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GOLDIE DIMENSION, DUAL KRULL DIMENSION AND SUBDIRECT IRREDUCIBILITY

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Abstract. In this survey paper we present some results relating the Goldie dimension, dual Krull dimension and subdirect irreducibility in modules, torsion theories, Grothendieck categories and lattices. Our interest in studying this topic is rooted in a nice module theoretical result of Carl Faith [*Commun. Algebra* **27** (1999), 1807–1810], characterizing Noetherian modules M by means of the finiteness of the Goldie dimension of all its quotient modules and the ACC on its subdirectly irreducible submodules. Thus, we extend his result in a dual Krull dimension setting and consider its dualization, not only in modules, but also in upper continuous modular lattices, with applications to torsion theories and Grothendieck categories.

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1. Introduction. A lovely 10 years old result of Carl Faith [**21**, **22**] states:

Faith's Theorem (FT). *A module is Noetherian if and only if it is* QFD *and satisfies the* ACC *on subdirectly irreducible submodules.*

Recall that a module M_R is called *quotient finite dimensional* (or *QFD*) [**17**], if any quotient module of M has finite Goldie (or uniform) dimension. If we denote for a module M by $\mathcal{L}(M)$ the lattice of all its submodules and by $\mathcal{S}(M)$ the subset of $\mathcal{L}(M)$ consisting of all subdirectly irreducible submodules of M, then the **FT** can be stated as follows:

$\mathcal{L}(M)$ *is a Noetherian poset* $\iff$ M *is* QFD *and* $\mathcal{S}(M)$ *is a Noetherian poset.*

Now observe that an arbitrary poset P is Noetherian if and only if it has dual Krull dimension $k^0(P) \leqslant 0$. Thus, the **FT** can be reformulated in a dual Krull dimension setting as follows:

$$\mathbf{FT}_0\colon\ k^0(\mathcal{L}(M)) \leqslant 0 \iff \mathcal{L}(M) \text{ is a QFD lattice and } k^0(\mathcal{S}(M)) \leqslant 0.$$

The following natural problems related to **FT** arise:

(1) Investigate whether the dual $\mathbf{FT}^0$ of the **FT** hold.
(2) Do the above reformulation $\mathbf{FT}_0$ of the **FT** hold for an arbitrary ordinal α instead of 0, i.e. is the following statement

$$\mathbf{FT}_\alpha\colon\ k^0(\mathcal{L}(M)) \leqslant \alpha \iff \mathcal{L}(M) \text{ is a QFD lattice and } k^0(\mathcal{S}(M)) \leqslant \alpha$$

true? A similar question for its dual $\mathbf{FT}^0_\alpha$.

Dedicated to Patrick F. Smith on his 65th birthday.

(3) Extend (2) from the lattice $\mathcal{L}(M)$ to an arbitrary upper continuous modular lattice L.
(4) Apply (3) to Grothendieck categories and to module categories equipped with hereditary torsion theories.

The aim of this survey paper is to present the answers, we know so far, to these four questions. We will also illustrate here a general strategy, which consists on putting a module-theoretical theorem, in our case the Faith's Theorem, in a latticial frame, in order to translate that theorem to module categories equipped with a hereditary torsion theory and to Grothendieck categories.

2. Subdirectly irreducible modules. The concept of *subdirectly irreducible* (SI) appears in various circumstances: universal algebras, rings, modules, lattices, posets, etc. Remember that a classical result of Birkhoff [**15**] states that any universal algebra is a subdirect product of SI algebras.

Loosely speaking, an object of a category with direct products is called *subdirectly irreducible* if it cannot be represented as a subdirect product of 'smaller' objects (i.e. proper epimorphic images). We shall illustrate below more precisely this concept for module categories.

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R the category of all unital right R-modules. The notation M_R will be used to designate a unital right R-module M. The lattice of all submodules of a module M_R will be denoted by $\mathcal{L}(M_R)$. We denote by $\mathbb{N}$ the set $\{0, 1, 2, \ldots\}$ of all natural numbers, by $\mathbb{Z}$ the ring of rational integers and by $\mathbb{R}$ the field of real numbers.

A module M_R is called *subdirectly irreducible* if any representation of M as a *subdirect product* of other modules is trivial, i.e. for every family $(M_i)_{i\in I}$ of right R-modules and for every monomorphism $\varepsilon : M \rightarrowtail \prod_{i\in I} M_i$ such that $\pi_j \circ \varepsilon$ is an epimorphism $\forall j \in I$, $\exists\, i \in I$ such that $\pi_i \circ \varepsilon$ is an isomorphism, where $\pi_j : \prod_{i\in I} M_i \twoheadrightarrow M_j$, $j \in I$, are the canonical projections. The concept of subdirectly irreducible module turns out to be the dual of that of cyclic module as we will see below.

Clearly, a module M_R is *cyclic* if and only if it satisfies the following condition:

$$\exists\, x_0 \in M,\ \forall\, N \in \text{Mod-}R,\ \forall f \in \text{Hom}_R(N, M) \quad \textit{with} \quad x_0 \in \text{Im}(f) \Longrightarrow f \textit{ is an epimorphism.}$$

Dually, a module M_R is said to be *cocyclic* if it satisfies the following condition:

$$\exists\, x_0 \in M,\ \forall\, N \in \text{Mod-}R,\ \forall g \in \text{Hom}_R(M, N) \quad \textit{with} \quad x_0 \notin \text{Ker}\,(g) \Longrightarrow g \textit{ is a monomorphism.}$$

To the best of our knowledge, the notion of cocyclic module appears for the first time in the literature in Fuchs [**23**, Section 3].

