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LIMITS OF FRACTIONAL DERIVATIVES AND COMPOSITIONS OF ANALYTIC FUNCTIONS

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Abstract

Suppose that the function f is analytic in the open unit disk Δ in the complex plane. For each $\alpha > 0$ a function $f^{[\alpha]}$ is defined as the Hadamard product of f with a certain power function. The function $f^{[\alpha]}$ compares with the fractional derivative of f of order α. Suppose that $f^{[\alpha]}$ has a limit at some point z_0 on the boundary of Δ. Then $w_0 = \lim_{z\to z_0} f(z)$ exists. Suppose that Φ is analytic in $f(\Delta)$ and at w_0. We show that if $g = \Phi(f)$ then $g^{[\alpha]}$ has a limit at z_0.

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1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let $\alpha > 0$. Suppose that the function f is analytic in Δ and, for $|z| < 1$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

We define $f^{[\alpha]}$ by

$$f^{[\alpha]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)} a_n z^n \tag{1.1}$$

for $|z| < 1$, where Γ denotes the gamma function. For $\beta > 0$ and $|z| < 1$ let

$$\frac{1}{(1-z)^\beta} = \sum_{n=0}^{\infty} A_n(\beta)\, z^n.$$

Then

$$A_n(\beta) = \frac{\Gamma(n+\beta)}{\Gamma(\beta)n!}$$

for each nonnegative integer n. Thus $f^{[\alpha]}$ is the Hadamard product of f with the function p where $p(z) = \Gamma(\alpha+1)/(1-z)^{\alpha+1}$ for $|z| < 1$. In [4] the authors obtained an integral formula for $f^{[\alpha]}$ in terms of f when $0 < \alpha < 1$.

The function $f^{[\alpha]}$ compares with the fractional derivative of f of order α. There are a number of definitions of fractional derivatives. One that applies to the Taylor series of a function analytic in Δ was introduced by Hadamard. It is defined by

$$f^{(\alpha)}(z) = z^{-\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n z^n \tag{1.2}$$

for $|z| < 1$. When α is a positive integer, $f^{(\alpha)}$ equals the usual derivative of f of order α. In general, a branch cut is needed to define an analytic branch of $f^{(\alpha)}$. The sequences

$$\left\{\frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}\right\} \quad \text{and} \quad \left\{\frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\right\}$$

have asymptotic expansions

$$n^{\alpha}\left\{c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right\}$$

as $n \to \infty$ with $c_0 \neq 0$. Hence certain facts about $f^{[\alpha]}$ are equivalent to facts about $f^{(\alpha)}$.

If $\operatorname{Re}\alpha > 0$ and n is a nonnegative integer then

$$\int_0^1 (1-t)^{\alpha-1} t^n \, dt = \frac{\Gamma(\alpha) n!}{\Gamma(n+1+\alpha)}.$$

This formula and (1.1) yield

$$f(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f^{[\alpha]}(tz) \, dt. \tag{1.3}$$

Also, if $0 < \alpha < \beta$ and $|z| < 1$ then

$$f^{[\alpha]}(z) = \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 t^{\alpha}(1-t)^{\beta-\alpha-1} f^{[\beta]}(tz) \, dt. \tag{1.4}$$

We are concerned with the limit of $f^{[\alpha]}$ as $z \to z_0$ where $|z| < 1$ and $|z_0| = 1$. Equation (1.4) and the Lebesgue convergence theorem imply that if $\lim_{z \to z_0} f^{[\beta]}(z)$ exists and $0 < \alpha < \beta$ then $\lim_{z \to z_0} f^{[\alpha]}(z)$ exists. A similar fact holds for fractional derivatives defined by (1.2).

For each positive integer m we have

$$f^{[m]}(z) = \frac{d^m}{dz^m}[z^m f(z)] = \sum_{k=0}^{m} \frac{(m!)^2}{(k!)(m-k)!} z^k f^{(k)}(z). \tag{1.5}$$

If n is a positive integer then by applying (1.5) successively for $m = n$, $n-1$, $n-2, \ldots, 3, 2, 1$, we see that $z^n f^{(n)}(z)$ is a linear combination of the functions $f^{[n]}, f^{[n-1]}, \ldots, f^{[1]}$, and f. Therefore $\lim_{z \to z_0} f^{[n]}(z)$ exists if and only if $\lim_{z \to z_0} f^{(n)}(z)$ exists.

