+3M\lambda^2)Q(\lambda)}I_3(-\lambda)\right) \\ &\quad \times(\mu^+ - \mu^-) \\ &= [-I_1(-\lambda) + I_2(-\lambda)]\frac{i\lambda}{Q(\lambda)} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\lambda^2 I_1(\lambda)B_1 + \lambda_1^2 I_1(\lambda_1)B_2 + \bar{\lambda}_1^2 I_1(\bar{\lambda}_1)B_3 - \lambda_1^2 I_2(\lambda_1)C_1 - \bar{\lambda}_1^2 I_2(\bar{\lambda}_1)C_2 + \lambda^3 I_3(\lambda)D_1 \\ &\quad + \lambda_1^3 I_3(\lambda_1)D_2 + \bar{\lambda}_1^3 I_3(\bar{\lambda}_1)D_3 + [-iI_1(-\lambda) + iI_2(-\lambda) - I_3(-\lambda)]\frac{\lambda^3}{Q(\lambda)}R \end{aligned}$$

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$$
\begin{aligned}
&+\left(-\frac{M}{\pi}\int_a^b\int_0^\infty \frac{t\xi^3\cos\xi t}{\sqrt{(t^2-a^2)(b^2-t^2)}\Delta(\xi)}\,d\xi\,dt+\frac{iM\lambda^3}{(1+3M\lambda^2)Q(\lambda)}I_3(-\lambda)\right)\\
&\times(\mu^+-\mu^-)\\
&=[-I_1(-\lambda)+I_2(-\lambda)]\frac{i\lambda^3}{Q(\lambda)},
\end{aligned}
\tag{3.18}
$$

where

$$
I_1(x)=\int_0^a \frac{te^{xt}}{\sqrt{(a^2-t^2)(b^2-t^2)}}\,dt,\quad I_2(x)=\int_b^\infty \frac{te^{xt}}{\sqrt{(a^2-t^2)(b^2-t^2)}}\,dt
$$

and

$$
I_3(x)=\int_a^b \frac{te^{xt}}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt.
$$

Also, the conditions (3.10) and (3.12) are simplified as

$$
\begin{aligned}
&\frac{(a\pi-J_1)}{J_2}\int_a^b \frac{h(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt-\frac{(a\pi-J_1)}{J_2}\int_0^a\int_x^a \frac{f_1(u)}{\sqrt{(a^2-x^2)(b^2-x^2)}}\,du\,dx\\
&-\frac{(a\pi-J_1)}{J_2}\int_b^\infty\int_b^x \frac{f_2(u)}{\sqrt{(a^2-x^2)(b^2-x^2)}}\,du\,dx+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}h(t)\,dt\\
&+\int_0^a\int_x^a\sqrt{\frac{a^2-x^2}{b^2-x^2}}f_1(u)\,du\,dx+\int_b^\infty\int_x^\infty\sqrt{\frac{a^2-x^2}{b^2-x^2}}f_2(u)\,du\,dx\\
&+\left(b-\frac{J_3}{\pi}\right)\int_b^\infty f_2(t)\,dt+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}th'(t)\,dt-\int_0^a\sqrt{\frac{a^2-t^2}{b^2-t^2}}tf_1(t)\,dt\\
&-\int_b^\infty\sqrt{\frac{a^2-t^2}{b^2-t^2}}tf_2(t)\,dt=0
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&\frac{(a\pi-J_1)}{J_2}\int_a^b \frac{h''(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt-\frac{(a\pi-J_1)}{J_2}\int_0^a\int_x^a \frac{f_1''(u)}{\sqrt{(a^2-x^2)(b^2-x^2)}}\,du\,dx\\
&-\frac{(a\pi-J_1)}{J_2}\int_b^\infty\int_b^x \frac{f_2''(u)}{\sqrt{(a^2-x^2)(b^2-x^2)}}\,du\,dx+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}h''(t)\,dt\\
&+\int_0^a\int_x^a\sqrt{\frac{a^2-x^2}{b^2-x^2}}f_1''(u)\,du\,dx+\int_b^\infty\int_x^\infty\sqrt{\frac{a^2-x^2}{b^2-x^2}}f_2''(u)\,du\,dx\\
&+\left(b-\frac{J_3}{\pi}\right)\int_b^\infty f_2''(t)\,dt+\int_a^b\sqrt{\frac{t^2-a^2}{b^2-t^2}}th'''(t)\,dt-\int_0^a\sqrt{\frac{a^2-t^2}{b^2-t^2}}tf_1''(t)\,dt\\
&-\int_b^\infty\sqrt{\frac{a^2-t^2}{b^2-t^2}}tf_2''(t)\,du=0,
\end{aligned}
\tag{3.20}
$$

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where $J_3 = \int_a^b \sqrt{(t^2 - a^2/b^2 - t^2)} \log |(b + t/b - t)| \, dt$. These equations are written as linear equations for the unknown constants, whose coefficients are provided in Appendix B.

