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GROUP RINGS WHICH ARE *v-HC* ORDERS AND KRULL ORDERS

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Let R be a ring and G a polycyclic-by-finite group. In this paper, it is determined, in terms of properties of R and G, when the group ring $R[G]$ is a prime Krull order and when it is a price v-HC order. The key ingredient in obtaining both characterizations is the first author's earlier study of height one prime ideals in the ring $R[G]$.

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1. Introduction

Let R be a prime Goldie ring and let G be a polycyclic-by-finite group. The purpose of this paper is to characterise those group rings $R[G]$ which are prime Krull orders in the sense of [**5**], and also those which are prime v-HC orders. The precise definition of the latter class is given in Section 2, but as a rough guide it is worth bearing in mind the following analogy: Noetherian v-H orders stand in the same relation to Noetherian maximal orders as do hereditary Noetherian prime rings to Dedekind prime rings. An account of their elementary properties can be found in [**7**], [**8**].

Recall that a subset X of G (or of $R[G]$) is (G-)*orbital* if X has only finitely many distinct G-conjugates; following [**1**], G is called *dihedral free* if it contains no orbital subgroup isomorphic to the infinite dihedral group $D=\langle a,b\colon a^{-1}ba=b^{-1},a^2=1\rangle$. The join of all the finite normal subgroups of G is denoted $\Delta^+(G)$. Our main results are as follows.

Theorem A. *Let R be a ring and let G be a polycyclic-by-finite group. Then $R[G]$ is a prime Krull order if and only if*

(i) *R is a prime Krull order,*

(ii) *$\Delta^+(G)=1$, and*

(iii) *G is dihedral free.*

Theorem B. *Let R and G be as in Theorem A. Then the following statements are equivalent.*

(a) *$R[G]$ is a prime v-HC order with enough v-invertible ideals.*

(b) (i) *R is a prime v-HC order with enough v-invertible ideals,*

(ii) $\Delta^+(G)=1$, *and*

(iii) *either G is dihedral free, or R has characteristic not equal to* 2.

The notion of a v-invertible ideal is recalled in Section 2. (Naturally, invertible ideals are v-invertible.) Both of these results are generalisations of [**1**, Theorem F], which gave Theorem A for commutative Noetherian coefficient rings R. Indeed the proof of Theorem A is accomplished by using a result from [**11**] to allow the replacement of R by its simple Artinian quotient ring Q (Proposition 2.7); then rather an easy argument permits us to replace Q by its centre, and so finally we may appeal to [**1**, Theorem F].

The proof of Theorem B requires first a refinement (Theorem 3.1) of the description of height one prime ideals of certain group rings given in [**1**, Theorem 3.2]. This result and a reduction from Section 2, in the spirit of that used for Theorem A, together serve to reduce the problem to the case of a group algebra of an abelian-by-finite group over a coefficient field, as is shown in Section 4. Finally this special case is treated in Sections 5 and 6, exploiting the fact that the algebra is a finitely generated Cohen–Macaulay module over its centre [**3**].

2. Definitions and reductions

All rings have 1, and all modules are unital. Unexplained notation and terminology is as in [**14**]. Let S be an order in a classical quotient ring $Q=Q(S)$. A right S-submodule I of Q is called a *(fractional) right S-ideal* provided I contains a unit of Q and $cI\subseteq S$ for some regular element c of S. If $I\subseteq S$, then it is said to be *integral. Left S-ideals* are defined analogously. By an *S-ideal* of Q we mean a left S-ideal which is also a right S-ideal. Let A, B be subsets of Q. We will use the notation: $(A{:}B)_l=\{q\in Q \mid qB\subseteq A\}$, $(A{:}B)_r=\{q\in Q \mid Bq\subseteq A\}$. In particular, we denote $(A{:}A)_l$ by $0_l(A)$. For any right S-ideal I, we define $I_v=(S{:}(S{:}I)_l)_r$, and if $I=I_v$, then it is called a *right v-ideal.* Similarly we define ${}_vJ=(S{:}(S{:}J)_r)_l$ for any left S-ideal J, and J is said to be a *left v-ideal if ${}_vJ=J$.* An S-ideal A is called a *v-ideal* or *reflexive*, if ${}_vA=A=A_v$. An integral *v-ideal* is simply called a *v-ideal* of S. Let $F(\sigma)$ be the right Gabriel topology cogenerated by $E(Q/S)$, the injective hull of the right S-module Q/S, in other words, σ is the idempotent kernel functor on right S-modules cogenerated by $E(Q/S)$. In an analogous way, we can define the idempotent kernel functor σ' on left S-modules. For a right S-ideal I, we define the *σ-closure* of I as $cl(I)=\{q\in Q \mid qC\subseteq I$ for some $C\in F(\sigma)\}$, and we say that I is *σ-closed* if $cl(I)=I$. *σ'-closed left S-ideals* are defined analogously. Consider the following three conditions.