In this paper, we prove a theorem about the existence of $\lim_{z\to z_0} g^{[\alpha]}(z)$ for the composition $g = \Phi(f)$ when $\lim_{z\to z_0} f^{[\alpha]}(z)$ exists and Φ is analytic. This generalizes a classical result about limits of nth derivatives of compositions.

In [2] Hardy and Littlewood obtain a number of results about fractional derivatives and fractional integrals of analytic functions. A survey of the history and development of the general theory of fractional calculus is contained in [5, 6].

2. The main results

THEOREM. *Let f be analytic in the open unit disk Δ and $\alpha > 0$, and suppose that $\lim_{z\to z_0} f^{[\alpha]}(z)$ exists for some $z_0 \in \partial\Delta$. Let $w_0 = \lim_{z\to z_0} f(z)$. Let $g(z) = \Phi(f(z))$ for $z \in \Delta$, where Φ is analytic in $f(\Delta)$ and at w_0. Then $\lim_{z\to z_0} g^{[\alpha]}(z)$ exists.*

The proof of this theorem relies on three lemmas. We first state an important corollary, which follows directly from the theorem and the following fact. If F is defined in Δ and $\lim_{z\to w} F(z)$ exists for all w on the boundary of Δ then F extends continuously to the boundary.

COROLLARY. *Suppose that f is analytic in Δ, $\alpha > 0$, and Φ is analytic in a neighborhood of $\overline{f(\Delta)}$. Let $g(z) = \Phi(f(z))$ for $|z| < 1$. If $f^{[\alpha]}$ extends continuously to $\overline{\Delta}$, then so does $g^{[\alpha]}$.*

3. Three lemmas

LEMMA 3.1. *Let φ be analytic and univalent in Δ and suppose that $\varphi(\Delta) \subseteq \Delta$. Suppose that $\varphi(\Delta)$ is a Jordan domain whose boundary contains a closed arc Λ on $\partial\Delta$. There is a closed arc Ψ on $\partial\Delta$ mapping onto Λ. If ζ_0 is in the interior of Ψ then $(1 - |\varphi(\zeta)|)/(1 - |\zeta|)$ is bounded in $N \cap \Delta$ where N is some neighborhood of ζ_0.*

PROOF. Since $\varphi(\Delta)$ is a Jordan domain, φ extends continuously to $\overline{\Delta}$ and φ is univalent in $\overline{\Delta}$. There is a closed arc Ψ on $\partial\Delta$ which is mapped bijectively onto Λ, and φ extends analytically in a neighborhood of each point ζ in the interior of Ψ, with $\varphi' \neq 0$ at every such point. By [3, Theorem 1.1] we have

$$\liminf_{\zeta\to\zeta_0} (1 - |\zeta|^2)\frac{|\varphi'(\zeta)|}{1 - |\varphi(\zeta)|^2} > 0.$$

Therefore

$$\lim_{\zeta\to\zeta_0} \frac{1 - |\varphi(\zeta)|^2}{1 - |\zeta|^2} \cdot \frac{1}{|\varphi'(\zeta)|}$$

exists, and so $\lim_{\zeta\to\zeta_0} ((1 - |\varphi(\zeta)|^2)/(1 - |\zeta|^2))$ exists. Hence there is a neighborhood N of ζ_0 such that $(1 - |\varphi(\zeta)|)/(1 - |\zeta|)$ is bounded in $N \cap \Delta$. □

Lemma 3.1 relates to the Julia–Carathéodory theorem (see [1, pages 23–32] and [7, pages 57–71]). Part of that theorem asserts that the nontangential derivative of φ exists at ζ_0 if and only if the nontangential limit of $(1 - |\varphi(\zeta)|)/(1 - |\zeta|)$ exists as $\zeta \to \zeta_0$, for suitable functions φ.

LEMMA 3.2. *Let* $0 < \alpha < \beta$, $|z_0| = 1$, *and let* N *be a neighborhood of* z_0. *Suppose that* f *is analytic in* Δ *and there is a constant* A *such that*

$$|f^{[\alpha]}(z)| \leq \frac{A}{(1-|z|)^{\beta}} \tag{3.1}$$

for $z \in N \cap \Delta$. *Then there exist a neighborhood* M *of* z_0 *and a constant* B *such that*

$$|f(z)| \leq \frac{B}{(1-|z|)^{\beta-\alpha}} \tag{3.2}$$

for $z \in M \cap \Delta$.