Nevertheless, (3.15) and (3.16) are not in a readily computable form. In order to make them suitable for computation, $g_2(u)$ is integrated on $[a, b]$ by multiplying u and u^3 separately. Then,

$$\int_a^b \frac{[t^3 - \frac{1}{2}(a^2 + b^2)t]h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt + bg_1'(b) - ag_1'(a) = 0$$

and

$$\int_a^b \frac{[t^5 - \frac{1}{2}(a^2 + b^2)t^3 - \frac{1}{8}(a^2 - b^2)^2 t]h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt - 6\int_a^b \frac{[t^3 - \frac{1}{2}(a^2 + b^2)t]h_1'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt$$
$$-a^3 g_1'(a) + b^3 g_1'(b) = 0.$$

Solving the above equations for $g_1'(a)$ and $g_1'(b)$ and using these in (3.15), (3.16), we get

$$-6\int_a^b \frac{t^3 h_1'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt - \left(\frac{3b^2 + a^2}{2}\right)\int_a^b \frac{t^3 h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt$$
$$+ \int_a^b \frac{t^5 h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt + (ab^2 - a^3)f_1'(a) = 0 \tag{3.21}$$

and

$$6\int_a^b \frac{t^3 h_1'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt + \left(\frac{3a^2 + b^2}{2}\right)\int_a^b \frac{t^3 h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt$$
$$- \int_a^b \frac{t^5 h_2'(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt - (b^3 - a^2 b)f_2'(b) = 0. \tag{3.22}$$

Further simplifying (3.21) and (3.22), we get

$$\int_0^a \frac{t^3 f_1''(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt - \int_b^\infty \frac{t^3 f_2''(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt$$
$$+ \int_a^b \frac{t^3 h'''(t)}{\sqrt{(t^2 - a^2)(b^2 - t^2)}} \, dt = 0 \tag{3.23}$$

and

$$6\int_0^a \frac{t^3 f_1(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt + \left(\frac{3a^2 + b^2}{2}\right)\int_0^a \frac{t^3 f_1''(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt$$
$$- \int_0^a \frac{t^5 f_1''(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt - 6\int_b^\infty \frac{t^3 f_2(t)}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} \, dt$$

$$
\begin{aligned}
&-\left(\frac{3a^2+b^2}{2}\right)\int_b^\infty \frac{t^3 f_2''(t)}{\sqrt{(t^2-a^2)(t^2-b^2)}}\,dt + \int_b^\infty \frac{t^5 f_2''(t)}{\sqrt{(t^2-a^2)(t^2-b^2)}}\,dt \\
&+6\int_a^b \frac{t^3 h'(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt + \left(\frac{3a^2+b^2}{2}\right)\int_a^b \frac{t^3 h'''(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt \\
&-\int_a^b \frac{t^5 h'''(t)}{\sqrt{(t^2-a^2)(b^2-t^2)}}\,dt = 0. \qquad (3.24)
\end{aligned}
$$

The constraints in (3.23) and (3.24) are expressed as linear equations and their coefficients are given in Appendix B.

Thus, the mixed boundary value problem has been reduced to solving a system of 10 linear equations for 10 unknown constants. Then, the unknown function $A(\xi)$ may be obtained explicitly, but not in the best computable form, from either (3.5) or (3.7). These equations are given by the relations (3.3), (3.4), (3.13), (3.14), (3.17)–(3.20), (3.23) and (3.24). The linear system may be represented, for clarity, in a matrix form $\mathbf{U}\mathbf{x}^T = \mathbf{v}^T$, where $\mathbf{U}$ is a 10×10 matrix; $\mathbf{x} = (B_1, B_2, B_3, C_1, C_2, D_1, D_2, D_3, R, (\mu^+ - \mu^-))$ and $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})$ are 1×10 vectors. The last four rows of the augmented matrix $[\mathbf{U} : \mathbf{v}^T]$ are listed in Appendix B.

Note that the solution of the mixed boundary value problem involves an arbitrary constant in either μ^+ or μ^-. This needs to be prescribed for the unique solution. Physically, for a floating membrane, it means that the membrane ends cannot be elastically supported by springs of zero vertical stiffness, as is done for a partial surface-piercing vertical barrier [10, 11]. This is supported by the argument that the method of solution determines the constant $(\mu^+ - \mu^-)$, and the constant may not necessarily be zero in the problem involving a vertical barrier with a gap. In other words, one may have freedom to clamp the membrane with a spring of zero stiffness on only one side of the barrier, while the membrane edge on the other side of the barrier must be supported with a spring of a certain stiffness for the existence of the solution. The measure of spring stiffness can be determined from the constant $(\mu^+ - \mu^-)$. Obviously, this quantity depends on the frequency of an incident wave.

However, for capillary–gravity waves, the difference between the edge slopes on both sides of the barrier is determined by the solution constant $(\mu^+ - \mu^-)$. Note that there is still an arbitrariness in the form of an edge-slope constant on one side of the barrier.

Finally, we point out that the difference between membrane stiffness or edge slope on both sides of the barrier, that is, $(\mu^+ - \mu^-)$, should be prescribed for the unique solution in the presence of a surface-piercing partial barrier. It essentially means that there are two arbitrary physical constants involved in the problem. However, in the present case of the barrier with a gap, there is only one arbitrary physical constant that needs to be prescribed for the unique solution.

3.1. Limiting cases The following limiting cases are considered here and they are fully in agreement with the known results.

(i) $S \to 0$ and a, b are fixed with $0 < a < b$: it may be noted that membrane-coupled gravity wave motion exists purely for nonzero tension, that is, when $S \neq 0$. In other words, $S \to 0$ is a singular limit of the boundary value problem, since the coefficient of the highest derivative in the surface boundary condition becomes zero. However, in the absence of tension $S = 0$, edge conditions and some of the derived conditions in the solution method are no longer valid. Hence, the required system of linear equations that are to be solved in this case are none other than the ones given by Chakrabarti et al. [3].

(ii) $a \to 0^+$ and $b\,(> 0)$ is fixed: this is mathematically equivalent to the scattering of membrane-coupled gravity waves by a submerged vertical barrier. In this case, some of the linear equations become redundant. Hence, the required system of equations that are to be solved for the scattering quantities become the ones associated with the submerged vertical barrier case of Manam [11].