(M) S is a maximal order in the sense of Asano [**13**], i.e., $O_l(A)=S=O_r(A)$ for any integral S-ideal A.

(K) ${}_v(A(S{:}A)_l)=O_l(A)$ for any integral S-*ideal* A with $A={}_vA$, $((S{:}B)_rB)_v=O_r(B)$ for any integral S-ideal B with $B=B_v$.

(C) S satisfies the ascending chain condition on integral σ-closed right S-ideals as well as on integral σ'-closed left S-ideals.

We say that S is a *Krull order* in the sense of [**5**] if it satisfies the conditions (M) and

(C), and S is said to be a *v-HC order* if it satisfies the conditions (K) and (C). Let C be a regular Ore set of S. Then we denote by S_C the partial quotient ring of S with respect to C. Let F be a division ring with centre K and let A be a K-algebra which is a prime Goldie ring with quotient ring $Q(A)$. We begin with two easy lemmas.

Lemma 2.1. *Let $S=A\otimes_K F$ and $T=Q(A)\otimes_K F$. Then $C=\{c\otimes 1 : c$ regular in $A\}$ is a regular Ore set of S and $T=S_C$.*

Proof. See the proof of Proposition 1.4 of [**15**].

Lemma 2.2 *With the same notation as Lemma 2.1, there is a one-to-one correspondence between the set of all non-zero (S,S)-bimodules B in T and the set of all non-zero (A,A)-bimodules $\mathbf{B}$ in $Q(A)$, which is given by*

$$B\to B\cap Q(A);\quad \mathbf{B}\to \mathbf{B}\otimes_K F.$$

Proof. See the proof of Case 1 of Theorem 7.3.9 of [**16**].

In particular, we see from Lemma 2.2 that T is a simple ring. If T is a Goldie ring, then S has a classical quotient ring $Q(S)$ which is a simple Artinian ring.

Lemma 2.3. *Assume that T is a Goldie ring and let $B=\mathbf{B}\otimes F$ be a non-zero (S,S)-bimodule in T, where $\mathbf{B}$ is an (A,A)-bimodule in $Q(A)$. Then*

(i) $(S:B)_l=(A:\mathbf{B})_l\otimes F$. *In particular, $B_v=\mathbf{B}_v\otimes F$.*

(ii) *There is a one-to-one correspondence between the set of all S-ideals B in T and the set of all A-ideals $\mathbf{B}$ in $Q(A)$.*

Proof. (i) It is clear that $(S:B)_l$ is an (S,S)-bimodule containing $(A:\mathbf{B})_l\otimes F$. Let $\alpha=\sum_1^n q_i\otimes d_i$ be any element in $(S:B)_l$, where $q_i\in Q(A)$ and $d_i\in F$. We may assume that $d_1,\ldots,d_n$ are linearly independent over K. Since $S\supseteq \alpha B\supseteq \alpha(\mathbf{B}\otimes 1)=\sum q_i\mathbf{B}\otimes d_i$, we have $q_i\mathbf{B}\subseteq A$ and so $q_i\in(A:\mathbf{B})_l$. Thus $\alpha\in(A:\mathbf{B})_l\otimes F$.

(ii) Since A is a prime Goldie ring, $\mathbf{B}$ is a right A-ideal if and only if $(A:\mathbf{B})_l\neq 0$ and similarly, B is a right S-ideal if and only if $(S:B)_l\neq 0$. Hence the assertion follows.