PROOF. Suppose that $-1 < \gamma < \beta - 1$. If $0 \leq r < 1$ we have

$$\begin{aligned}\int_0^1 (1-t)^{\gamma} \frac{1}{(1-rt)^{\beta}}\,dt &= \int_0^1 (1-t)^{\gamma} \sum_{n=0}^{\infty} A_n(\beta) t^n r^n\,dt \\ &= \sum_{n=0}^{\infty} A_n(\beta) \int_0^1 (1-t)^{\gamma} t^n\,dt r^n \\ &= \sum_{n=0}^{\infty} A_n(\beta) \frac{\Gamma(\gamma+1)\Gamma(n+1)}{\Gamma(n+2+\gamma)} r^n \\ &= \Gamma(\gamma+1) \sum_{n=0}^{\infty} \frac{A_n(\beta)}{\Gamma(\gamma+2)A_n(\gamma+2)} r^n.\end{aligned}$$

There is a constant C such that $A_n(\beta)/A_n(\gamma+2) \leq CA_n(\beta-\gamma-1)$ for every nonnegative integer n. Therefore

$$\int_0^1 \frac{(1-t)^{\gamma}}{(1-rt)^{\beta}}\,dt \leq \frac{C}{\gamma+1} \sum_{n=0}^{\infty} A_n(\beta-\gamma-1) r^n = \frac{C}{(\gamma+1)(1-r)^{\beta-\gamma-1}}. \tag{3.3}$$

The continuity of $f^{[\alpha]}$ and (3.1) imply that such an inequality also holds for $z \in S = \{re^{i\theta} : 0 \leq r < 1, |\theta - \theta_0| < \eta\}$, where $z_0 = e^{i\theta_0}$, $\eta > 0$ and η is sufficiently small. Suppose that $z \in S$. Then $tz \in S$ for $0 \leq t \leq 1$. Hence (1.3) implies that

$$|f(z)| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} \frac{A}{(1-tr)^{\beta}}\,dt,$$

and (3.3) implies that (3.2) holds for $z \in S$. Hence there is a neighborhood M of z_0 such that (3.2) holds for $z \in M \cap \Delta$. □

LEMMA 3.3. *Suppose that* $\alpha > 0$ *and* α *is not an integer. Let* p *denote the greatest integer in* α *and let* $q = p + 1$. *Suppose that* f *is analytic in* Δ *and let* $|z_0| = 1$. *Then* $\lim_{z\to z_0} \int_0^1 t^{\alpha}(1-t)^{q-\alpha-1} f^{(q)}(tz)\,dt$ *exists if and only if* $\lim_{z\to z_0} f^{[\alpha]}(z)$ *exists.*

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PROOF. Equation (1.4) implies that

$$f^{[\alpha]}(z) = \frac{1}{\Gamma(q-\alpha)} \int_0^1 t^{\alpha}(1-t)^{q-\alpha-1} f^{[q]}(tz)\,dt. \tag{3.4}$$

Suppose that $\lim_{z\to z_0} f^{[\alpha]}(z)$ exists. Then $\lim_{z\to z_0} \int_0^1 t^{\alpha}(1-t)^{q-\alpha-1} f^{[q]}(tz)\,dt$ exists. There are constants $c_0, c_1, \ldots c_{q-1}$ such that

$$z^q f^{(q)}(z) = f^{[q]}(z) + \sum_{k=1}^{q} c_{q-k} f^{[q-k]}(z) \tag{3.5}$$

for $|z| < 1$. Let $F_k(z) = \int_0^1 t^{\alpha}(1-t)^{q-\alpha-1} f^{[q-k]}(tz)\,dt$ for $|z| < 1$ and $k = 0, 1, \ldots, q$. To show that

$$\lim_{z\to z_0} \int_0^1 t^{\alpha}(1-t)^{q-\alpha-1} f^{(q)}(tz)\,dt \tag{3.6}$$

exists, it is sufficient to show that for each $k = 0, 1, \ldots, q-1$ the existence of $\lim_{z\to z_0} F_k(z)$ implies the existence of $\lim_{z\to z_0} F_{k+1}(z)$. From Equation (1.4) with $\alpha = q-k-1$ and $\beta = q-k$ we obtain $f^{[q-k-1]}(z) = \int_0^1 s^{q-k-1} f^{[q-k]}(sz)\,ds$. This implies $F_{k+1}(z) = \int_0^1 s^{q-k-1} F_k(sz)\,ds$, which yields our conclusion.