(iii) $b \to \infty$ and $a > 0$ is fixed: this is the case of the surface-piercing barrier problem for membrane-coupled gravity waves. For similar reasons cited in the previous case, the reduced conditions are associated with the piercing barrier case of Manam [11].

In the presence of partial barriers, numerical results for membrane-coupled gravity waves have been obtained for very small values of S, say of the order 10^{-7}, although these are physically unrealistic. However, results could converge exactly to those for free surface gravity waves only in the case of a submerged barrier, while they could show only signs of convergence in the case of a piercing barrier due to the singular limit. One may expect such difficulties in the case of a vertical barrier with a gap as well. In fact, coefficients in the linear system comprise singular as well as highly oscillatory improper integrals when the value of the surface or membrane tension is unrealistically small. Therefore, we consider only physically realistic values for tension in the numerical computation that is carried out in the next section.

4. Numerical results

We solve the linear system numerically for the unknowns using *MATHEMATICA*. A nondimensional quantity for the membrane tension is introduced by the parameter $\beta = MK^2$. The reflection coefficient $|R|$ is plotted against the wave parameter Kh for different values of the nondimensional structure parameters β and δ, where δ is the ratio of gap width to its mean depth h. The parameter δ must satisfy $a = h(1 - (\delta/2))$ and $b = h(1 + (\delta/2))$, so that $0 < \delta < 2$ [16]. The values a and b are fixed for specific values of δ and h, respectively. Consequently, a change in the value of δ makes a uniform variation in the gap width around h.

Figure 2 shows the reflection coefficient $|R|$ against the nondimensional wavenumber Kh for different values of the membrane tension β when $\delta = 1.3, 1.5, 1.7$. Computations are carried out beyond the value $\delta = 1.0$. This is justified for the reason that the gap width becomes narrower for smaller values of δ. Consequently, the barrier reflects most of the incident wave energy. The solid curve represents the reflection

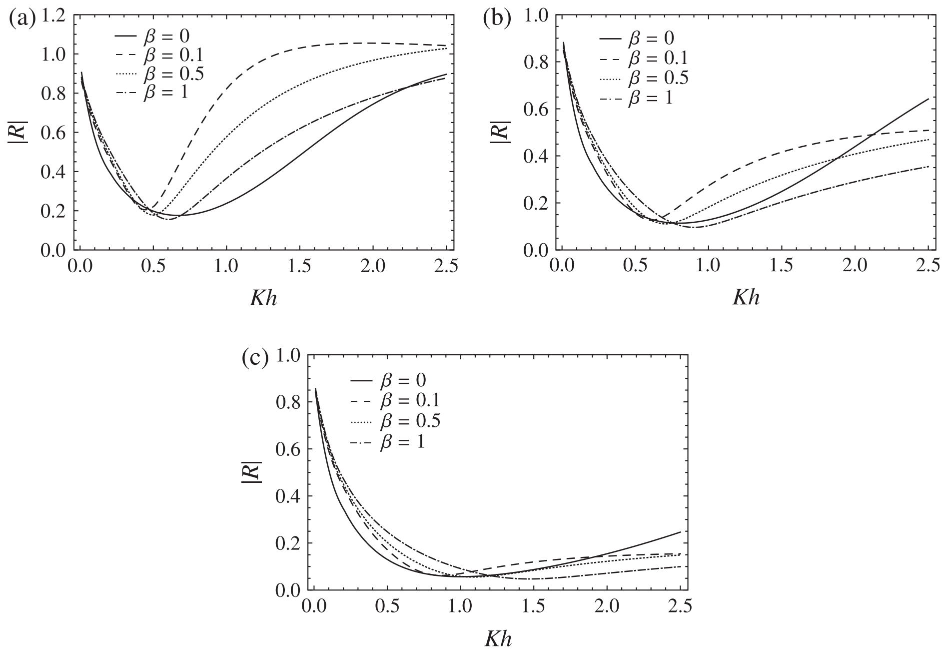

Figure 2. Reflection coefficient $|R|$ for different values of the membrane tension when (a) $\delta = 1.3$, (b) $\delta = 1.5$, (c) $\delta = 1.7$.

coefficient for free surface gravity waves, that is, when $\beta = 0$. It may be observed from Figure 2(a)–(c) that $|R|$ decreases with an increase in δ when the value of β is fixed. It demonstrates that wave transmission rises as the barrier gap becomes wider. The same observation was made by Porter [16] in the context of free surface gravity waves. Variation in the membrane tension has hardly any effect on the reflection of low-frequency membrane-coupled gravity waves. Reflection is observed to increase negligibly with an increase in the membrane tension. This trend continues to apply for waves up to intermediate frequencies at bigger gap widths. Beyond those frequencies, an opposite trend is seen with the rate of increase being significantly higher. They reveal, as one would expect, that the membrane with higher tension produces lower reflection for relatively shorter waves. Short membrane-coupled gravity waves are observed to reflect almost completely when the tension $\beta = 0.1$ and $\delta = 1.3$. It can be seen that the amount of reflection in shorter membrane-coupled gravity waves is smaller than the same in free surface gravity waves when the membrane is under high tension. This suggests that the tensional membrane spreads some amount of shorter wave energy uniformly below the mean surface. Thus, highly tensional membrane-coupled gravity waves with short wavelength transmit through a wider gap better than long waves.

