

ASYMPTOTICALLY UNBIASED INFERENCE FOR ISING MODELS

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In his recent paper on statistical inference for binary Markov fields Pickard (1982) writes that ‘maximum likelihood inference for the interaction parameter cannot be performed for large sized lattices’. The reason for this is that the solution of the asymptotic normal equations has a bias which is of the same or higher order as the random fluctuations. We should like to show here that this bias can be removed by a simple modification of the normal equations. The same modification was proposed by Guyon (1982) in the case of Gaussian random fields; it consists essentially in replacing biased sample product moments by unbiased ones. We take as our estimator the solution of the equations

$$(1) \quad DH(\mathbf{a})_\nu = \alpha_\nu / |\mathcal{C}_\nu| Q_\nu(\mathbf{x}) \quad (\nu = 0, \dots, 1).$$

On p. 351 Pickard discusses the version with N^{-1} instead of $\alpha_\nu / |\mathcal{C}_\nu|$. The difference between α_ν and $|\mathcal{C}_\nu|/N$ is of the order $O(N^{-1/d})$.

If no phase transition occurs, then we have by general results from statistical mechanics that

$$(2) \quad DH(\mathbf{a})_\nu = \alpha_\nu \mathcal{E}_\infty \left(\prod_{i \in \theta} x_i \right)$$

where θ is any clan of the equivalence class \mathcal{C}_ν and \mathcal{E}_∞ denotes the expectation with respect to the infinite Gibbs distribution P_∞ . Because the number of terms in $Q_\nu(\mathbf{x})$ is just $|\mathcal{C}_\nu|$, we obtain from (1) and (2)

$$(3) \quad DH(\hat{\mathbf{a}})_\nu - DH(\mathbf{a})_\nu = \alpha_\nu N / |\mathcal{C}_\nu| \cdot N^{-1} (Q_\nu(\mathbf{x}) - \mathcal{E}_\infty(Q_\nu(\mathbf{x}))).$$

where $\hat{\mathbf{a}}$ is the solution of (1).

If we are sufficiently far away from phase transition, then under P_∞ $N^{-\frac{1}{2}}(\mathbf{Q}(\mathbf{x}) - \mathcal{E}_\infty(\mathbf{Q}(\mathbf{x}))) \xrightarrow{\mathcal{D}} \mathbf{N}(\mathbf{0}, \mathbf{D}^2H)$, see e.g. Dobrushin and Tirozzi (1977), Section 1.3, or Künsch (1982), Section 4. Clearly this implies that under P_∞ $N^{\frac{1}{2}}(\hat{\mathbf{a}} - \mathbf{a}) \xrightarrow{\mathcal{D}} \mathbf{N}(\mathbf{0}, (\mathbf{D}^2H)^{-1})$.

What happens if we replace P_∞ by the sequence of laws P_Λ ? From (3) we obtain

$$(4) \quad DH(\hat{\mathbf{a}})_\nu - DH(\mathbf{a})_\nu = \alpha_\nu N / |\mathcal{C}_\nu| \cdot N^{-1} (Q_\nu(\mathbf{x}) - \mathcal{E}_\Lambda(Q_\nu(\mathbf{x}))) \\ + \alpha_\nu / |\mathcal{C}_\nu| \sum_{\theta \in \mathcal{C}_\nu} \left(\mathcal{E}_\Lambda \left(\prod_{i \in \theta} x_i \right) - \mathcal{E}_\infty \left(\prod_{i \in \theta} x_i \right) \right).$$

The asymptotic normality of the first term follows from Theorem 1 of Pickard. In order to estimate the second term, we need to know how fast $\mathcal{E}_\Lambda(\prod_{i \in \theta} x_i)$ converges to $\mathcal{E}_\infty(\prod_{i \in \theta} x_i)$ for a fixed clan θ . But at least if the interaction is weak enough, it follows

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from Corollary 2.4 of Künsch (1982) that this convergence is exponential, i.e.

$$(5) \quad \left| \mathcal{E}_\Lambda \left(\prod_{i \in \theta} x_i \right) - \mathcal{E}_\infty \left(\prod_{i \in \theta} x_i \right) \right| \leq C_1 \exp(-C_2 d(\theta, \Lambda^c))$$

where $d(\theta, \Lambda^c)$ denotes the distance between θ and the region outside Λ . From (5) it follows that the second term in (4) is of the order $O(N^{-1})$ and therefore the asymptotic behaviour of $\hat{\mathbf{a}}$ is the same under P_Λ as under P_∞ .

It is now clear that we can construct confidence ellipsoids and acceptance regions at an asymptotically correct level, provided \mathbf{DH} and \mathbf{D}^2H are known and we use the solution of the equations (1).

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