Conversely, suppose that (3.6) holds. There are constants $d_0, d_1, \ldots, d_{q-1}$ such that

$$f^{[q]}(z) = z^q f^{(q)}(z) + \sum_{k=1}^{q} d_{q-k} z^{q-k} f^{(q-k)}(z)$$

for $|z| < 1$. Let $G_k(z) = \int_0^1 t^{\alpha}(1-t)^{\beta} f^{(q-k)}(tz)\,dt$ for $|z| < 1$ and $k = 0, 1, \ldots, q$, where $\beta = q-\alpha-1$. To show that $\lim_{z\to z_0} f^{[q]}(z)$ exists, it is sufficient to show that for each $k = 0, 1, \ldots, q-1$ the existence of $\lim_{z\to z_0} G_k(z)$ implies the existence of $\lim_{z\to z_0} G_{k+1}(z)$.

Let $H_k(z) = f^{(q-k)}(z)$. Integrating along the line segment from 0 to z yields $f^{(q-k-1)}(z) = f^{(q-k-1)}(0) + z\int_0^1 H_k(sz)\,ds$. Hence there is a constant b such that

$$G_{k+1}(z) = b + z\int_0^1 I_k(sz)\,ds \tag{3.7}$$

where $I_k(z) = \int_0^1 t^{\alpha+1}(1-t)^{\beta} H_k(tz)\,dt$. We claim that

$$I_k(z) = G_k(z) + c\int_0^1 u^{\alpha+\beta+1} G_k(uz)\,du \tag{3.8}$$

where $|z| < 1$ and c is a constant. Let

$$G_k(z) = \sum_{n=0}^{\infty} A_n z^n, \quad I_k(z) = \sum_{n=0}^{\infty} B_n z^n,$$

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and $H_k(z) = \sum_{n=0}^{\infty} C_n z^n$ for $|z| < 1$. Using the formula

$$\int_0^1 t^{z-1}(1-t)^{w-1}\,dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

for z and w in the right half-plane we find that

$$A_n = \frac{\Gamma(\alpha+n+1)\Gamma(\beta+1)}{\Gamma(\alpha+n+\beta+2)}C_n \quad \text{and} \quad B_n = \frac{\Gamma(\alpha+n+2)\Gamma(\beta+1)}{\Gamma(\alpha+n+\beta+3)}C_n$$

for every nonnegative integer n. This implies that

$$B_n = \frac{n+\alpha+1}{n+\alpha+\beta+2}A_n.$$

Setting $c = (\alpha+\beta+2)/(\alpha+1)$, we see that

$$\frac{n+\alpha+1}{n+\alpha+\beta+2} = 1 + c\frac{1}{n+\alpha+\beta+2}$$

and obtain

$$I_k(z) = \sum_{n=0}^{\infty} A_n z^n + c\sum_{n=0}^{\infty}\frac{1}{n+\alpha+\beta+2}A_n z^n,$$

which yields (3.8).

Suppose that $\lim_{z\to z_0} G_k(z)$ exists. Then (3.8) implies that $\lim_{z\to z_0} I_k(z)$ exists. Hence (3.7) implies that $\lim_{z\to z_0} G_{k+1}(z)$ exists. □

4. Proof of the main theorem

Case I. Suppose that $\alpha = n$ is a positive integer. Faà di Bruno's formula for the nth derivative of a composition is

$$g^{(n)} = n!\sum_{m=1}^{n}\Phi^{(m)}\bigg\{\sum\prod_{k=1}^{n}\frac{1}{j_k!}\bigg[\frac{f^{(k)}}{k!}\bigg]^{j_k}\bigg\} \tag{4.1}$$

where the sum inside the braces is over all combinations of nonnegative integers $j_1, j_2, \ldots, j_n$ such that

$$\sum_{k=1}^{n} k j_k = n \quad \text{and} \quad \sum_{k=1}^{n} j_k = m.$$

Suppose that $\lim_{z\to z_0} f^{[n]}(z)$ exists. Then $\lim_{z\to z_0} f^{(k)}(z)$ exists for $k = 1, 2, \ldots, n$. Hence the analyticity of Φ and (4.1) imply that $\lim_{z\to z_0} g^{(n)}(z)$ exists. Therefore $\lim_{z\to z_0} g^{[n]}(z)$ exists.