The next result (see e.g. [**42**, 14.8]) provides various characterizations of cocyclic modules, which will naturally lead below to the most general concept of a subdirectly irreducible poset (see Definition 4.1).

PROPOSITION 2.1. *The following statements are equivalent for a non-zero module* M_R.

(1) M *is cocyclic.*
(2) $\bigcap_{0\neq X\leqslant M} X \neq 0$.

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(3) *The poset* $\mathcal{L}(M) \setminus \{0\}$*, ordered by inclusion, has a least element.*
(4) M *has a simple essential socle.*
(5) M *is subdirectly irreducible.* □

For any module M_R we have obviously $M = \sum_{C \in \mathcal{C}(M)} C$, and dually, less obviously $0 = \bigcap_{X \in \mathcal{S}(M)} X$, where $\mathcal{C}(M) := \{ C \leqslant M \mid C \text{ is cyclic}\}$ and $\mathcal{S}(M) := \{X \leqslant M \mid M/X \text{ is cocyclic}\}$.

Note that the elements of $\mathcal{S}(M)$ are called *subdirectly irreducible submodules of* M in Faith [**21**, **22**]. So, $X \leqslant M$ is a subdirectly irreducible submodule of M if and only if the module M/X is subdirectly irreducible.

3. FT^0. In this section we present a statement **FT**^{0} dual to that of Faith's Theorem **FT**, which gives a characterization of Artinian modules M_R in terms of submodules of M which behave dually to the submodules in $\mathcal{S}(M)$. Note that Artinian modules are precisely those modules having Krull dimension $\leqslant 0$, hence it seems natural to ask also for similar characterizations of modules having Krull dimension at most a given ordinal $\alpha \geqslant 0$. This Krull dimension setting will be discussed in the subsequent sections.

We will denote by $\mathcal{F}(M)$ the set of all finitely generated submodules of a module M.

THEOREM 3.1 (THE DUAL FAITH THEOREM **FT**^{0}) ([**8**, Theorems 1.12 and 1.13]). *The following statements are equivalent for a module* M_R.
(1) M *is Artinian.*
(2) M *is* QFD *or* SFD^0*, and* C *is Artinian for any* $C \in \mathcal{C}(M)$.
(3) M *is* QFD *or* SFD^0*, and* $\mathcal{C}(M)$ *is an Artinian poset.*
(4) M *is* QFD *or* SFD^0*, and* F *is Artinian for any* $F \in \mathcal{F}(M)$.
(5) M *is* QFD *or* SFD^0*, and* $\mathcal{F}(M)$ *is an Artinian poset.* □

Recall that a module M_R is called *sub finite dual dimensional* (or *SFD*^{0}), if any submodule of M has finite dual uniform (or dual Goldie, or hollow) dimension. So, the term of a SFD^0 module is dual to that of a QFD module. The reader is referred to [**27**, **38**, **40**] for the concept of *dual uniform dimension* of modules and modular lattices.

As it is well known, any module with Krull (or dual Krull dimension) is QFD, but a module with Krull dimension is not necessarily SFD^0: the Abelian group $\mathbb{Z}$ is a Noetherian $\mathbb{Z}$-module, but it does not have dual Goldie dimension. However, if M_R satisfies the property AB5* (this means that the lattice $\mathcal{L}(M_R)$ of all submodules of M is *lower continuous*), in particular if M is *linearly compact*, then M is QFD if and only if M is SFD^0, by [**28**, Lemma 6]. Notice that any SFD^0 module is QFD, by [**41**, Proposition 12].

4. Latticial background. For a partially ordered set, shortly poset, $(P, \leqslant)$ and elements $a \leqslant b$ in P we write

$$b/a := [a, b] = \{x \in P \mid a \leqslant x \leqslant b\},$$
$$[a, b[:= \{x \in P \mid a \leqslant x < b\},$$
$$]a, b] := \{x \in P \mid a < x \leqslant b\}.$$

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All posets considered in this paper are assumed to have a least element denoted by 0 and a last element denoted by 1. If $x < y$ are elements of a poset P and there is no $z \in P$ such that $x < z < y$, then we say that x is *covered* by y, and we write $x \prec y$. An element $a \in P$ is said to be an *atom* of P if $0 \prec a$.

We denote by $\mathcal{L}$ (resp. $\mathcal{M}$, $\mathcal{C}$, $\mathcal{U}$) the class of all lattices with 0 and 1 (resp. modular lattices with 0 and 1, complete lattices, upper continuous lattices). Throughout this paper a lattice will always mean a member of $\mathcal{L}$, and $(L, \leqslant, \wedge, \vee, 0, 1)$, or more simply, just L, will always denote such a lattice. The opposite lattice of L will be denoted by L^0. If $L \in \mathcal{C}$, then for every subset S of L we denote $\bigwedge S = \bigwedge_{x \in S} x$ and $\bigvee S = \bigvee_{x \in S} x$. An element e of a lattice L is said to be *essential* in L if $e \wedge x \neq 0$ for each $0 \neq x \in L$. Dually, an element $s \in L$ is *small* in L if s is essential in L^0, i.e. if $s \vee x \neq 1$ for every $x \neq 1$ in L. If $L \in \mathcal{C}$, then the *socle* $\mathrm{Soc}(L)$ of L is the join of all atoms of L. A lattice L is said to be *semi-Artinian* if for any $1 \neq x \in L$, the lattice $1/x$ has at least an atom. An element c of a lattice $L \in \mathcal{C}$ is *compact in* L if whenever $c \leqslant \bigvee_{x \in A} x$ for a subset A of L, there is a finite subset F of A such that $c \leqslant \bigvee_{x \in F} x$. The lattice L is *compact* if 1 is a compact element in L, and *compactly generated* if every element of L is a join of compact elements.