In Figure 3(a) and (b), the reflection coefficient $|R|$ is depicted against the wavenumber Kh for free surface and capillary–gravity waves, respectively, for

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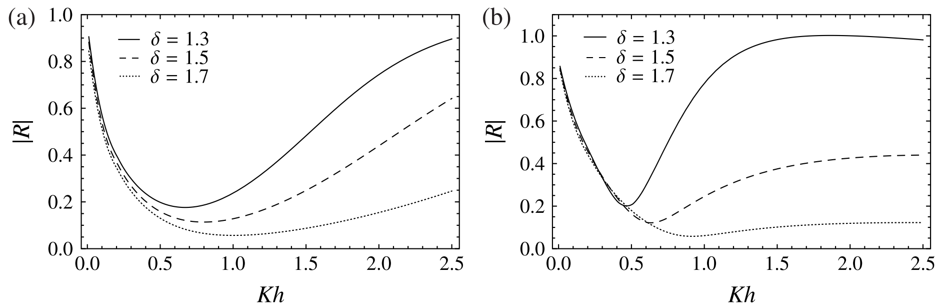

FIGURE 3. Reflection coefficient $|R|$ for different values of the parameter δ in the case of free surface with surface tension (a) $\beta = 0$, (b) $\beta = 0.074$.

different values of the gap width parameter δ. It is well known that $\beta = 0.074$ is a typical value for water at room temperature, and it is chosen in the computation of the reflection coefficient for capillary–gravity waves. As expected, a narrow gap produces higher overall reflection in both types of waves. Spreading of energy concentration for long capillary–gravity waves is evident, as reflection is found to be less for these waves than that for long free surface gravity waves. Also, it can be seen from Figure 3(b) that capillary–gravity waves with intermediate wavelength reflect better than free surface gravity waves of similar nature when the gap is relatively narrow. But, when the gap becomes sufficiently wider, lower reflection of very short capillary–gravity waves is seen, because the concentration of energy is more situated across the gap. This reveals that energy concentration for shorter capillary–gravity waves also undergoes some amount of wave energy spreading below the mean surface.

Finally, note that sharp rises or dips in the membrane-coupled gravity wave reflection in the presence of partial barriers are no longer seen in the case of the complete barrier with a gap. Perhaps this is due to the complementary nature of the partial vertical barriers that are involved in the present problem.

5. Conclusions

A mixed boundary value problem for the Laplace equation with a higher-order boundary condition has been tackled for an analytical solution. This is associated with capillary as well as membrane-coupled gravity wave scattering in deep water against a vertical barrier with a gap. The solution method involves the utilization of an orthogonal mode-coupling relation and the bounded solution of a weakly singular integral equation. We show limiting cases to produce known results in the problems of scattering against vertical barriers. The degree of arbitrariness in the membrane edge stiffness in the present problem is found to be different from the extent of that arbitrariness for the case of a surface-piercing vertical barrier problem.

The reflection coefficient for capillary and membrane-coupled gravity waves is plotted against a nondimensional wavenumber for different values of nondimensional parameters involved. Results are justified while comparing with free surface gravity waves. We find that for moderate gap widths, membrane tension plays a significant role in the reflection. In addition, membrane-coupled gravity waves with intermediate wavelengths reflect better than free surface gravity waves of similar nature. Further, uniform spreading of the long membrane coupled gravity wave energy below the mean surface is greater than that of the free surface gravity wave energy to feel the barrier gap. Moreover, there is also evidence that the membrane spreads some amount of shorter wave energy below the mean surface so as to get better transmission.

Some amount of capillary gravity wave energy spreading is found, although it appears insignificant. This is evident from the observation that capillary gravity waves with intermediate to short wavelengths undergo higher reflection than free surface gravity waves of similar nature, until the barrier gap becomes sufficiently wider. Besides, shorter waves feel the barrier gap so much that they achieve higher transmission.

Further extension of the solution method is possible for a mixed boundary value problem associated with the scattering of capillary or membrane-coupled gravity waves by a complete vertical barrier with a finite number of gaps in it. Also, the general method of solution can be applied to similar kinds of problems in other areas of wave structure interactions.

Appendix A. Evaluation of integrals

The following is a list of integrals that can be evaluated by standard contour integration techniques:

(i) $$\int_a^b \frac{t}{\sqrt{(t^2-a^2)(b^2-t^2)}(u^2-t^2)}\,dt = \begin{cases} \dfrac{-\pi}{2\sqrt{(a^2-u^2)(b^2-u^2)}} & \text{for } u \in (0,a), \\ \dfrac{\pi}{2\sqrt{(a^2-u^2)(b^2-u^2)}} & \text{for } u \in (b,\infty). \end{cases}$$

(ii) $$\int_a^b \frac{t}{u^2-t^2}\sqrt{\frac{t^2-a^2}{b^2-t^2}}\,dt = \frac{\pi}{2}\left(\sqrt{\frac{a^2-u^2}{b^2-u^2}}-1\right), \quad u \in (0,a)\cup(b,\infty).$$

(iii) $$\int_a^b \frac{t\sqrt{(t^2-a^2)(b^2-t^2)}}{(t^2-u^2)}\,dt = -\frac{\pi}{2}\left[u^2-\frac{1}{2}(a^2+b^2)\right], \quad a<u<b.$$

(iv) $$\int_a^b \frac{t^3\sqrt{(t^2-a^2)(b^2-t^2)}}{(t^2-u^2)}\,dt = -\frac{\pi}{2}\left[u^4-\frac{1}{2}(a^2+b^2)u^2-\frac{1}{8}(a^2-b^2)^2\right], \quad a<u<b,$$

(v) $$\int_a^b \frac{t^3}{\sqrt{(t^2-a^2)(b^2-t^2)}(u^2-t^2)}\,dt = \begin{cases} \dfrac{-\pi}{2}\left(1+\dfrac{u^2}{\sqrt{(a^2-u^2)(b^2-u^2)}}\right) \\ \quad\text{for } u \in (0,a), \\ \dfrac{-\pi}{2}\left(1-\dfrac{u^2}{\sqrt{(a^2-u^2)(b^2-u^2)}}\right) \\ \quad\text{for } u \in (b,\infty). \end{cases}$$