Lemma 2.4. *Assume that T is a Goldie ring. Then*

(i) *A satisfies the condition (K) if and only if S satisfies (K).*

(ii) *A is a maximal order in $Q(A)$ if and only if S is a maximal order in $Q(S)$.*

Proof. (i) Let $B=\mathbf{B}\otimes F$ be any ideal of S, where $\mathbf{B}$ is an ideal of A. Note that ${}_vB=B$ is equivalent to ${}_v\mathbf{B}=\mathbf{B}$ and that $(S:B)_l=(A:\mathbf{B})_l\otimes F$ by Lemmas 2.2 and 2.3. It follows that ${}_v(B(S:B)_l)={}_v((\mathbf{B}\otimes F)((A:\mathbf{B})_l\otimes F))={}_v(\mathbf{B}(A:\mathbf{B})_l\otimes F)={}_v(\mathbf{B}(A:\mathbf{B})_l)\otimes F$. Hence $O_l(B)={}_v(B(S:B)_l)\Leftrightarrow 1\in O_l(B)\Leftrightarrow 1\in O_l(\mathbf{B})\Leftrightarrow O_l(\mathbf{B})={}_v(\mathbf{B}(A:\mathbf{B})_l)$.

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(ii) It follows, from similar arguments to (i), that $O_l(B)=O_l(\mathbf{B})\otimes F$. Hence S is a maximal order if and only if A is a maximal order.

To apply Lemma 2.4 to group rings, let $Q=(F)_n$ be a simple Artinian ring, where F is a division ring with centre K and let G be a polycyclic-by-finite group. Then it is well known that the group ring $Q[G]$ is Noetherian and has a classical quotient ring which is an Artinian ring [**18**]. Furthermore $Q[G]$ is a prime ring if and only if $\Delta^+(G)=\langle 1\rangle$ (Theorem 4.2.10 of [**16**]). Since $F[G]\cong F\otimes K[G]$, we have:

Corollary 2.5. *Assume that $\Delta^+(G)=\langle 1\rangle$. Then*

(i) *$Q[G]$ is a Krull order if and only if $K[G]$ is a Krull order.*

(ii) *$Q[G]$ is a v-H order if and only if $K[G]$ is a v-H order.*

Let S be an order in $Q(S)$ and let X be any v-ideal in $Q(S)$. We say that X is v-invertible if $(X(S:X)_r)_v=S={}_v((S:X)_lX)$. In this case, it follows that $(S:X)_r=(S:X)_l$, and we denote it by X^{-1}. A ring S is said *to have enough v-invertible ideals* if any v-ideal of S contains a v-invertible ideal of S. A v-ideal A of S is called *v-idempotent* if ${}_v(A^2)=A=(A^2)_v$. We say that A is *eventually v-idempotent* provided there exists $n\geqq 1$ such that $(A^n)_v=(A^{n+1})_v$. Let R be a v-HC order with enough v-invertible ideals. Then the following hold.

(i) $A_v={}_vA$ for any non-zero ideal A of R (Lemma 1.2 of [**7**]).

(ii) Any prime v-ideal of R is a maximal v-ideal of R (i.e. a v-ideal maximal amongst the v-ideals of R) (Lemma 1.2 of [**9**]).

(iii) If A is any v-ideal of R, then $A=(XB)_v$, for some v-invertible ideal X of R and eventually v-idempotent ideal B of R (Proposition 3 of [**10**] and (ii)).

(iv) Suppose B is eventually v-idempotent and $M_1,\dots,M_n$ is the full set of all maximal v-ideals of R containing B. Then $(B^n)_v=((M_1\cap\dots\cap M_n)^n)_v$ and is v-idempotent (Proposition 1.4 of [**8**]).

Suppose that R is a v-HC order with enough v-invertible ideals and that any maximal v-ideal of R is v-invertible. Then any v-ideal of R is v-invertible from (ii), (iii) and (iv). Let A be any ideal of R. Then it follows that $R\subseteq O_l(A)\subseteq O_l(A_v)=R$, because A_v is a v-ideal of R by (i) and so is v-invertible. Hence $O_l(A)=R$ and similarly, $O_r(A)=R$. This implies that R is a maximal order. Thus we have:

Proposition 2.6. *Suppose that R is a v-HC order with enough v-invertible ideals. Then R is a Krull order if and only if every maximal v-ideal of R is v-invertible.*

Before giving a further reduction, note that R satisfies the condition (C) if and only if $R[G]$ does by [**12**, Theorem].