Suppose that $\alpha > 0$ and $\lim_{z\to z_0} f^{[\alpha]}(z)$ exists. Let w_0, Φ, and g be defined as in the theorem. For $z \in \Delta$ let

$$h(z) = f^{[\alpha]}(z).$$

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There is a neighborhood N of z_0 such that h is bounded in $N \cap \Delta$. Let φ be a conformal mapping of Δ onto $N \cap \Delta$ and let Ψ denote the closed arc on $\partial\Delta$ such that $\varphi(\Psi) = \partial N \cap \partial\Delta$. For $|\zeta| < 1$ let

$$k(\zeta) = h(\varphi(\zeta)). \tag{4.2}$$

Then k is analytic and bounded in Δ. By the Schwarz–Pick lemma there is a constant C such that

$$|k'(\zeta)| \le \frac{C}{1 - |\zeta|} \tag{4.3}$$

for $|\zeta| < 1$. Henceforth we use C to denote a generic constant, and it is not the same constant each time. From (4.2) we obtain

$$k'(\zeta) = h'(\varphi(\zeta))\varphi'(\zeta). \tag{4.4}$$

Let $|\zeta_0| = 1$ such that $\varphi(\zeta_0) = z_0$. Since φ extends analytically to a neighborhood of ζ_0 and is univalent there, it follows that there exist a neighborhood M of ζ_0 and a positive constant σ such that

$$|\varphi'(\zeta)| \ge \sigma \tag{4.5}$$

for $\zeta \in M \cap \Delta$. Hence (4.3)–(4.5) imply that

$$|h'(z)| \le \frac{C}{1 - |\zeta|} \tag{4.6}$$

where $z = \varphi(\zeta)$ and $\zeta \in M \cap \Delta$. Lemma 3.1 and (4.6) imply that

$$|h'(z)| \le \frac{C}{1 - |z|} \tag{4.7}$$

for $z \in P \cap \Delta$ where P is some neighborhood of z_0. Let $z_0 = e^{i\theta_0}$. Then (4.7) implies that such an inequality holds for some constant where $z \in S$ and $S = \{z = re^{i\theta} : 0 \le r < 1, |\theta - \theta_0| < \delta\}$ for some $\delta > 0$.

For $z \in \Delta$ let $F(z) = (f')^{[\alpha]}(z) = \sum_{n=0}^{\infty} b_n z^n$ and $G(z) = (f^{[\alpha]})'(z) = \sum_{n=0}^{\infty} c_n z^n$. Then $b_n = ((n+1)/(n+1+\alpha))c_n$ for each nonnegative integer n. Since $(n+1)/(n+1+\alpha) = 1 - \alpha/(n+1+\alpha)$ this implies that

$$F(z) = G(z) - \alpha \int_0^1 t^{\alpha} G(tz)\, dt. \tag{4.8}$$

From (4.7) and (4.8) we conclude that

$$|(f')^{[\alpha]}(z)| \le \frac{C}{1 - |z|} \tag{4.9}$$

where $z \in S$. Such an inequality also holds for $z \in P \cap \Delta$ where P is some neighborhood of z_0.

Case II. Suppose that $0 < \alpha < 1$. Then (4.9) and Lemma 3.2 imply that

$$|f'(z)| \le \frac{C}{(1 - |z|)^{1-\alpha}} \tag{4.10}$$

for $z \in Q \cap \Delta$ where Q is some neighborhood of z_0. Hence such an inequality also holds for $z \in T$ where $T = \{z = re^{i\theta} : 0 \le r < 1, |\theta - \theta_0| < \epsilon\}$ for some $\epsilon > 0$.

