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CONSTRUCTIONS AND APPLICATIONS OF RIGID SPACES III

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Introduction. One often encounters problems that are difficult as they are, but become manageable when translated to a different category. Thus very often, problems on Boolean algebras are answered by first transferring them to problems on Boolean spaces. (See, for example, [**7**]). It is with this spirit that we approach in this paper two problems on Boolean algebras. These problems are two decades old, and are considered to be outstanding problems in the field. We solve them completely by making use of the results of [**4**] and [**5**].

The numbering system in this paper is a continuation of that in our two papers [**4**] and [**5**]. The results in these two papers will be frequently quoted here, with due reference to the sections in which they occur. Both the major results of this paper have been announced by us in [**6**] in the year 1971.

4.1. Cardinality of rigid Boolean algebras. A Boolean algebra is said to be *rigid* if it admits no automorphism different from the identity map. G. Birkhoff [**5**, Problem 74] asks: Does there exist a Boolean algebra without any proper automorphism? This question has been answered in the affirmative by M. Katetov [**7**] and later by several others. In this connection, J. DeGroot and R. H. McDowell [**3**] ask the following: Do there exist rigid Boolean algebras of arbitrarily large cardinality? This again has been answered in the affirmative, by F. W. Lozier [**8**] and later we have proved a stronger theorem (see § 3.4.). Going still further, J. DeGroot asks in [**2**] the following question: What can we say about the cardinalities of rigid Boolean algebras? We answer this question in this section.

Here we consider the following set-theoretic axiom GCH: For each infinine cardinal m, it is true that $m^+ = 2^m$.

THEOREM 4.1.1. *Assume* GCH. *Let m be any uncountable cardinal number. Then there exists a rigid Boolean algebra with cardinality m.*

Proof. Case 1: Let there exist a cardinal number n such that $m = 2^n$.

Let X be the space constructed by c-process (see chapter 1 of [**4**] for the definition of this and related terms) from a c-system satisfying the following conditions:

(i) Each base is got from the sum of two copies of a maximal non-discrete space of cardinality n, by identifying the two limit points; and

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(ii) no two distinct base spaces are homeomorphic.
(See § 3.4 in [**5**] for the existence of such a system).

Let βX be the Stone-Čech compactification of X. Then βX is a rigid space. (See § 3.4 in [**5**] for a proof of this assertion.)

Let $B(X)$ be the Boolean algebra of all clopen (that is, both open and closed) subsets of βX. Then it is well-known that the automorphism group of $B(X)$ is isomorphic to the homeomorphism group of βX. Hence $B(X)$ is a rigid Boolean algebra.

We now compute the cardinality of $B(X)$. This is easily done by successively showing that the following spaces have at least 2^n clopen subsets:

(i) any maximal non-discrete space Y of cardinality n;
(ii) any base space of X;
(iii) the space X;
(iv) the space βX.

If Y is a maximal non-discrete space of cardinality n, it has a unique accumulation point y_0. If A is any subset of $Y/\{y_0\}$, then one and only one of the two sets A and $Y/\{y_0\}/A$ is clopen in Y. If follows that among subsets of $Y/\{y_0\}$, there are as many sets clopen in Y, as there are sets non-clopen in Y. Noting that n is infinite (since m is infinite) and that $Y/\{y_0\}$ has exactly 2^n subsets, we get that Y has at least 2^n clopen subsets.

Let Z be the space obtained from the sum of two copies of Y by identifying the two limit points. If we take a clopen subset A of one of the copies of Y such that A does not contain the unique accumulation point, then clearly A is clopen in Z also. Thus Z has at least 2^n clopen subsets.

Thus each base-space of X has at least 2^n clopen subsets. Take a clopen subset A of the first base-space and look at A^*. (This A^* is the set of all points of X lying above some point of A. See § 1.1 in [**4**].) Then A^* is a clopen subset of X. (See § 1.4. in [**4**].) Also, if A and B are distinct subsets of the first base space, then A^* and B^* are distinct. It follows that X has at least 2^n clopen subsets.

Since X satisfies some special conditions (see § 2.4. in [**4**]), βX is zero-dimensional. It is well-known that the map $W \rightarrow W \cap X$ is a bijection between the family of all clopen subsets of βX and the family of all clopen subsets of X. Hence X and βX have the same number of clopen subsets.