Proposition 2.7. *The group ring $R[G]$ is a prime Krull order if and only if*

(i) *R is a prime Krull order in a quotient ring Q,*

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(ii) $\Delta^+(G)=\langle 1\rangle$, *and*

(iii) $Q[G]$ *is a Krull order.*

Proof. Suppose that $R[G]$ is a prime Krull order. Then it easily follows that R is a Krull order. By Lemma 2.2 of [**5**], $Q[G]$ is also a Krull order. Since $R[G]$ is prime, (ii) follows from Theorem 4.2.10 of [**16**]. Suppose that (i), (ii) and (iii) hold. Then $R[G]$ is prime. Let p be a maximal v-ideal of R. Note that it is a maximal v-invertible ideal of R, because R is Krull. Then $(R/p)[G]\cong R[G]/p[G]$ implies that $p[G]$ is a prime and v-invertible ideal of $R[G]$ by Theorem 4.2.10 of [**16**]. Hence $R[G]$ is a v-HC order with enough v-invertible ideals by Theorem 1.15 of [**11**].

To prove that $R[G]$ is a Krull order, let P be a maximal v-ideal of $R[G]$.

Case 1. Suppose that $p=P\cap R\neq 0$. Then p is a prime v-invertible ideal and hence so is $p[G]$, because $\Delta^+(G)=\langle 1\rangle$. By Lemma 1.2 of [**9**], $p[G]$ is a maximal v-ideal and so $P=p[G]$, which is v-invertible.

Case 2. Suppose that $P\cap R=0$. Then $P'=PQ[G]$ is v-invertible by (iii) and so, in particular, $\bigcap_{n=1}^{\infty}(P'^n)_v=0$. On the other hand, either P is v-idempotent or v-invertible. But since $(P^n)_v\subseteq(P'^n)_v$ for all $n\geqq 1$, P must be v-invertible. Hence $R[G]$ is a Krull order by Proposition 2.6.

The following is almost a special case of Theorem 1.15 of [**11**].

Proposition 2.8. *The group ring $R[G]$ is a prime v-HC order with enough v-invertible ideals if and only if*

(i) *R is a prime v-HC order in a quotient ring Q, with enough v-invertible ideals,*

(ii) $\Delta^+(G)=\langle 1\rangle$, *and*

(iii) $Q[G]$ *is a v-H order with enough v-invertible ideals.*

Proof. The necessity follows from Theorem 4.2.10 of [**16**] and Theorem 1.15 of [**11**]. Conversely, assume that (i), (ii) and (iii) hold. Then $R[G]$ is a prime ring by Theorem 4.2.10 of [**16**]. Let p be a maximal v-invertible ideal of R. Then it is a semiprime ideal by Theorem 1.13 of [7]. So it follows that the ideal $p[G]$ is semiprime and v-invertible by (ii) and Theorems 4.2.12, 4.2.13 of [**16**]. Hence $R[G]$ is a v-HC order with enough v-invertible ideals by Theorem 1.15 of [**11**].

Proof of Theorem A. By Proposition 2.7 and Corollary 2.5 R may be assumed to be a field. In this case the result is given by [**1**, Theorem F].

Let $S=\Sigma\oplus S_x(x\in G)$ be a strongly graded ring of type G and let A be an ideal of S_1, the part of degree 1. We say that A is a G-ideal if $S_{x^{-1}}AS_x\subseteq A$ for any $x\in G$. By a G-v-ideal we mean a G-ideal which is a v-ideal. We close this section with the following easy lemma used in the next section.

Lemma 2.9. *Let R be a Krull order and let G be a polycyclic-by-finite group with*

$\Delta^+(G)=1$. *Let* A *be any integral* $R[G]$*-ideal such that* $A=A_v$. *Then* A *contains a* v*-invertible ideal.*

Proof. By Lemma 10.2.5 of [**16**], G has a normal subgroup N of finite index which is poly-infinite cyclic. From the exact sequence

$$1\to N\to G\xrightarrow{\pi} G/N\to 1,$$

where π is the canonical map, we derive the following strongly graded ring of type G/N; $R[G]=\Sigma\oplus S_{\pi(g)}(g\in G)$, where $S_{\pi(g)}=\Sigma\oplus Rx(x\in\pi(g))$ and $S_{\pi(1)}=R[N]$, a Krull order by Corollary 3.9 of [**15**]. Since N has finite index in G, $Q(R[G])=Q(R[N])\otimes_{R[N]}R[G]$ by Lemma 13.3.5 of [**16**]. Hence $R[N]\cap A$ is a non-zero G-v-invertible ideal of $R[N]$. Hence $((R[N]\cap A)R[G])_v$ is a v-invertible ideal contained in A by Lemma 1.3 of [**11**].