Suppose that $z \in T$ and $0 \le t < 1$. Then $tz \in T$ and $f(z) - f(tz) = \int_L f'(w)\,dw$ where L is the line segment from tz to z. Since L is given by $w = (1-s)tz + sz$ for $0 \le s \le 1$, we obtain, using (4.10),

$$\begin{aligned}|f(z) - f(tz)| &\le (1-t)\int_0^1 |f'[(1-s)tz + sz]|\,ds \\ &\le C(1-t)\int_0^1 \frac{1}{[1-\{(1-s)t+s\}]^{1-\alpha}}\,ds \\ &= C(1-t)^\alpha \int_0^1 \frac{1}{(1-s)^{1-\alpha}}\,ds.\end{aligned}$$

Since $0 < \alpha < 1$ the last integral exists, and so

$$|f(z) - f(tz)| \le C(1-t)^\alpha \tag{4.11}$$

for $z \in T$ and $0 \le t \le 1$.

There is a number $\rho > 0$ such that Φ is analytic in a neighborhood of $\{w : |w - w_0| \le \rho\}$. If $|w - w_0| < \rho$ then

$$\Phi(w) = \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\Phi(\zeta)e^{i\theta}}{\zeta - w}\,d\theta \tag{4.12}$$

where $\zeta = w_0 + \rho e^{i\theta}$. Let $0 < \eta < \rho$. Then there exist a neighborhood N of z_0 and a number τ such that $0 < \tau < 1$ and

$$|f(tz) - w_0| \le \eta \tag{4.13}$$

for $z \in N \cap \Delta$ and $\tau \le t \le 1$. If $z \in N \cap \Delta$ then (4.12) gives

$$g(z) = \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\Phi(\zeta)e^{i\theta}}{\zeta - f(z)}\,d\theta \tag{4.14}$$

and hence

$$g'(z) = \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\Phi(\zeta)e^{i\theta}}{(\zeta - f(z))^2} f'(z)\,d\theta. \tag{4.15}$$

Let $H(z) = \int_\tau^1 t^\alpha (1-t)^{-\alpha} g'(tz)\,dt$ for $z \in \Delta$. Then (4.15) yields

$$H(z) = \int_\tau^1 t^\alpha (1-t)^{-\alpha} \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\Phi(\zeta)e^{i\theta}}{(\zeta - f(tz))^2} f'(tz)\,d\theta\,dt.$$

By writing

$$\frac{1}{(\zeta - f(tz))^2} = \left\{\frac{1}{(\zeta - f(tz))^2} - \frac{1}{(\zeta - f(z))^2}\right\} + \frac{1}{(\zeta - f(z))^2}$$

we have $H(z) = I(z) + J(z)$ where $I(z) = (\rho/2\pi)\int_\tau^1 \int_0^{2\pi} I(\theta, t, z)\,d\theta\,dt$,

$$I(\theta, t, z) = \frac{\Phi(\zeta)e^{i\theta} t^\alpha (1-t)^{-\alpha}[f(tz) - f(z)][2\zeta - f(z) - f(tz)]f'(tz)}{[\zeta - f(tz)]^2[\zeta - f(z)]^2},$$

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and

$$J(z) = \frac{\rho}{2\pi}\int_0^{2\pi} \frac{\Phi(\zeta)e^{i\theta}}{(\zeta - f(z))^2}\,d\theta \int_\tau^1 t^\alpha(1-t)^{-\alpha} f'(tz)\,dt \quad \text{for } z \in \Delta.$$

From (4.10), (4.11), and (4.13) we conclude that

$$|I(\theta, t, z)| \le \frac{C}{(1-t)^{1-\alpha}} \tag{4.16}$$

for $0 \le \theta \le 2\pi$, $\tau \le t < 1$, and $z \in N \cap \Delta$. Since $0 < \alpha < 1$ the integral $\int_0^1 (1/(1-t)^{1-\alpha})\,dt$ exists. By considering $I(z)$ as a double integral we find that the conditions hold for applying the Lebesgue convergence theorem. We conclude that $\lim_{z\to z_0} I(z)$ exists. Also, $\lim_{z\to z_0} J(z)$ exists because $J(z)$ is the product of two integrals, each of which has a limit. The second integral has a limit as a consequence of Lemma 3.3. We have shown that $\lim_{z\to z_0} H(z)$ exists. Lemma 3.3 implies that $\lim_{z\to z_0} g^{[\alpha]}(z)$ exists. This completes the proof of the Theorem when $0 < \alpha < 1$.