Thus the cardinality of $B(X)$ is at least 2^n. On the other hand, since X has cardinality n, it has at most 2^n (clopen) subsets; therefore so does βX. It follows that the cardinality of $B(X)$ is exactly 2^n, which is the same as m.

Case 2: Let Case 1 not hold. Then GCH implies that m is a limit cardinal, that is, one without a predecessor.

Now we proceed to construct a topological space. Let n be an infinite isolated cardinal number (that is the one having a predecessor, and hence by GCH, of the form 2^p for some cardinal p) less than m. Let D be a discrete space of cardinality n and let βD be its Stone-Čech compactification. Observe that in

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$\beta D/D$, the set F of limit points of subsets of smaller (than n) cardinality, has a smaller (than $|\beta D/D|$ cardinality). If we choose points p in $\beta D/D$ that are not in F then the space $D \cup \{p\}$ with the relative topology (from βD), is a maximal non-discrete topological space, with density character n. With this special choice of maximal non-discrete spaces, we can employ the method described earlier (in the proof of case 1) to construct a zero dimensional Hausdorff rigid space X_n of cardinality n. Then every point of X_n will have tightness n. (The *tightness* at a point of a topological space X is by definition the smallest cardinality n_0 such that whenever $A \subset X$ and $X \in \bar{A}$ there is $B \subset A$ such that $x \in \bar{B}$ and $|B| \leqq n_0$.)

Thus for each isolated $n < m$, choose a space X_n having the following properties:

(i) X_n is a zero dimensional Hausdorff rigid space.

(ii) X_n has cardinality n, tightness n, and has 2^n clopen subsets.

Let X be the one-point compactification of the disjoint sum of the Stone-Čech compactification βX_n of these space X_n. Then

$$X = [\oplus\{\beta X_n \mid n < m; n \text{ isolated}\}] \cup \{\infty\}$$

where ∞ is the extra point in the one point compactification. when X is clearly a zero-dimensional compact Hausdorff space. The following facts are needed for the later claims:

(1) The density character of βX_n, is n.

(2) If W is any clopen subset of βX_n, then the density character of W is n.

Since (2) can be proved exactly as (1), we sketch a proof of (1) alone. Let C be the set of clopen subsets of βX_n. Let D be a dense subset of X_n. Let $\mathscr{P}(D)$ be the power set of D. Then $W = \overline{W \cap D}$ for all $W \in C$. Consequently the map $W \to W \cap D$ from C into $\mathscr{P}(D)$ is one-to-one. Therefore D has at least 2^n subsets (since βX_n has at least 2^n clopen subsets). It follows that the cardinality of D is at least n.

Now we claim that X is rigid. Let $h: X \to X$ be a homeomorphism. Let there exist two points x, y in X such that $x \in \beta X_{n1}$, $y \in \beta X_{n2}$, $n_1 \neq n_2$ and $h(x) = y$. Then there are clopen neighbourhoods W_1 of x and W_2 of y such that $W_1 \subset \beta X_{n1}$, $W_2 \subset \beta X_{n2}$ and $h(W_1) = W_2$. Now the density character of W_1 is n_1, whereas that of W_2 is n_2. (By fact (2) noted above.) This contradicts the fact that h is a homeomorphism. Thus we have proved that h cannot take a point of βX_{n1} to a point of βX_{n2} unless $n_1 = n_2$. Next, we claim that no point of $X/\{\infty\}$ can be mapped to ∞ by h. If possible let $x \in X/\{\infty\}$ be such that $h(x) = \infty$. There is a unique n_1 such that $x \in \beta X_{n1}$. Since $X/\beta X_{n1}$ is a neighbourhood of ∞, one can find clopen neighbourhoods W_1 of x and W_2 of ∞ such that $W_1 \subset \beta X_{n1}$, $W_2 \subset X/\beta X_{n1}$ and $h(W_1) = W_2$. This implies that h takes some points of βX_{n1} to points of βX_{n2} with $n_1 \neq n_2$. This has already been proved to be impossible. Hence our claim is proved. Combining all these, we conclude that $h(\beta X_n) \subset \beta X_n$ for every n. Since h is onto, it then follows that $h(\beta X_n) = \beta X_n$ for each n. Since βX_n is rigid for each n, we have that h is identity on each βX_n and hence on the whole of X. Thus X is rigid.