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(vi) $$\int_a^b \frac{t^5}{\sqrt{(t^2-a^2)(b^2-t^2)(u^2-t^2)}}\,dt = \begin{cases} \mathcal{K}_1(u) & \text{for } u \in (0,a), \\ \mathcal{K}_2(u) & \text{for } u \in (b,\infty), \end{cases}$$

where

$$\mathcal{K}_1(u) = \frac{-\pi}{2}\Big(u^2 + \frac{1}{2}(a^2+b^2) + \frac{u^4}{\sqrt{(a^2-u^2)(b^2-u^2)}}\Big),$$
$$\mathcal{K}_2(u) = \frac{-\pi}{2}\Big(u^2 + \frac{1}{2}(a^2+b^2) - \frac{u^4}{\sqrt{(a^2-u^2)(b^2-u^2)}}\Big).$$

Appendix B. Coefficients of linear equations

The coefficients in the linear equations (3.19), (3.20), (3.23) and (3.24) are represented as the 7th, 8th, 9th and 10th rows of the augmented matrix $[\mathbf{U} : \mathbf{v}^T]$, respectively, and are listed here in a computable form:

$$a_{70} = -\frac{(a\pi - J_1)}{\lambda J_2}\int_0^a \frac{e^{\lambda a} - e^{\lambda t}}{s_2(t)}\,dt + \frac{1}{\lambda}\int_0^a s_4(t)(e^{\lambda a} - e^{\lambda t})\,dt - \int_0^a s_4(t)te^{\lambda t}\,dt,$$

$$a_{71} = -\frac{(a\pi - J_1)}{\lambda_1 J_2}\int_0^a \frac{e^{\lambda_1 a} - e^{\lambda_1 t}}{s_2(t)}\,dt + \frac{1}{\lambda_1}\int_0^a s_4(t)(e^{\lambda_1 a} - e^{\lambda_1 t})\,dt - \int_0^a s_4(t)te^{\lambda_1 t}\,dt,$$

$$a_{72} = \bar{a}_{71},$$

$$a_{73} = -\frac{(a\pi - J_1)}{\lambda_1 J_2}\int_b^\infty \frac{e^{\lambda_1 t} - e^{\lambda_1 b}}{s_2(t)}\,dt - \frac{1}{\lambda_1}\int_b^\infty (1+\lambda_1 t)s_4(t)e^{\lambda_1 t}\,dt + \Big(\frac{J_3}{\pi} - b\Big)\frac{e^{\lambda_1 b}}{\lambda_1},$$

$$a_{74} = \bar{a}_{73}, \quad a_{75} = \frac{(a\pi - J_1)}{J_2}\int_a^b \frac{e^{\lambda t}}{s_1(t)}\,dt + \int_a^b (1+\lambda t)s_3(t)e^{\lambda t}\,dt,$$

$$a_{76} = \frac{(a\pi - J_1)}{J_2}\int_a^b \frac{e^{\lambda_1 t}}{s_1(t)}\,dt + \int_a^b (1+\lambda_1 t)s_3(t)e^{\lambda_1 t}\,dt, \quad a_{77} = \bar{a}_{76},$$

$$\begin{aligned} a_{78} = {} & \frac{(a\pi - J_1)}{J_2 Q(\lambda)}\int_a^b \frac{e^{-\lambda t}}{s_1(t)}\,dt + \frac{i(a\pi - J_1)}{J_2 Q(\lambda)}\int_0^a \frac{e^{-\lambda t} - e^{-\lambda a}}{s_2(t)}\,dt \\ & + \frac{i(a\pi - J_1)}{J_2 Q(\lambda)}\int_b^\infty \frac{e^{-\lambda b} - e^{-\lambda t}}{s_2(t)}\,dt + \frac{1}{Q(\lambda)}\int_a^b (1-\lambda t)s_3(t)e^{-\lambda t}\,dt \\ & - \frac{i}{Q(\lambda)}\int_0^a s_4(t)(e^{-\lambda t} - e^{-\lambda a})\,dt - \frac{i}{Q(\lambda)}\int_b^\infty (1-\lambda t)s_4(t)e^{-\lambda t}\,dt \\ & + \frac{i\lambda}{Q(\lambda)}\int_0^a s_4(t)te^{-\lambda t}\,dt + \frac{i}{Q(\lambda)}\Big(\frac{J_3}{\pi} - b\Big)e^{-\lambda b}, \end{aligned}$$

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$$a_{79} = -\frac{iM(a\pi - J_1)}{J_2 Q(\lambda)(1+3M\lambda^2)} \int_a^b \frac{e^{-\lambda t}}{s_1(t)}\,dt + \frac{(a\pi - J_1)}{J_2}\frac{M}{\pi}\int_a^b \int_0^\infty \frac{\sin \xi t}{s_1(t)\Delta(\xi)}\,d\xi\,dt$$
$$- \frac{iM}{Q(\lambda)(1+3M\lambda^2)} \int_a^b (1-\lambda t)s_3(t)e^{-\lambda t}\,dt + \frac{M}{\pi}\int_a^b \int_0^\infty \frac{\sin \xi t}{\Delta(\xi)} s_3(t)\,d\xi\,dt$$
$$+ \frac{M}{\pi}\int_a^b \int_0^\infty \frac{t\xi \cos \xi t}{\Delta(\xi)} s_3(t)\,d\xi\,dt,$$