For all undefined notation and terminology on lattices, the reader is referred to [**16**, **18**, **26**] and/or [**39**].

The next definition is inspired by Proposition 2.1.

DEFINITION 4.1. A poset P is said to be *subdirectly irreducible*, abbreviated SI, if $P \neq \{0\}$ and the set $P \setminus \{0\}$ has a least element; i.e. there exists an element $0 \neq x_0 \in P$ such that $x_0 \leqslant x$ for every $0 \neq x \in P$. An element $s \in P$ is said to be a *subdirectly irreducible element of* P if the interval $1/s$ is a subdirectly irreducible poset, and the set of all subdirectly irreducible elements of P will be denoted by $\mathcal{S}(P)$. □

Observe that a module M_R is subdirectly irreducible if and only if the lattice $\mathcal{L}(M_R)$ of all submodules of M_R is subdirectly irreducible, and the poset $\mathcal{S}(M_R)$ defined just after Proposition 2.1 is exactly $\mathcal{S}(\mathcal{L}(M_R))$.

DEFINITIONS 4.2. (a) A lattice L is said to be *co-irreducible* or *uniform* (resp. *completely co-irreducible* or *completely uniform*) if $L \neq \{0\}$ and $x \wedge y \neq 0$ for any non-zero elements $x, y \in L$ (resp. $\bigwedge_{i \in I} x_i \neq 0$ for any non-empty family $(x_i)_{i \in I}$ of non-zero elements $x_i \in L$).

(b) An element x of a lattice L is said to be *irreducible*, (resp. *completely irreducible*, abbreviated *CI*) if $x \neq 1$ and whenever $x = a \wedge b$ for $a, b \in L$, then $x = a$ or $x = b$ (resp. whenever $x = \bigwedge_{i \in I} a_i$ for a non-empty family $(a_i)_{i \in I}$ of elements of L, then $x = a_j$ for some $j \in I$). □

Clearly, an element $x \in L$ is irreducible (resp. completely irreducible) if and only if the lattice $1/x$ is co-irreducible (resp. completely co-irreducible). For any lattice L we denote by $\mathcal{I}(L)$ the set of all irreducible elements of L, and by $\mathcal{I}^c(L)$ the set of all completely irreducible elements of L. For any module M_R, we set $\mathcal{I}(M_R) := \mathcal{I}(\mathcal{L}(M_R))$ and $\mathcal{I}^c(M_R) := \mathcal{I}^c(\mathcal{L}(M_R))$.

If $L \in \mathcal{C}$, then clearly $s \in L$ is a *subdirectly irreducible element of* L if and only if s is *completely irreducible*, so $\mathcal{S}(L) = \mathcal{I}^c(L)$. In the sequel, for the term of subdirectly irreducible element of any lattice, we will occasionally use the more suggestive term of completely irreducible (CI) element.

The next result is a lattice extension of Proposition 2.1.

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PROPOSITION 4.3 ([**2**, Proposition 0.5]). *The following statements are equivalent for a lattice* $L \in \mathcal{C}$, $L \neq \{0\}$.

(1) L *is subdirectly irreducible.*
(2) $\bigwedge_{x \in L \setminus \{0\}} x \neq 0$.
(3) L *is completely co-irreducible.*
(4) L *has an atom* a *that is essential in* L.
(5) L *is co-irreducible and* $\mathrm{Soc}(L) \neq 0$. □

For a poset P we denote by $k(P)$ (resp. $k^0(P)$) the *Krull dimension* (resp. the *dual Krull dimension*) of P (see also [**10**, Section 3]). The notation $k(P) \leqslant \alpha$ means that P has Krull dimension, and this is $\leqslant$ than the ordinal α. A nice result due to Lemonnier [**31**, Corollaire 6] states that an arbitrary poset P has Krull dimension if and only if it has dual Krull dimension. Any poset having Krull dimension has also Gabriel dimension, but in general, not conversely.

For the definition and basic properties of the Krull dimension and dual Krull dimension (resp. Gabriel dimension) of a poset the reader is referred to [**31**] or [**33**] (resp. to [**1**] or [**35**]).

5. QFD lattices. Let $L \in \mathcal{L}$ be a lattice. Recall that a set S of non-zero elements of L is said to be *independent* if for every finite subset F of S and for each $s \in S \setminus F$, one has $s \wedge (\bigvee_{x \in F} x) = 0$.

DEFINITION 5.1. One says that a lattice $L \in \mathcal{L}$ has *finite Goldie* (or *uniform*) *dimension* if there is no infinite independent subset of L. The lattice L is said to have *finite dual Goldie dimension* if L^0 has finite Goldie dimension. The lattice L is called *QFD* (i.e. quotients have finite Goldie dimension) if $1/x$ has finite Goldie dimension for every $x \in L$. □

The reader is referred to the survey paper [**36**] in this Proceedings for more about Goldie dimension of modular lattices.

The next result, originally proved for modules by Lemonnier (see [**32**, Lemme 1.1]), is an important tool for studying the QFD property of upper continuous modular lattices.

LEMMA 5.2 (LEMONNIER'S LEMMA) ([**10**, Lemma 3.4]). *Let* $\mathbb{P}$ *be a property of the class* $\mathcal{M} \cap \mathcal{U}$ *of all upper continuous modular lattices, which satisfies the following condition:*

(∗) *If* L *is a lattice having* $\mathbb{P}$, *then there exist* $a < b$ *in* L *and* $c_1, c_2 \in b/a$ *with* $c_1 \vee c_2 = b$, $c_1 \wedge c_2 = a$, $c_1 \neq a$, *and the lattice* c_2/a *again having the property* $\mathbb{P}$.