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Now we introduce the following notations to compute the number of clopen subsets of X. Let n be a fixed isolated infinite cardinal $< m$. Then

A_n = the set of all isolated cardinals $< n$.
$\bar{F}_n$ = the family of all clopen subsets of X_n.
Y_n = the disjoint union of all X_p's with p in A_n.
$\bar{F}_{n]}$ = the family of all clopen subsets of X contained in Y_n.
$(p(Y_n))^{A_n}$ = the set of all functions from A_n to the set of all subsets of Y_n.
$\bar{F}_1$ = the family of all clopen subsets of X not containing ∞.
$\bar{F}$ = the family of all clopen subsets of X.

The following are easily noted for each n in A_m:

(i) $|X_n| = n$.
(ii) $|Y_n| = \text{Sup}_{p \leqq n}|X_p|$.
(iii) $|A_n| \leqq n$.
(iv) $|(p(Y_n))^{A_n}| \leqq (2^n)^n$ (by the above three facts) $= 2^n$.

If V is any member of $\bar{F}_{n]}$ we define $f_V \in (p(Y_n))^{A_n}$ by the rule $f_V(p) = V \cap X_p$ for each p in A_n. Then obviously the map $V \to f_V$ is one-to-one. Hence we have

(v) $|\bar{F}_n| \leqq (p(Y_n))^{A_n}$.

Now every clopen set not containing ∞ meets only a finite number of X's and hence is contained in Y_n for some n in A_m. In other words:

(vi) $\bar{F}_1 \subset \bigcup_{n \in A_m} \bar{F}_{n]}$

Now

$$\begin{aligned}
|\bar{F}_1| &\leqq \sum_{n \in A_m} |\bar{F}_{n]}| \quad \text{by (vi)} \\
&\leqq \sum_{n \in A_m} (p(Y_n))^{A_n} \quad \text{by (v)} \\
&= \sum_{n \in A_m} 2^n \quad \text{by (iv)} \\
&\leqq \sum_{n \in A_m} m \quad (\text{since } 2^n < m \text{ for each } n \text{ in } A_m) \\
&\leqq m.m \quad \text{by (iii)} \\
&= m.
\end{aligned}$$

Thus we have

(vii) $|F_1| \leqq m$.

Finally, if V is any clopen subset of X, either $V \in F_1$ or its complement $\in F_1$. In other words

(viii) $F \subset F_1 \cup \{V \subset X \mid X/V \in \bar{F}_1\}$.

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Therefore

$$|F| \leqq |F_1| + |F_1| \leqq m + m \quad \text{(by (vii))} = m.$$

Thus

(ix) $|F| \leqq m$.

On the other hand, for each n in A_m, $F_n \subset F$ and therefore

$$|\bar{F}| \geqq \sup_{n \in A_m} |\bar{F}_n|$$

$$= \sup_{n \in A_m} 2^n \quad \text{(by what we have proved in case 1)}$$

$$= m.$$

Thus

(x) $|F| = m$.

Now the proof of the theorem is complete, by the observation that F is a Boolean algebra under usual operations and has the same automorphism group as X.

COROLLARY 4.1.2. *Assume GCH. Let m be a cardinal number. Then there exists a rigid Boolean algebra with cardinality m if and only if either $m \leqq 2$ or m is uncountable.*

Proof. If $m \leqq 2$, then any Boolean algebra of cardinality m is easily seen to be rigid. If m is uncountable, the above theorem applies. Conversely, let m be a cardinaly number such that there is a rigid Boolean algebra of cardinality m. If m is finite and > 2, then $m = 2^n$ for some positive integer $\geqq 2$ and the Boolean algebra corresponding to it (namely, the power set of a set having n elements) is easily seen to be nonrigid. If m is countable, and if B is the rigid Boolean algebra corresponding to it, then its Stone-space X cannot be finite (since then B would be finite), nor can it be uncountable (since then B would also be so). X is therefore a countable compact Hausdorff space. It therefore has plenty of isolated points (this is a consequence of Baire category theorem), contradicting the fact that X is rigid.

4.2. Rigid σ-complete Boolean algebras. While answering Birkhoff's problem 74, Katetov [**7**] asks whether there exist σ-complete Boolean algebras without any nontrivial automorphism. The purpose of the present section is to show that such Boolean algebras exist in plenty.

THEOREM 4.2.1. *Every Boolean algebra can be embedded in a rigid σ-complete Boolean algebra.*

Proof. Step 1: Let us start the proof by looking at a special kind of Stone space. Consider the spaces constructed by c-process in § 2.1 of [**4**]. To recall,

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each of the base spaces is of the form P_m for some infinite cardinal number. Here P_m denotes the set of all ordinal numbers not exceeding the initial ordinal of m with the topology that is the join of the following two topologies:

(i) the usual order topology.