$$v_7 = \frac{i(a\pi - J_1)}{J_2 Q(\lambda)} \int_0^a \frac{e^{-\lambda t} - e^{-\lambda a}}{s_2(t)}\,dt + \frac{i(a\pi - J_1)}{J_2 Q(\lambda)} \int_b^\infty \frac{e^{-\lambda b} - e^{-\lambda t}}{s_2(t)}\,dt$$
$$- \frac{i}{Q(\lambda)} \int_0^a s_4(t)(e^{-\lambda t} - e^{-\lambda a})\,dt - \frac{i}{Q(\lambda)} \int_b^\infty (1-\lambda t)s_4(t)e^{-\lambda t}\,dt$$
$$+ \frac{i\lambda}{Q(\lambda)} \int_0^a s_4(t)te^{-\lambda t}\,dt + \frac{i}{Q(\lambda)}\Big(\frac{J_3}{\pi} - b\Big)e^{-\lambda b},$$

$$a_{80} = -\frac{\lambda(a\pi - J_1)}{J_2} \int_0^a \frac{e^{\lambda a} - e^{\lambda t}}{s_2(t)}\,dt + \lambda \int_0^a s_4(t)(e^{\lambda a} - e^{\lambda t})\,dt - \lambda^2 \int_0^a s_4(t)te^{\lambda t}\,dt,$$

$$a_{81} = -\frac{\lambda_1(a\pi - J_1)}{J_2} \int_0^a \frac{e^{\lambda_1 a} - e^{\lambda_1 t}}{s_2(t)}\,dt + \lambda_1 \int_0^a s_4(t)(e^{\lambda_1 a} - e^{\lambda_1 t})\,dt$$
$$-\lambda_1^2 \int_0^a s_4(t)te^{\lambda_1 t}\,dt,$$

$$a_{82} = \bar{a}_{81},$$

$$a_{83} = -\frac{\lambda_1(a\pi - J_1)}{J_2} \int_b^\infty \frac{e^{\lambda_1 t} - e^{\lambda_1 b}}{s_2(t)}\,dt - \lambda_1 \int_b^\infty (1+\lambda_1 t)s_4(t)e^{\lambda_1 t}\,dt + \Big(\frac{J_3}{\pi} - b\Big)\lambda_1 e^{\lambda_1 b},$$

$$a_{84} = \bar{a}_{83}, \quad a_{85} = \frac{\lambda^2(a\pi - J_1)}{J_2} \int_a^b \frac{e^{\lambda t}}{s_1(t)}\,dt + \lambda^2 \int_a^b (1+\lambda t)s_3(t)e^{\lambda t}\,dt,$$

$$a_{86} = \frac{\lambda_1^2(a\pi - J_1)}{J_2} \int_a^b \frac{e^{\lambda_1 t}}{s_1(t)}\,dt + \lambda_1^2 \int_a^b (1+\lambda_1 t)s_3(t)e^{\lambda_1 t}\,dt, \quad a_{87} = \bar{a}_{86},$$

$$a_{88} = \frac{\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)} \int_a^b \frac{e^{-\lambda t}}{s_1(t)}\,dt + \frac{i\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)} \int_0^a \frac{e^{-\lambda t} - e^{-\lambda a}}{s_2(t)}\,dt$$
$$+ \frac{i\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)} \int_b^\infty \frac{e^{-\lambda b} - e^{-\lambda t}}{s_2(t)}\,dt + \frac{\lambda^2}{Q(\lambda)} \int_a^b (1-\lambda t)s_3(t)e^{-\lambda t}\,dt$$
$$- \frac{i\lambda^2}{Q(\lambda)} \int_0^a s_4(t)(e^{-\lambda t} - e^{-\lambda a})\,dt - \frac{i\lambda^2}{Q(\lambda)} \int_b^\infty (1-\lambda t)s_4(t)e^{-\lambda t}\,dt$$
$$+ \frac{i\lambda^3}{Q(\lambda)} \int_0^a s_4(t)te^{-\lambda t}\,dt + \frac{i\lambda^2}{Q(\lambda)}\Big(\frac{J_3}{\pi} - b\Big)e^{-\lambda b},$$

$$
\begin{aligned}
a_{89} = &-\frac{iM\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)(1+3M\lambda^2)}\int_a^b \frac{e^{-\lambda t}}{s_1(t)}\,dt - \frac{(a\pi - J_1)}{J_2}\frac{M}{\pi}\int_a^b\int_0^\infty \frac{\xi^2\sin\xi t}{s_1(t)\Delta(\xi)}\,d\xi\,dt\\
&-\frac{iM\lambda^2}{Q(\lambda)(1+3M\lambda^2)}\int_a^b (1-\lambda t)s_3(t)e^{-\lambda t}\,dt - \frac{M}{\pi}\int_a^b\int_0^\infty \frac{\xi^2\sin\xi t}{\Delta(\xi)}s_3(t)\,d\xi\,dt\\
&-\frac{M}{\pi}\int_a^b\int_0^\infty \frac{t\xi^3\cos\xi t}{\Delta(\xi)}s_3(t)\,d\xi\,dt,
\end{aligned}
$$