Then, any $L \in \mathcal{M} \cap \mathcal{U}$ *having* $\mathbb{P}$ *is not* QFD. □

We are now going to characterize QFD compactly generated modular lattices, which generalizes a result on QFD modules due to Camillo [**17**].

DEFINITION 5.3. Let L be a lattice. We say that a lattice L verifies Condition (C) (or *Camillo's Condition*) if for every $m \in L$ there exists a compact element t of L such that $t \leqslant m$ and $[t, m[$ has no maximal element. □

The next result, whose proof essentially uses the Lemonnier's Lemma, is the latticial version of [**17**, Theorem].

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THEOREM 5.4 ([**4**, Theorem 2.8]). *A modular, compactly generated lattice L is* QFD *if and only if L verifies Condition (C).* □

For arbitrary upper continuous modular lattices, we have the following characterization of the QFD condition.

THEOREM 5.5 ([**4**, Theorem 4.1]). *A modular upper continuous lattice L is* QFD *if and only if for every directed set $D \subseteq L$ there exists $d_0 \in D$ such that, for every $d \in D$ with $d \in [d_0, \bigvee D[$, d is small in $(\bigvee D)/d_0$.* □

We say that a non-trivial poset P is *dense* if for any $a, b \in P$ with $a < b$ there exists $c \in P$ such that $a < c < b$.

THEOREM 5.6 ([**4**, Theorem 4.8]). *Let L be an upper continuous modular lattice and denote $\varphi(x) = \bigvee\{\ k \mid k \in x/0,\ k$ compact in $L\}$, $x \in L$. Assume that for every $a \in L$ with $a > \varphi(a)$ the interval $a/\varphi(a)$ is a dense poset. Then, L is* QFD *if and only if L verifies Condition (C) and does not contain a sublattice isomorphic to the lattice $[0, 1]^{\mathbb{N}}$ considered with the componentwise order, where $[0, 1] \subseteq \mathbb{R}$.* □

We end this section by presenting the following latticial extension of a nice characterization of QFD modules in terms of finite meet irreducible decompositions we recently learned from Patrick Smith in a private communication, Ankara, August 2008:

PROPOSITION 5.7. *A lattice $L \in \mathcal{M} \cap \mathcal{U}$ is* QFD *if and only every element $1 \neq x \in L$ can be written as a finite meet of irreducible elements of L.* □

6. $\mathbf{FT}_\alpha$ and $\mathbf{FT}^0_\alpha$. Taking into account that a poset P is Artinian if and only if $k(P) \leqslant 0$, a part of Theorem 3.1 can be reformulated as follows:

$$k(M_R) \leqslant 0 \iff M \text{ is QFD and } k(\mathcal{C}(M)) \leqslant 0 \iff M \text{ is QFD and } k(\mathcal{F}(M)) \leqslant 0.$$

Here, $k(\mathcal{C}(M))$ (resp. $k(\mathcal{F}(M))$) means the Krull dimension of the poset $\mathcal{C}(M)$ (resp. $\mathcal{F}(M)$) of all cyclic (resp. finitely generated) submodules of M, ordered by inclusion. This reformulation suggested the following extension for an arbitrary ordinal α:

THEOREM 6.1 (THE DUAL α-FAITH THEOREM $\mathbf{FT}^0_\alpha$) ([**8**, Theorem 1.17]). *For any M_R and any ordinal $\alpha \geqslant 0$*

$$k(M_R) \leqslant \alpha \iff M \text{ is QFD and } k(\mathcal{F}(M)) \leqslant 0.$$ □

Because the compact elements of the lattice $\mathcal{L}(M)$ of all submodules of a module M_R are exactly the finitely generated submodules of M, it is natural to ask whether a latticial extension of Theorem 6.1 is true; see Problem 8.2.

We are now going to discuss the validity of the *Latticial α-Faith Theorem* $\mathbf{FT}_\alpha$, that is,

$$\mathbf{FT}_\alpha\colon\ k^0(L) \leqslant \alpha \iff L \text{ is a QFD lattice and } k^0(\mathcal{S}(L)) \leqslant \alpha,$$

for an arbitrary lattice $L \in \mathcal{M} \cap \mathcal{U}$ and an arbitrary ordinal $\alpha \geqslant 0$, where $\mathcal{S}(L)$ is the set of all subdirectly elements of L, also called CI elements of L.

An essential tool in establishing our results is the following extension to posets of the dual of a result due to Goodearl and Zimmermann-Huisgen [**25**] concerning the relationship between the Krull dimension of a module and the length of reverse well-ordered chains of its submodules.

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THEOREM 6.2 ([**5**, Theorem 1.11]). *The following conditions are equivalent for a poset P.*

(a) *$k^0(P)$ exists and is countable.*

(b) *$\lambda(P)$ is countable.*

In case conditions (a) *and* (b) *are met and $k^0(P) = \alpha > 0$, then*

$$\omega^\beta < \lambda(P) \leqslant \omega^{\alpha+1} \quad \text{for all ordinals } \beta < \alpha.$$

In case α is finite, the lower bound for $\lambda(P)$ can be improved to $\omega^\alpha < \lambda(P)$. □

For a poset P we have denoted by $\lambda(P)$ the so called *codepth* of P; i.e. the least ordinal that does not embed in P. See also [**30**], where the term of *depth* of P, denoted by $\delta(P)$, has been defined as the least ordinal that does not embed in P^0. We have also denoted by ω the first transfinite ordinal, which is the order type of the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. For basic properties of the arithmetic of ordinal numbers, the reader is referred to [**37**].