Case III. Suppose that $\alpha > 1$ and α is not an integer. Let p denote the greatest integer in α and let $q = p + 1$. Since k is analytic and bounded in Δ, we have, for each positive integer j,

$$|k^{(j)}(\zeta)| \le \frac{C}{(1-|\zeta|)^j} \tag{4.17}$$

for $|\zeta| < 1$. Since $k(\zeta) = h(\varphi(\zeta))$ for $|\zeta| < 1$ and ζ near ζ_0, Faà di Bruno's formula gives

$$k^{(j)}(\zeta) = K_j(\zeta) + h^{(j)}(\varphi(\zeta))[\varphi'(\zeta)]^j \tag{4.18}$$

where $K_j(\zeta)$ is the sum of the first $j-1$ terms in that formula. Because φ is analytic in a neighborhood of ζ_0 and $|\varphi'(\zeta)| \ge \sigma > 0$ there, (4.18) and (4.17) provide an inductive step for concluding that for each integer $n > 0$,

$$|h^{(n)}(\varphi(\zeta))| \le \frac{C}{(1-|\zeta|)^n} \tag{4.19}$$

for $|\zeta| < 1$ and ζ near ζ_0. In particular,

$$|h^{(q)}(\varphi(\zeta))| \le \frac{C}{(1-|\zeta|)^q}$$

for $|\zeta| < 1$ and ζ near ζ_0. With $z = \varphi(\zeta)$, Lemma 3.1 yields

$$|h^{(q)}(z)| \le \frac{C}{(1-|z|)^q} \tag{4.20}$$

for $|z| < 1$ and z near z_0.

For $z \in \Delta$ let $F(z) = (f^{(q)})^{[\alpha]}(z) = \sum_{n=0}^\infty b_n z^n$ and $G(z) = (f^{[\alpha]})^{(q)}(z) = \sum_{n=0}^\infty c_n z^n$. Then

$$b_n = \frac{(n+1)(n+2)\cdots(n+q)}{(n+q+\alpha)(n+q-1+\alpha)\cdots(n+1+\alpha)} c_n \tag{4.21}$$

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for each nonnegative integer n. There are constants $d_1, d_2, \ldots, d_q$ such that

$$\frac{(n+1)(n+2)\cdots(n+q)}{(n+q+\alpha)(n+q-1+\alpha)\cdots(n+1+\alpha)} = 1 + \frac{d_1}{n+q+\alpha} + \frac{d_2}{n+q-1+\alpha} + \cdots + \frac{d_q}{n+1+\alpha}.$$

Hence

$$F(z) = G(z) + \sum_{j=1}^{q} d_j \int_0^1 t^{q-j+\alpha} G(tz)\, dt.$$

Therefore (4.20) implies that

$$|(f^{(q)})^{[\alpha]}(z)| \le \frac{C}{(1-|z|)^q}$$

for $z \in \Delta$ and z near z_0. Lemma 3.2 implies that

$$|f^{(q)}(z)| \le \frac{C}{(1-|z|)^{q-\alpha}} \tag{4.22}$$

for $z \in \Delta$ and z near z_0. Since $q - \alpha < 1$, this implies that $f^{(j)}(z)$ is bounded for each $j = 1, 2, \ldots, q-1$ and $z \in N \cap \Delta$, where N is a neighborhood of z_0. In particular, the boundedness of f' yields

$$|f(z) - f(tz)| \le C(1-t) \tag{4.23}$$

for $z \in T$ and $0 \le t \le 1$ as shown previously.