$$
\begin{aligned}
v_8 = &\frac{i\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)}\int_0^a \frac{e^{-\lambda t} - e^{-\lambda a}}{s_2(t)}\,dt + \frac{i\lambda^2(a\pi - J_1)}{J_2 Q(\lambda)}\int_b^\infty \frac{e^{-\lambda b} - e^{-\lambda t}}{s_2(t)}\,dt\\
&-\frac{i\lambda^2}{Q(\lambda)}\int_0^a s_4(t)(e^{-\lambda t} - e^{-\lambda a})\,dt - \frac{i\lambda^2}{Q(\lambda)}\int_b^\infty s_4(t)e^{-\lambda t}\,dt\\
&+\frac{i\lambda^3}{Q(\lambda)}\int_0^a s_4(t)te^{-\lambda t}\,dt + \frac{i\lambda^2}{Q(\lambda)}\Big(\frac{J_3}{\pi} - b\Big)e^{-\lambda b} + \frac{i\lambda^3}{Q(\lambda)}\int_b^\infty s_4(t)te^{-\lambda t}\,dt,
\end{aligned}
$$

$$
a_{90} = \lambda^3\int_0^a \frac{t^3 e^{\lambda t}}{s_2(t)}\,dt,\quad a_{91} = \lambda_1^3\int_0^a \frac{t^3 e^{\lambda_1 t}}{s_2(t)}\,dt,\quad a_{92} = \bar{a}_{91},\quad a_{93} = -\lambda_1^2\int_b^\infty \frac{t^3 e^{\lambda_1 t}}{s_2(t)}\,dt,
$$

$$
a_{94} = \bar{a}_{93},\quad a_{95} = \lambda^3\int_a^b \frac{t^3 e^{\lambda t}}{s_1(t)}\,dt,\quad a_{96} = \lambda_1^3\int_a^b \frac{t^3 e^{\lambda_1 t}}{s_1(t)}\,dt,\quad a_{97} = \bar{a}_{96},
$$

$$
a_{98} = -\frac{i\lambda^3}{Q(\lambda)}\int_0^a \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda^3}{Q(\lambda)}\int_b^\infty \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt - \frac{\lambda^3}{Q(\lambda)}\int_a^b \frac{t^3 e^{-\lambda t}}{s_1(t)}\,dt,
$$

$$
a_{99} = \frac{iM\lambda^3}{Q(\lambda)(1+3M\lambda^2)}\int_a^b \frac{t^3 e^{-\lambda t}}{s_1(t)}\,dt - \frac{M}{\pi}\int_a^b\int_0^\infty \frac{t^3\xi^3\cos\xi t}{\Delta(\xi)s_1(t)}\,d\xi\,dt,
$$

$$
v_9 = -\frac{i\lambda^3}{Q(\lambda)}\int_0^a \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda^3}{Q(\lambda)}\int_b^\infty \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt,
$$

$$
a_{100} = \Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_0^a \frac{t^3 e^{\lambda t}}{s_2(t)}\,dt - \lambda^2\int_0^a \frac{t^5 e^{\lambda t}}{s_2(t)}\,dt,
$$

$$
a_{101} = \Big(6 + \frac{\lambda_1^2(3a^2+b^2)}{2}\Big)\int_0^a \frac{t^3 e^{\lambda_1 t}}{s_2(t)}\,dt - \lambda_1^2\int_0^a \frac{t^5 e^{\lambda_1 t}}{s_2(t)}\,dt,\quad a_{102} = \bar{a}_{101},
$$

$$
a_{103} = -\Big(6 + \frac{\lambda_1^2(3a^2+b^2)}{2}\Big)\int_b^\infty \frac{t^3 e^{\lambda_1 t}}{s_2(t)}\,dt + \lambda_1^2\int_b^\infty \frac{t^5 e^{\lambda_1 t}}{s_2(t)}\,dt,\quad a_{104} = \bar{a}_{103},
$$

$$
a_{105} = \lambda\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_a^b \frac{t^3 e^{\lambda t}}{s_1(t)}\,dt - \lambda^3\int_a^b \frac{t^5 e^{\lambda t}}{s_1(t)}\,dt,
$$

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$$a_{106} = \lambda_1\Big(6 + \frac{\lambda_1^2(3a^2+b^2)}{2}\Big)\int_a^b \frac{t^3 e^{\lambda_1 t}}{s_1(t)}\,dt - \lambda_1^3\int_a^b \frac{t^5 e^{\lambda_1 t}}{s_1(t)}\,dt, \quad a_{107} = \bar{a}_{106},$$

$$\begin{aligned} a_{108} = &-\frac{i\lambda}{Q(\lambda)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_0^a \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda^3}{Q(\lambda)}\int_0^a \frac{t^5 e^{-\lambda t}}{s_2(t)}\,dt \\ &- \frac{i\lambda^3}{Q(\lambda)}\int_b^\infty \frac{t^5 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda}{Q(\lambda)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_b^\infty \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt \\ &- \frac{\lambda}{Q(\lambda)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_a^b \frac{t^3 e^{-\lambda t}}{s_1(t)}\,dt + \frac{\lambda^3}{Q(\lambda)}\int_a^b \frac{t^5 e^{-\lambda t}}{s_1(t)}\,dt, \end{aligned}$$

$$\begin{aligned} a_{109} = &\frac{iM\lambda}{Q(\lambda)(1+3M\lambda^2)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_a^b \frac{t^3 e^{-\lambda t}}{s_1(t)}\,dt \\ &+ \frac{6M}{\pi}\int_a^b\int_0^\infty \frac{t^3\xi\cos\xi t}{\Delta(\xi)s_1(t)}\,d\xi\,dt - \frac{M(3a^2+b^2)}{2\pi}\int_a^b\int_0^\infty \frac{t^3\xi^3\cos\xi t}{\Delta(\xi)s_1(t)}\,d\xi\,dt \\ &- \frac{iM\lambda^3}{Q(\lambda)(1+3M\lambda^2)}\int_a^b \frac{t^5 e^{-\lambda t}}{s_1(t)}\,dt + \frac{M}{\pi}\int_a^b\int_0^\infty \frac{t^5\xi^3\cos\xi t}{\Delta(\xi)s_1(t)}\,d\xi\,dt, \end{aligned}$$