Note that for any module M_R, the lattice $\mathcal{L}(M)$ of all submodules of M has the property that for each $N < P$ in $\mathcal{L}(M)$, the quotient module P/N has a subdirectly quotient module P/Q, so we may say that the lattice $\mathcal{L}(M)$ is 'rich in subdirectly irreducibles'. We take this property as definition for an arbitrary lattice or poset.

DEFINITION 6.3. A lattice L is said to be *rich in subdirectly irreducibles*, abbreviated *RSI*, if for every $a < b$ in L, the interval b/a has a subdirectly irreducible quotient interval $b/c \subseteq b/a$. □

The property of a lattice L being RSI is related to the property of L being a lattice with *completely irreducible decomposition*, which means that every $1 \neq a \in L$ can be written as a meet of a family, not necessarily finite, of CI elements of L (see [**2**, Remarks 0.15]). Other recent results on completely irreducible submodules and their connections with primal submodules, primary submodules and their meet decompositions may be found in [**12**], [**13**].

The next result characterizes RSI lattices in terms of Gabriel dimension, so providing large classes of such lattices.

PROPOSITION 6.4 ([**5**, Proposition 1.2]). *A lattice $L \in \mathcal{M} \cap \mathcal{U}$ is* RSI *if and only if for each $a < b$ in L there exist $x < y$ in b/a such that y/x has Gabriel dimension. So, if L has Gabriel dimension, then L is* RSI. *In particular, if L is Artinian, semi-Artinian, Noetherian, or has* (*dual*) *Krull dimension, then L is* RSI. □

Now, we are going to show that the Latticial $\mathbf{FT}_\alpha$ holds for any finite ordinal α and for any upper continuous modular lattice L which is RSI. In doing so, we have to characterize the existence and magnitude of $k^0(L)$ in terms of the existence and size of $k^0(\mathcal{S}(L))$. This task is achieved by the following very technical Lemma.

LEMMA 6.5 ([**5**, Lemmas 1.13 and 1.17]). *Let $L \in \mathcal{M} \cap \mathcal{U}$ be a* QFD *lattice which is* RSI. *Then, the following assertions hold.*

(1) *If L contains a chain of order type ω^α for some ordinal $\alpha \geqslant 1$, then $\mathcal{S}(L)$ contains a chain of order type ω^α.*

(2) *If $\mathcal{S}(L)$ has countable dual Krull dimension, then L has* (*dual*) *Krull dimension.* □

THEOREM 6.6 ([**5**, Theorem 1.15]). *Let $L \in \mathcal{M} \cap \mathcal{U}$, be such that $k^0(L) = \alpha$ is a countable ordinal.*

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(1) *If* $1 \leqslant \alpha < \omega$ *or if* α *is a limit ordinal, then* $k^0(\mathcal{S}(L)) = \alpha$.
(2) *If* $\alpha = \delta + 1$ *for some* $\delta \geqslant -1$, *then* $k^0(\mathcal{S}(L)) = \delta$ *or* $k^0(\mathcal{S}(L)) = \delta + 1$.

Proof. Clearly $k^0(\mathcal{S}(L))$ exists and $k^0(\mathcal{S}(L)) = \gamma \leqslant \alpha$ for some ordinal γ. The conclusions of the theorem are trivial for $\alpha = 0$; so we assume that $\alpha \geqslant 1$. By Theorem 6.2, we have $\lambda(\mathcal{S}(L)) \leqslant \omega^{\gamma+1} \leqslant \omega^{\alpha+1}$, $\omega^\beta < \lambda(L)$ for each $\beta < \alpha$, and $\omega^\alpha < \lambda(L)$ if α is finite. Thus L contains a chain of type ω^β for each $\beta < \alpha$ and a chain of type ω^α if α is finite. By Lemma 6.5 (1), $\mathcal{S}(L)$ also has chains of these types. Hence $\omega^\beta < \lambda(\mathcal{S}(L)) \leqslant \omega^{\gamma+1} \leqslant \omega^{\alpha+1}$ for each $\beta < \alpha$, and $\omega^\alpha < \lambda(\mathcal{S}(L))$ if α is finite. Now, (1) and (2) follow easily. □

THEOREM 6.7 ([**5**, Theeorem 1.18]). *The following conditions are equivalent for* $L \in \mathcal{M} \cap \mathcal{U}$.
(1) $k^0(L)$ *exists and is countable.*
(2) L *is both* QFD *and* RSI, *and* $k^0(\mathcal{S}(L))$ *exists and is countable.*
If conditions (1) *or* (2) *are met and* $-1 \leqslant k^0(\mathcal{S}(L)) = \alpha$, *then* $k^0(L) = \alpha$ *or* $k^0(L) = \alpha + 1$. *Moreover,* $k^0(L) = \alpha$ *if* $0 \leqslant \alpha < \omega$.

Proof. (1) $\Longrightarrow$ (2) is trivial, and (2) $\Longrightarrow$ (1) follows by combining Lemma 6.5 and Theorem 6.2. The final assertions are immediate from Theorem 6.6. □

COROLLARY 6.8 (THE LATTICIAL $\mathbf{FT}_n$). *For any* $L \in \mathcal{M} \cap \mathcal{U}$ *and* $n \in \mathbb{N}$,

$$k^0(L) \leqslant n \Longleftrightarrow L \text{ is both QFD and RSI, and } k^0(\mathcal{S}(L)) \leqslant n. \qquad \square$$

Since the lattice $\mathcal{L}(M_R)$ is always RSI, we obtain at once the $\mathbf{FT}_n$ for any module and any $n \in \mathbb{N}$:

COROLLARY 6.9 ($\mathbf{FT}_n$) ([**5**, Corollary 1.19]). *Let* M_R *be a module, and let* $n \in \mathbb{N}$. *Then*

$$k^0(M_R) \leqslant n \Longleftrightarrow M_R \text{ is QFD and } k^0(\mathcal{S}(M_R)) \leqslant n. \qquad \square$$

In particular, for $n = 0$, Corollary 6.9 gives precisely the Faith's Theorem **FT**.