3. Height one primes of group rings

For the proof of Theorem B we need a variant of [**1**, Theorem 3.2]. Recall that if I is an ideal of a group ring $R[G]$, then $I^+=\{g\in G:(g-1)\in I\}$, a normal subgroup of G.

Theorem 3.1. *Let* R *be a commutative Noetherian domain, and let* H *be a finitely generated abelian-by-finite group with* $\Delta^+(H)=1$. *Set* $B=\Delta(H)$, *the largest abelian normal subgroup of* H. *Let* P *be a height one prime ideal of* $R[H]$.

(a) *Then either* (*i*) $P=(P\cap R[B])R[H]$; *or* (*ii*) *there exists an isolated orbital dihedral subgroup* D *of* H, *and a prime ideal* I *of* $R[D]$, *with*

$$P=\bigcap_{x\in H} I^x R[H].$$

(b) *Suppose that* Γ *is a group of operators on* H, *and that* P *is* Γ*-orbital. Then in* (*a*)(*i*), *we may take* $B=\Delta_\Gamma(H)=\{h\in H:|\Gamma:C_\Gamma(h)|<\infty\}$, *and in* (*a*)(*ii*), D *and* I *are* Γ*-orbital. In either case,* $P\cap K[B]\neq 0$.

Proof. (a) There is a prime Q of $R[B]$ with $P\cap R[B]=\bigcap_{x\in H}Q^x$, and $ht(Q)=1$ [**17**, Section 8.1 and Corrigendum].

Case 1. Suppose that Q^+ is finite. Then, as in the proof of Case 1 of [**1**, Theorem 3.2], $P=(P\cap R[B])R[H]$.

Case 2. Suppose that Q^+ is infinite. Let $E=N_H(Q)$, and let P_1 be a prime ideal of $R[E]$ with $P_1\cap R[B]=Q$. Thus P_1 has height one, and, since $Q^+\subseteq P_1^+$, P_1^+ is infinite. As $|H:E|<\infty$, P_1 is H-orbital; so also is P_1^+, and with it $\Delta^+(P_1^+)$. But then $\Delta^+(P_1^+)\subseteq\Delta^+(H)$, and so $\Delta^+(P_1^+)=1$. Since the Hirsch number of P_1^+ is a lower bound for the height of P_1, P_1^+ must have Hirsch length 1, and so, by [**1**, Lemma 2.1], P_1^+ is either infinite cyclic or infinite dihedral. In the former case, the proof of [**1**, Theorem 3.2] shows that $P=(P\cap R[B])R[H]$.

Suppose then that P_1^+ is dihedral. Let D be the isolator in E of P_1^+. Then D is also dihedral, by [**1**, Lemma 2.1] again, and

$$P_1=(P_1\cap R[D])R[E]$$

by [17, Corollary 22]. Since D is a normal subgroup of E, it follows that $P_1\cap R[D]$ is semiprime, and there is a prime ideal I of $R[D]$, with $P_1\cap R[D]=\bigcap_{x\in E}I^x$. Thus

$$P_1=\bigcap_{x\in E}I^xR[E].$$

By [6, Theorem 1.7].

$$P=\bigcap_{x\in H}(P_1R[H])^x.$$

Combining these two expressions we deduce that

$$P_1=\bigcap_{x\in H}I^xR[H],$$

as required.

(b) The first statement follows from (a) by applying [17, Theorem D], exactly as in the proof of [1, Theorem 3.2]. Moreover, it is clear that D and I are Γ-orbital in (ii), if P is. The final statement follows from the facts that $|D{:}D\cap B|=2$, and B is torsion free abelian.

4. The proof of Theorem B—reduction to abelian-by-finite groups

This is achieved by means of the following lemma. Let G be a polycyclic-by-finite group. A *plinth* of G is a torsion-free abelian orbital subgroup A of G such that $A\otimes_Z\mathbb{Q}$ is an irreducible $\mathbb{Q}[T]$-module for every subgroup T of finite index in G. The characteristic subgroup generated by the plinths of G is denoted by $P(G)$, and the largest normal subgroup of G containing $P(G)$ as a subgroup of finite index is denoted by $S(G)$.

For any ideal I of a ring R, $\mathscr{C}(I)$ is the set of all elements c of R regular modulo I. If $\mathscr{C}(I)$ is an Ore set then R_I will denote the ring of quotients of R with respect to $\mathscr{C}(I)$.