$$\begin{aligned} v_{10} = &-\frac{i\lambda}{Q(\lambda)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_0^a \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda^3}{Q(\lambda)}\int_0^a \frac{t^5 e^{-\lambda t}}{s_2(t)}\,dt \\ &- \frac{i\lambda^3}{Q(\lambda)}\int_b^\infty \frac{t^5 e^{-\lambda t}}{s_2(t)}\,dt + \frac{i\lambda}{Q(\lambda)}\Big(6 + \frac{\lambda^2(3a^2+b^2)}{2}\Big)\int_b^\infty \frac{t^3 e^{-\lambda t}}{s_2(t)}\,dt, \end{aligned}$$

where

$$s_1(t) = \sqrt{(t^2-a^2)(b^2-t^2)}, \quad s_2(t) = \sqrt{(a^2-t^2)(b^2-t^2)},$$
$$s_3(t) = \sqrt{(t^2-a^2)/(b^2-t^2)} \quad \text{and} \quad s_4(t) = \sqrt{(a^2-t^2)/(b^2-t^2)}.$$

Appendix C. Reducing computational complexity of a double integral

The double integrals appearing in Appendix B are still numerically difficult to handle. So, there is a need to reduce their computational complexity. Some of the standard contour integration techniques can be applied to make the numerical evaluation of these double integrals a little easier. The procedure is shown here for one typical double integral, and the same is applied for the other 11 such double integrals. The integral

$$\mathcal{I} = \int_a^b\int_0^\infty \frac{tx\cos xt}{s_1(t)[x^2(1-Mx^2)^2+K^2]}\,dx\,dt, \quad M, K > 0$$

is considered for its evaluation. The inner integral

$$\int_0^\infty \frac{x\cos xt}{[x^2(1-Mx^2)^2+K^2]}\,dx, \quad a \le t \le b$$

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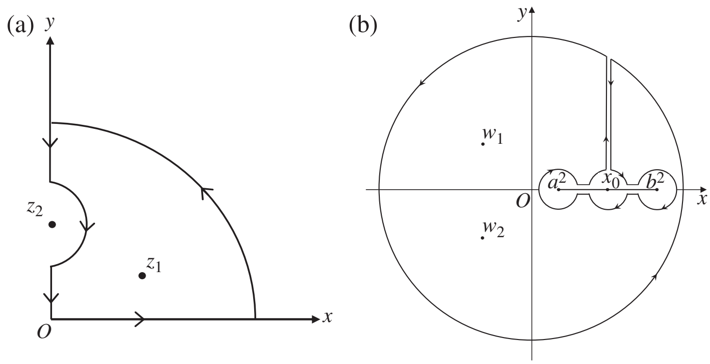

FIGURE 4. Contours (a) in the quarter plane and (b) around branch cut.

can be simplified by considering the function

$$f(z) = \frac{ze^{izt}}{z^2(1 - Mz^2)^2 + K^2}$$

over the contour shown in Figure 4(a). The function $f(z)$ has four complex zeros distributed in each of the quadrants and two purely imaginary zeros one each in the upper and lower half planes. The only root that contributes inside the closed contour is denoted as z_1, as seen in Figure 4(a). It may be obtained that

$$\int_0^\infty \frac{x\cos xt}{x^2(1 - Mx^2)^2 + K^2}\,dx = \mathrm{Re}\Big(\frac{i2\pi z_1 e^{iz_1 t}}{6M^2 z_1^5 - 8Mz_1^3 + 2z_1}\Big) - \int_0^\infty \frac{ye^{-yt}}{K^2 - y^2(1 + My^2)^2}\,dy.$$

Hence, the double integral becomes

$$\mathcal{I} = \int_a^b \frac{t}{s_1(t)}\,\mathrm{Re}\Big(\frac{i2\pi z_1 e^{iz_1 t}}{6M^2 z_1^5 - 8Mz_1^3 + 2z_1}\Big)\,dt - \int_a^b \int_0^\infty \frac{tye^{-yt}}{[K^2 - y^2(1 + My^2)^2]s_1(t)}\,dy\,dt.$$

The second term in the above is not so suitable for numerical computation. It may be noted that the transformation $yt = s$ and $t^2 = u$ makes it equivalent to

$$\int_{a^2}^{b^2} \int_0^\infty \frac{u^2 se^{-s}}{2[u^3K^2 - s^2(u + Ms^2)^2)]s_1(\sqrt{u})}\,ds\,du.$$

This can be further simplified by evaluating the following integral, by the aid of the contour shown in Figure 4(b), as given by

$$\int_{a^2}^{b^2} \frac{t^2}{[t^3K^2 - s^2(t + Ms^2)^2)]s_1(\sqrt{t})}\,dt$$
$$= \frac{\pi(w_1 - w_2)x_0^2 s_5(x_0) + (w_2 - x_0)w_1^2 s_5(w_1) + (x_0 - w_1)w_2^2 s_5(w_2)}{K^2(w_1 - w_2)(w_2 - x_0)(x_0 - w_1)},$$

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where

$$s_5(t) = \frac{\exp[-i\{\mathrm{Arg}(ts^2 - a^2) + \mathrm{Arg}(ts^2 - b^2)\}/2]}{\sqrt{|(ts^2 - a^2)(ts^2 - b^2)|}};$$

$x_0 > 0$ is a pole of the integrand that lies either inside or outside the branch cut, and w_1, w_2 are the poles inside the closed contour.

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