The next result provides an evaluation of the dual Krull dimension $k^0(L)$ of a lattice L in terms of irreducible and completely irreducibles elements of L. Note that the proof of (1) $\Longleftrightarrow$ (2) in theorem below is based on two main ingredients: the Lemonnier's Lemma (see Lemma 5.2) and a corrected version of [**10**, Proposition 3.10] (see [**11**]) involving the subclass

$$\rangle\!\rangle \mathcal{K}^0_\alpha \langle\!\langle := \{\, X \in \mathcal{M} \mid \forall\, a < b \text{ in } X,\ \exists\, c \in [a, b[\,,\ k^0(c/a) \leqslant \alpha \,\}$$

of the class $\mathcal{M}$ of all modular lattices with 0 and 1.

THEOREM 6.10 ([**5**, Theorems 1.21 and 1.22]). *The following statements are equivalent for an arbitrary ordinal* $\alpha \geqslant 0$ *and a lattice* $L \in \mathcal{M} \cap \mathcal{U}$.
(1) $k^0(L) \leqslant \alpha$.
(2) L *is* QFD, L *is* RSI, *and* $k^0(1/x) \leqslant \alpha$ *for all* $x \in \mathcal{S}(L)$.
(3) L *is* QFD *and* $k^0(1/x) \leqslant \alpha$ *for all* $x \in \mathcal{I}(L)$. □

COROLLARY 6.11 ([**5**, Corollary 1.24]). *The following statements are equivalent for a module* M_R *and an arbitrary ordinal* $\alpha \geqslant 0$.

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(1) $k^0(M_R) \leqslant \alpha$.
(2) M_R *is* QFD *and* $k^0(M/N) \leqslant \alpha$ *for every* (*completely*) *irreducible submodule* N *of* M. □

Using Theorem 5.5 we obtain the following evaluation of $k^0(L)$ in terms of small elements of L.

THEOREM 6.12 ([**5**, Theorem 1.28]). *The following statements are equivalent for a lattice* $L \in \mathcal{M} \cap \mathcal{U}$ *and an arbitrary ordinal* $\alpha \geqslant 0$.
(1) $k^0(L) \leqslant \alpha$.
(2) *For every* $x \in L$, *there exists* $s \leqslant x$ *such that* s *is small in* $x/0$ *and* $k^0(x/s) \leqslant \alpha$. □

An immediate consequence of Theorem 6.12 is the following dual of a result due to Huynh, Dung and Smith [**29**, Lemma 6] (see also [**19**, Theorem 6.3]) which answers in the positive an Open Problem raised by Albu and Rizvi [**8**, p. 1923]:

COROLLARY 6.13 ([**5**, Corollary 1.29]). *The following statements are equivalent for a module* M_R *and an ordinal* $\alpha \geqslant 0$.
(1) $k^0(M) \leqslant \alpha$.
(2) *Every submodule* X *of* M *has a small submodule* S *with* $k^0(X/S) \leqslant \alpha$. □

7. Applications to torsion theories and Grothendieck categories. Throughout this section $\tau = (\mathcal{T}, \mathcal{F})$ will be a fixed hereditary torsion theory on Mod-R. The set $F_\tau := \{ I \leqslant R_R \mid R/I \in \mathcal{T} \}$ is called the *Gabriel topology* associated with τ. For any M_R we denote $\mathrm{Sat}_\tau(M) = \{ N \mid N \leqslant M,\ M/N \in \mathcal{F} \}$, and for any $N \leqslant M$ we denote by $\overline{N} = \bigcap \{ C \mid N \leqslant C \leqslant M,\ M/C \in \mathcal{F} \}$ the τ-*closure* of N in M. It is known that $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice for any M_R (see [**39**, Chapter 9, Proposition 4.1]).

As in [**4**, **5**], a module M_R is said to be τ-QFD if the lattice $\mathrm{Sat}_\tau(M)$ is QFD. More generally, if $\mathbb{P}$ is any property on lattices, we say that a module M_R is/has τ-$\mathbb{P}$ if the lattice $\mathrm{Sat}_\tau(M)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of a τ-*Artinian* module, τ-*Noetherian* module, τ-*RSI* module, etc. The τ-*Krull dimension* $k_\tau(M)$ (resp. τ-*dual Krull dimension* $k^0_\tau(M)$) of M is defined as the Krull dimension (resp. dual Krull dimension) of the lattice $\mathrm{Sat}_\tau(M)$.

For all undefined notation and terminology on torsion theories the reader is referred to [**7**, **24**] and/or [**39**].

An important problem in Module Theory appeared about 40 years ago is to *relativize* a certain property, that is,

Given a property $\mathbb{P}$ *in the lattice* $\mathcal{L}(M_R)$, *investigate the property* $\mathbb{P}$ *in the lattice* $\mathrm{Sat}_\tau(M_R)$;

in other words,

Having a theorem $\mathbb{T}$ *on modules, investigate its relativization* τ-$\mathbb{T}$.