Lemma 4.1. *Let K be a field and let G be a polycyclic-by-finite group with $\Delta^+(G)=1$. Suppose that $K[S(G)]$ is a v-H order with enough v-invertible ideals. Then $K[G]$ is a v-H order with enough v-invertible ideals.*

Proof. Let P be a prime v-ideal of $K[G]$. We show first that

$$ht(P)=1 \tag{1}$$

By [1, Theorem F] there exists a normal subgroup N of finite index in G such that $K[N]$ is a maximal order. Since $K[G]$ is prime, by [16, Theorem 4.2.10], $P\cap K[N]\neq 0$. Since $Q(K[G])=Q(K[N])\otimes_{K[N]}K[G]$, it is easy to see that $P\cap K[N]$ is a v-ideal. Let Y

be a prime ideal of $K[N]$ with $P \cap K[N] = \bigcap_{g \in G} Y^g$. Thus Y is a v-ideal, and so has height one by [**14**, Proposition 5.1.9]. Now (1) follows from [**17**, Lemma 29].

In view of (1), $P = (P \cap K[S(G)])K[G]$ by Theorem A of [**1**]. Now

$$P \cap K[S(G)] \textit{ is a } v\textit{-ideal,} \tag{2}$$

by [**11**, Lemma 1.1]. Thus $P \cap K[S(G)] = \bigcap_{g \in G} I^g$, where I is a prime, G-orbital v-ideal of $K[S(G)]$. Let J be the maximal v-invertible ideal of $K[S(G)]$ contained in I. Thus, for all $g \in G$, J^g is the maximal v-invertible ideal in I^g. Set $J_0 = \bigcap_{g \in G} J^g$ and $\hat{J} = J_0 K[G]$. Then $\hat{J}$ is semiprime, since J_0 is an intersection of G-orbital primes and $S(G)$ is isolated in G. The v-invertibility of J_0 implies that $\hat{J}$ is v-invertible in $K[G]$. By [**7**, Proposition 2.7], $K[G]_{\hat{J}}$ exists and is an HNP ring.

Let Z be the centre of $K[G]$ and T the partial quotient ring of $K[G]$ obtained by inverting the non-zero elements of Z. We claim that

$$T \textit{ has no proper } v\textit{-ideals.} \tag{3}$$

For, let A be a non-zero prime v-ideal of T. It is easy to see that $A \cap K[G]$ is a non-zero prime v-ideal of $K[G]$. By (2) and Theorem 3.1(b), there exists a non-zero element c in $A \cap K[\Delta(G)]$. Thus $\prod_{g \in V} c^g$, where V is a transversal to $C_G(c)$ in G, is a non-zero element of $A \cap Z$, and so $A = T$, as claimed.

Let $\mathbf{J}$ denote the collection of ideals J of $K[S(G)]$ which are maximal among v-invertible G-invariant ideals. Set $\hat{J} = JK[G]$ for any J in $\mathbf{J}$, and $\hat{\mathbf{J}} = \{\hat{J} : J \in \mathbf{J}\}$. Next we show that

$$K[G] = \bigcap_{\hat{\mathbf{J}}} K[J]_{\hat{J}} \cap T. \tag{4}$$

Let $x \in \bigcap K[G]_{\hat{J}} \cap T$. Since $x \in T$, there is a v-invertible ideal X of $K[G]$ with $Xx \subseteq K[G]$; we can choose X maximal with this property. Suppose that $X \neq K[G]$. Then X is contained in a maximal v-invertible ideal E, and $E \in \hat{\mathbf{J}}$ by the first part of the proof. There exists $c \in \mathscr{C}(E)$ with $xc \in K[G]$. Thus $X \subsetneq E^{-1}X \subseteq K[G]$, and so

$$EE^{-1}Xx \subseteq K[G] \cap Ex \subseteq K[G] \cap Ec^{-1} \subseteq E.$$

Therefore, $E^{-1}Xx \subseteq O_r(E) = K[G]$. This contradicts the maximality of X, and from this contradiction we deduce that $X = K[G]$, proving (4).

By (3) and [**13**, Proposition I.3.1], T is a maximal order. Since each of the rings $K[G]_{\hat{J}}$ is HNP, it follows from (4) that $K[G]$ is a prime Noetherian v-H order, as in [**7**, Theorem 2.23].