Let $\zeta = w_0 + \rho e^{i\theta}$. There is a neighborhood M of z_0 such that (4.13) holds and $f^{(j)}(z)$ is bounded in $M \cap \Delta$ for $j = 1, 2, \ldots, q-1$. We have

$$\frac{d}{dz}(\zeta - f(z))^{-1} = \frac{f'(z)}{(\zeta - f(z))^2},$$
$$\frac{d^2}{dz^2}(\zeta - f(z))^{-1} = \frac{f''(z)}{(\zeta - f(z))^2} + \frac{2(f'(z))^2}{(\zeta - f(z))^3},$$

and in general $(d^n/dz^n)(\zeta - f(z))^{-1}$ is given by Faà di Bruno's formula for each positive integer n. For $n = q$ this gives

$$\frac{d^q}{dz^q}(\zeta - f(z))^{-1} = \frac{f^{(q)}(z)}{(\zeta - f(z))^2} + R_q(z, \zeta) \tag{4.24}$$

where $R_q(z, \zeta)$ denotes the sum of the remaining $q-1$ terms in that formula. Because $f^{(j)}(z)$ is bounded for $z \in M \cap \Delta$ and $j = 1, 2, \ldots, q-1$ and $|f(z) - w_0| \le \eta$, we conclude that

$$|R_q(z, \zeta)| \le C$$

for $0 \le \theta \le 2\pi$ and $z \in M \cap \Delta$. By replacing M by a smaller neighborhood of z_0 we also have

$$|R_q(tz, \zeta)| \le C \tag{4.25}$$

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for $\tau \le t \le 1$. From (4.14) and (4.24) we obtain

$$g^{(q)}(z) = \frac{\rho}{2\pi}\int_0^{2\pi} \Phi(\zeta)e^{i\theta}\Big\{\frac{f^{(q)}(z)}{(\zeta - f(z))^2} + R_q(z,\zeta)\Big\}\,d\theta \tag{4.26}$$

for $z \in M \cap \Delta$.

Let $H(z) = \int_\tau^1 t^\alpha(1-t)^{q-\alpha-1} g^{(q)}(tz)\,dt$ for $z \in \Delta$. Then (4.26) yields

$$H(z) = \int_\tau^1 t^\alpha(1-t)^{q-\alpha-1}\frac{\rho}{2\pi}\int_0^{2\pi} \Phi(\zeta)e^{i\theta}\Big\{\frac{f^{(q)}(tz)}{(\zeta - f(tz))^2} + R_q(tz,\zeta)\Big\}\,d\theta\,dt.$$

By writing

$$\frac{1}{(\zeta - f(tz))^2} = \Big\{\frac{1}{(\zeta - f(tz))^2} - \frac{1}{(\zeta - f(z))^2}\Big\} + \frac{1}{(\zeta - f(z))^2}$$

we obtain $H(z) = I(z) + J(z) + K(z)$ where $I(z) = (\rho/2\pi)\int_\tau^1\int_0^{2\pi} I(\theta,t,z)\,d\theta\,dt$,

$$I(\theta,t,z) = \frac{\Phi(\zeta)e^{i\theta}t^\alpha(1-t)^{q-\alpha-1}[f(tz)-f(z)][2\zeta - f(z) - f(tz)]f^{(q)}(tz)}{[\zeta - f(tz)]^2[\zeta - f(z)]^2},$$

$$J(z) = \frac{\rho}{2\pi}\int_0^{2\pi}\frac{\Phi(\zeta)e^{i\theta}}{(\zeta - f(z))^2}\,d\theta\int_\tau^1 t^\alpha(1-t)^{q-\alpha-1}f^{(q)}(tz)\,dt,$$

and

$$K(z) = \frac{\rho}{2\pi}\int_\tau^1 t^\alpha(1-t)^{q-\alpha-1}\int_0^{2\pi}\Phi(\zeta)e^{i\theta}R_q(tz,\zeta)\,d\theta\,dt \quad \text{for } z \in \Delta.$$

From (4.23), (4.22), and (4.13) we conclude that $I(\theta,t,z)$ is bounded for $0 \le \theta \le 2\pi$, $\tau \le t < 1$, and $z \in M \cap \Delta$. Considering $I(z)$ as a double integral, we see that we can apply the Lebesgue convergence theorem to conclude that $\lim_{z\to z_0} I(z)$ exists. Also, $J(z)$ is the product of two integrals each of which has a limit. The second integral has a limit as a consequence of Lemma 3.3. Hence $\lim_{z\to z_0} J(z)$ exists. From (4.25) and the existence of $\int_\tau^1 (1-t)^{q-\alpha-1}\,dt$ we also conclude that $\lim_{z\to z_0} K(z)$ exists. We have shown that $\lim_{z\to z_0} H(z)$ exists. Lemma 3.3 implies that $\lim_{z\to z_0} g^{[\alpha]}(z)$ exists.

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