The best illustration of this problem is the relativization of the renowned *Hopkins–Levitzki Theorem*, abbreviated H-LT:

H-LT: *Any right Artinian ring* R *with identity is right Noetherian.*

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The relativization of this theorem with respect to a hereditary torsion theory τ on Mod-R:

τ-H-LT: *Any right τ-Artinian ring R with identity is right τ-Noetherian*,

is also known as the *Teply–Miller Theorem*.

Let us mention that the module-theoretical proofs available in the literature of the Relative **H-LT** (**τ-H-LT**), namely the original one in 1979 due to Miller and Teply [**34**, Theorem 1.4], and another one in 1982 due to Faith [**20**, Theorem 7.1 and Corollary 7.2], are very long and complicated; so, the relativization of a result on modules is not always a simple job, and sometimes it may be even impossible. A very simple and natural approach to the **τ-H-LT** is to formulate and prove it in the most general latticial setting of an arbitrary modular lattice with 0 and 1, and then to apply it for the lattice $\mathrm{Sat}_\tau(R_R)$. This has been done by Albu and Smith [**9**, Theorem 1.9]. For a very thorough discussion on the various aspects of the **H-LT** and the connections between them, see the survey paper [**3**].

As we have already seen, our characterizations of upper continuous modular lattices L with $k^0(L) \leqslant \alpha$ require the lattice L to be RSI. This condition is automatically satisfied for the lattice of submodules of any module M. But, for an arbitrary hereditary torsion theory τ on Mod-R, the lattice $\mathrm{Sat}_\tau(M)$ may fail to be RSI. Therefore, we first look for sufficient conditions on τ to insure that, for any module M_R, the lattice $\mathrm{Sat}_\tau(M)$ is RSI, i.e. any module M_R is τ-RSI.

We denote by $\mathrm{Max}_\tau(R)$ the set of all maximal elements of the poset

$$(\mathrm{Sat}_\tau(R_R) \setminus \{R\},\ \subseteq).$$

Note that we may have $\mathrm{Max}_\tau(R) = \varnothing$ (see, e.g. [**6**, Remarques 2.5 (2)]). However, if the Gabriel topology F_τ has a basis of finitely generated right ideals, then, the poset $(\mathrm{Sat}_\tau(R_R) \setminus \{R\},\ \subseteq)$ is inductive, and so, a *τ-relative Krull Lemma* holds:

$$(\dagger) \qquad \forall\, I \in \mathrm{Sat}_\tau(R_R) \setminus \{R\},\ \exists\, J \in \mathrm{Max}_\tau(R) \text{ such that } I \subseteq J,$$

and, in particular, we have $\mathrm{Max}_\tau(R) \neq \varnothing$. Recall that by a *basis* of the Gabriel topology F_τ we mean a subset B of F_τ such that every right ideal in F_τ contains some $J \in B$. For such torsion theories τ satisfying the condition (†), any module M_R is τ-RSI by [**5**, Proposition 2.5]. On the contrary, if $\mathrm{Max}_\tau(R) = \varnothing$, then the module R_R is not τ-RSI.

The latticial results from the previous sections can be now easily specialized from an arbitrary upper continuous modular lattice L to lattices of type $\mathrm{Sat}_\tau(M_R)$. We present below only three of them, and leave to the reader the pleasure to do it for the remaining ones.

THEOREM 7.1 (**τ-FT**) ([**5**, Corollary 2.9]). *Assume that the Gabriel topology F_τ for the hereditary torsion theory τ has a basis of finitely generated right ideals. Then the following assertions are equivalent for a module M_R.*

(1) *M is τ-Noetherian.*

(1) *M is τ-*QFD *and* $\mathrm{Sat}_\tau(M_R)$ *has* ACC *on its subdirectly irreducible elements.* □

COROLLARY 7.2 ([**5**, Corollary 2.10]). *A ring R is right τ-Noetherian if and only if the following conditions are satisfied.*

(a) *R_R is τ-*QFD*;*

(b) $\mathrm{Sat}_\tau(R_R)$ *has* ACC *on its subdirectly irreducible elements;*

(c) *the Gabriel topology F_τ has a basis of finitely generated right ideals.* □

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THEOREM 7.3 (τ-**FT**$_n$). *Let τ be a hereditary torsion theory on* Mod-R *satisfying the τ-Krull Lemma condition* (†). *Then, for any module M_R and any $n \in \mathbb{N}$ we have:*

$$k^0_\tau(M_R) \leqslant n \Longleftrightarrow M_R \text{ is } \tau\text{-QFD and } k^0(\mathcal{S}(\mathrm{Sat}_\tau(M)) \leqslant n. \qquad \square$$

For the remaining of this section $\mathcal{G}$ will denote a fixed *Grothendieck category*, that is, an Abelian category with exact direct limits and with a generator. For any object $X \in \mathcal{G}$, $\mathcal{L}(X)$ will denote the lattice of all subobjects of X. It is well known that $\mathcal{L}(X)$ is an upper continuous modular lattice (see e.g. [**39**, Chapter 4, Proposition 5.3, and Chapter 5, Section 1]. For all undefined notation and terminology on Abelian categories the reader is referred to [**7**] and/or [**39**].