5. Sufficient conditions in the abelian-by-finite case

Our proof that certain group algebras are v-H orders proceeds by way of a local-global result whose proof requires the following concept ([**2**, page 78]). A Noetherian ring S which is a finite module over a central subring C is *centrally Macaulay* if S is a Cohen–Macaulay C-module. The key property we require concerning such rings is stated in the next lemma, which, though implicit in [**2**], [**3**], [**4**], is not stated explicitly there.

Proposition 5.1. *Let S be a prime Noetherian centrally Macaulay ring with centre Z. Then*

(i) *$S=\bigcap_p S_p$, where the intersection is over the height one primes P of S, and, for each such P, $p=P\cap Z$ and $S_p=S\otimes_Z Z_p$.*

(ii) *If c is a regular element of S, then $cS_p=S_p$ for all but finitely many p as in (i).*

(iii) *Suppose that each S_p is hereditary. Let A be an ideal of S. Then $\bigcap_p AS_p$ is the smallest right [resp. left] reflexive ideal of S containing A.*

(iv) *S is a prime Noetherian v-H order with enough v-invertible ideals if and only if each S_p is hereditary.*

Proof (i) This is a special case of [**2**, Theorem 4.13], together with Lemma 5.1 of [**3**].

(ii) Let c be a regular element of S. Since S is a finite Z-module, $0\neq cS\cap Z$. Let $I=cS\cap Z$. There are at most finitely many height one primes of S containing I. If P is any height one prime not containing I, then $I\nsubseteq p$, so $IS_p=S_p$, and so $cS_p=S_p$.

(iii) Let A_v be the smallest right reflexive ideal containing A. Since S_p is hereditary, AS_p is reflexive, and so $A_vS_p=AS_p$. Thus

$$A_v\subseteq\bigcap_p AS_p:=B.$$

For the converse, let $T=\{q\in Q(S):qA\subseteq S\}$. Then

$$TB\subseteq\bigcap_p TAS_p=\bigcap_p S_p=S,$$

so that $B\subseteq A_v$. Similar remarks apply on the left.

(iv) This is immediate from [7, Theorem 2.23] and (i), (ii) and (iii).

We now apply the preceding result to group algebras. Let K be a field and let G be a finitely generated abelian-by-finite group with $\Delta^+(G)=1$. Then $S=K[G]$ is prime Noetherian, by [**16**, Theorem 4.2.10 and Corollary 10.2.8], so S satisfies the hypotheses for (i) and (ii) of Proposition 5.2, by [4, Theorems 6.4 and 3.4]. These observations yield (i) of the following proposition.

Proposition 5.2. *Let K be a field and let G be a finitely generated abelian-by-finite group with $\Delta^+(G)=1$. For each height one prime P of $K[G]$, let $p=P\cap Z$ where Z is the* *centre of* $K[G]$, *and let* $K[G]_p$ *denote the localisation of* $K[G]$ *at* p (*i.e.* $K[G]_p = KG \otimes_Z Z_p$). *Then*

(i) $K[G]$ *satisfies* (*i*) *and* (*ii*) *of Proposition 5.1.*

(ii) *Suppose that char* $K \neq 2$. *Then for all height one primes* P *of* $K[G]$, $K[G]_p$ *is a semilocal HNP ring.*

Proof. (ii) It only remains to prove that, when char $K \neq 2$, $K[G]_p$ is hereditary, where $p = P \cap Z$ and P is a height one prime. With this notation, by Theorem 3.1, either (a) $P = (P \cap K[B])K[G]$, where B is the FC-subgroup of G, or (b) there is a prime ideal I of the group algebra of an isolated orbital dihedral subgroup D of G, with $P = \bigcap_{x \in G} I^x K[G]$.

If (a) applies then P is principal since B is free abelian (see [**16**, Lemma 4.1.6]). Suppose that (b) holds. Then there are elements $x_1, \ldots, x_t$ of G such that, as right modules,

$$K[G]_p/P_p \cong \sum_{i=1}^{t} \oplus K[G]_p/I^{x_i}K[G]_p,$$

since $K[G]_p/P_p$ is simple Artinian. Consider the ith summand on the right hand side of this isomorphism. Writing I for I^{x_i} and D for D^{x_i}, for convenience, we have

$$K[G]_p/IK[G]_p = (K[D]/I) \otimes_{K[D]} K[G]_p. \tag{5}$$

But D has an infinite cyclic subgroup of index 2, and char $K \neq 2$, so $K[D]$ is hereditary by [**16**, Theorem 10.3.9]. Therefore, $K[D]/I$ has projective dimension 1. Since $K[G]_p$ is a flat $K[D]$-module, (5) shows that $K[G]_p/IK[G]_p$ has dimension 1 as a $K[G]_p$-module. That is, the unique irreducible $K[G]_p$-module has projective dimension 1, and $K[G]_p$ is hereditary.