We say that an object $X \in \mathcal{G}$ is *subdirectly irreducible*, abbreviated SI, if the lattice $\mathcal{L}(X)$ is subdirectly irreducible. More generally, if $\mathbb{P}$ is any property on lattices, we say that an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. Thus, we obtain the concepts of *co-irreducible* (*uniform*) object, object *rich in subdirectly irreducibles* (*RSI*), object of *finite Goldie dimension*, object with (*dual*) *Krull dimension*, *QFD* object, etc. If X has Krull dimension (resp. dual Krull dimension), we write $k(X) := k(\mathcal{L}(X))$ (resp. $k^0(X) := k^0(\mathcal{L}(X))$). Similarly, a subobject Y of an object $X \in \mathcal{G}$ is/has $\mathbb{P}$ if the element Y of the lattice $\mathcal{L}(X)$ is/has $\mathbb{P}$. We denote by

$$\mathcal{S}(X) := \mathcal{S}(\mathcal{L}(X)) = \{\, Y \leqslant X \mid X/Y \text{ is SI} \,\}$$

the set of all CI subobjects of an object $X \in \mathcal{G}$, which were called 'subdirectly irreducible subobjects' of X in [**5**].

The existence of CI subobjects of an object $X \in \mathcal{G}$ is intimately related to the existence of simple objects of $\mathcal{G}$. It may happen that $\mathcal{G}$ has no simple object (see e.g. [**14**, p. 1539]). For such a category $\mathcal{G}$, the only object in $\mathcal{G}$ having (dual) Krull dimension is the zero object of $\mathcal{G}$, and no non-zero object of $\mathcal{G}$ is RSI.

The next result characterizes those Grothendieck categories $\mathcal{G}$ having a finitely generated generator. Recall that an object $C \in \mathcal{G}$ is called *finitely generated* if C is a compact element of the lattice $\mathcal{L}(C)$ of all subobjects of C.

PROPOSITION 7.4 ([**5**, Proposition 2.12]). *The following assertions are equivalent for a Grothendieck category $\mathcal{G}$.*

(1) *$\mathcal{G}$ has a finitely generated generator.*
(2) *There exists a unital ring A and a hereditary torsion theory $\chi = (\mathcal{H}, \mathcal{E})$ on* Mod-A *such that $\mathcal{G} \simeq$* Mod-$A/\mathcal{H}$ *and the Gabriel topology F_χ has a basis of finitely generated right ideals of A.* $\square$

By Proposition 7.4, any Grothendieck category $\mathcal{G}$ having a finitely generated generator has simple objects, and any $X \in \mathcal{G}$ is RSI. A recent result of Albu and Van Den Berg [**14**, p. 1545]) provides an example of an indecomposable non-locally finitely generated Grothendieck category with a single simple object, and answers in the negative a sharper form of Question 2.14 raised by Albu *et al.* [**5**] asking whether a Grothendieck category having simple objects has a finitely generated generator.

We end this paper by presenting specializations of a few latticial results of the previous sections from an arbitrary upper continuous modular lattice L to lattices of type $\mathcal{L}(X)$, X object of a Grothendieck category $\mathcal{G}$ having a finitely generated generator. As observed just after Proposition 7.4, any object of such a category is RSI, so the three results below are immediate consequences of the corresponding latticial results. Notice that there is a slight change of terminology in the next two results

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when comparing them with the original ones of [**5**]: instead of the term of 'subdirectly irreducible subobject' we use the more appropriate term of 'completely irreducible subobject', abbreviated CI.

THEOREM 7.5 ([**5**, Corollary 2.18]). *The following assertions are equivalent for a Grothendieck category $\mathcal{G}$ having a finitely generated generator, an object $X \in \mathcal{G}$, and an ordinal $\alpha \geqslant 0$.*

(1) $k^0(X) \leqslant \alpha$.

(2) *X is* QFD *and $k^0(X/Y) \leqslant \alpha$ for every (completely) irreducible subobject Y of X.* □

THEOREM 7.6 (CATEGORICAL **FT**) ([**5**, Corollary 2.19]). *The following assertions are equivalent for a Grothendieck category $\mathcal{G}$ having a finitely generated generator and an object $X \in \mathcal{G}$.*

(1) *X is Noetherian.*

(2) *X is* QFD *and has ACC on its* CI *subobjects.* □

THEOREM 7.7 (CATEGORICAL $\mathbf{FT}_n$). *Let $\mathcal{G}$ be a Grothendieck category having a finitely generated generator. Then, for any object $X \in \mathcal{G}$ and any $n \in \mathbb{N}$ we have:*

$$k^0(X) \leqslant n \Longleftrightarrow X \text{ is QFD} \quad \text{and} \quad k^0(\mathcal{S}(X)) \leqslant n. \qquad \square$$

8. Some open questions.

PROBLEM 8.1. Can $\mathcal{F}(M)$ be replaced in Theorem 6.1 by its subset $\mathcal{C}(M)$?

PROBLEM 8.2 (THE LATTICIAL $\mathbf{FT}^0_\alpha$). Let $L \in \mathcal{M} \cap \mathcal{U}$, and denote by $\mathcal{C}(L)$ the set of all compact elements of L. Then

$$k(L) \leqslant \alpha \iff L \text{ is QFD} \quad \text{and} \quad k(\mathcal{C}(L)) \leqslant \alpha.$$

In particular, L is Artinian $\iff L$ is QFD and $\mathcal{C}(L)$ is an Artinian poset.

PROBLEM 8.3 (THE LATTICIAL $\mathbf{FT}_\alpha$). For any $L \in \mathcal{M} \cap \mathcal{U}$ and any ordinal $\alpha \geqslant 0$,

$$k^0(L) \leqslant \alpha \Longleftrightarrow L \text{ is both QFD and RSI, and } k^0(\mathcal{S}(L)) \leqslant \alpha.$$

PROBLEM 8.4. Do dual characterizations to those in Theorem 6.10 hold for $L \in \mathcal{M} \cap \mathcal{U}$ having $k(L) \leqslant \alpha$?

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