From Proposition 5.1(iv) and 5.2 we have:

Corollary 5.3. *Let* K *be a field of characteristic not equal to* 2, *and let* G *be a finitely generated abelian-by-finite group with* $\Delta^+(G) = 1$. *Then* $K[G]$ *is a prime Noetherian v-H order with enough v-invertible ideals.*

6. Necessary conditions, and proof of Theorem B

Lemma 6.1. *Let* G *be a polycyclic-by-finite group with* $\Delta^+(G) = 1$, *and let* R *be a commutative Krull domain. Let* D *be an isolated orbital dihedral subgroup of* G. *Put* $P = \bigcap_{x \in G} \mathbf{d}^x$. *Then* P *is a prime v-ideal of* $R[G]$.

Proof. Let $D = \langle a, b \mid a^{-1}ba = b^{-1},\ a^2 = 1 \rangle$. Routine arguments allow us to assume that R is a field. It follows from the proof of Lemma 1.3 of [**19**] that

$(b-1)^{-1}(a-1)\in O_l(\mathbf{d})\subseteq(R[D]:\mathbf{d})_l$. Since $(b-1)^{-1}(a-1)\notin R[D]$, $\mathbf{d}_v\subsetneqq R[D]$, and since $\mathbf{d}$ is a maximal ideal we conclude that $\mathbf{d}$ is reflexive.

By [1, Lemma 2.2], P is a prime ideal. Let $\alpha\in P_v$, so that $(R[G]:P)_l\alpha\subseteq R[G]$. For $x\in G$, $P\subseteq\mathbf{d}^xR[G]$, and so

$$(R[D^x]:\mathbf{d}^x)_l\subseteq(R[G]:P)_l.$$

In particular, writing $\alpha=\sum_{t\in T}\alpha_t t$, where T is a right transversal to D^x in G, and $\alpha_t\in R[D^x]$ for all $t\in T$, we conclude from the reflexivity of $\mathbf{d}^x$ in $R[D^x]$ that $\alpha_t\in\mathbf{d}^x$ for all $t\in T$. Hence $\alpha\in\bigcap_{x\in G}\mathbf{d}^xR[G]=P$, and so $P=P_v$.

Lemma 6.2. *Let K be a field of characteristic 2 and let G be a polycyclic-by-finite group. Suppose that $\Delta^+(G)=1$ and G is not dihedral free. Then $K[G]$ is not a $v-HC$ order.*

Proof. There is an orbital dihedral subgroup D in G, and by Lemma 2.1 of [1] we may assume that D is isolated. Let $P=\bigcap_{x\in G}\mathbf{d}^x$; P is a prime v-ideal by Lemma 6.1, and P is localisable by Lemma 2.2(v) of [1]. By Lemma 2.9, P contains a maximal v-invertible ideal, N. (In fact, it is easy to see that $N=P$.)

Suppose now that $K[G]$ is a v-Hc order, (so it has enough v-invertible ideals, by Lemma 2.9). By Proposition 2.7 of [7], $K[G]_N$ and its localisation $K[G]_P$ are HNP rings. Since $K[G]_P$ is local, its Jacobson radical P_P is principal. This contradicts Lemma 2.2(v) of [1]. Thus $K[G]$ is not a v-HC order.

Proof of Theorem B. Suppose that R and G satisfy hypotheses (i), (ii) and (iii) of Theorem B(b). Let Q be the quotient ring of R, and let K be its centre. By Proposition 2.8 and Corollary 2.5, we may replace R by K. If G is dihedral free, then $K[G]$ is a maximal order by Theorem F of [1]. Otherwise, the characteristic of K is not 2, and $K[S(G)]$ is a prime Noetherian v-H order with enough v-invertible ideals, by Corollary 5.3. (a) then follows from Lemma 4.1.

Conversely if $R[G]$ is a prime v-HC order with enough v-invertible ideals, then it is easy to deduce (i) (or see Theorem 1.15 of [11]), (ii) follows from Connell's theorem [16, Theorem 4.2.10], and (iii) follows from Proposition 2.8, Corollary 2.5 and Lemma 6.2.

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