(ii) the smallest topology in which every subset of cardinality $< m$, is closed. Further distinct base spaces, by their choice, have distinct cardinalities. Let us make an extra requirement that each base space is uncountable.

We have proved in § 2.1 of [**4**] that such spaces are zero-dimensional Hausdorff spaces. Let X be one such space. Look at βX, its Stone-Čech compactification. We have already shown in § 2.1 of [**4**] that for every x in X, $X/\{x\}$ is not c^*-embedded in βX and hence that βX is also rigid.

Now consider the Boolean algebra $B(X)$ of all clopen subsets of βX. Clearly $B(X)$ is also rigid for automorphisms. We claim that $B(X)$ is σ-complete. Since clopen subsets of X are precisely the intersections of those of βX with X, we may regard $B(X)$ as the Boolean algebra of all clopen subsets of X. To show that $B(X)$ is σ-complete, we therefore show that if $V_1, V_2, \ldots, V_n, \ldots$ is a sequence of clopen subsets of X, then there is a largest clopen set contained in each of them; in fact, we prove that $\bigcap_{n=1}^{\infty} V_n$ is itself clopen.

Let us denote by (P) the property that the intersection of a countable number of clopen sets is clopen. We make the following observations:

(1) If m is uncountable, P_m has (P). For let $V_1, V_2, \ldots, V_n, \ldots$ be a countable sequence of clopen subsets of P_m and let W be their intersection. If the unique limit point is not in W, then W is obviously open. If the unique limit point is in W, then it is in each V_n; therefore the cardinality of $P_m \backslash V_i$ is less than m; therefore $P_m \backslash W$ has cardinality $< m$; therefore W is open. The closedness of W follows from the fact that it is the intersection of closed sets $\{V_i\}$.

(2) The property (P) is preserved by sums. That is, if $X = \bigoplus_{\alpha w J} X_\alpha$ is a disjoint sum of topological spaces and if each X_α has (P), then X has (P) (where J is any set).

(3) The property (P) is preserved by quotients. That is, if $f: X \rightarrow Y$ is a quotient map and if X has (P), then Y has (P).

It follows from (2) and (3) and Remark 1.4, that (P) is preserved by c-process. Therefore it follows from (1) that the space X constructed above has (P).

Thus $B(X)$ is a σ-complete rigid Boolean algebra.

Step 2: Recall that in the choice of cardinal numbers in the construction of X discussed above, we have plenty of freedom but for some minor conditions. In particular, we can choose then as large as we please. We fix an uncountable cardinal number m_0. and construct a space X_{m_0}, exactly as above, with the

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following single extra conditions: the first base space P_{m_0} chosen has car dinality $m_1 > m_0$.

Let A be an initial segment of P_{m_1} having cardinality m_0. For each subset B of A, consider the subset B^* of X_{m_0}, namely the set of all points in X_{m_0} that lie above some element of B. Since A is discrete, open and closed in P_{m_1}, it follows that each such B^* is a clopen subset of X_{m_0}. Further, the map $B \rightarrow B^*$ from $\mathscr{P}(A)$ (where $\mathscr{P}(A)$ is the power set of A) to $B(X_{m_0})$ can be checked to be a Boolean algebra isomorphism (not onto) in the following sense: it preserves unions, intersections and all relative complements.

Thus it is possible to embed the Boolean algebra 2^{m_0} in $B(X_{m_0})$.

Step 3: Let B be any Boolean algebra. Then it is well-known (see [**1**]) that B can be embedded in 2^{m_0} for some m_0. It follows from Step 2 that B can be embedded in the rigid σ-complete Boolean algebra $B(X_{m_0})$.

Remark 4.2.2: (a) In our notion of embedding of Boolean algebras, the bound elements 0 and 1 need not be preserved.

(b) Our methods in fact prove the following stronger result: Let m be any infinite cardinal number. Call a Boolean algebra m-complete if any collection of its elements, having cardinality $< m$, has infimum and supremum. (Thus σ-completeness is the same as $\aleph_0$-completeness.) Then there are plenty of m-complete rigid Boolean algebras, however large this m may be.

To prove this assertion, we have only to require that each base space has cardinality $> m$; for the rest, we can imitate the proof of the theorem.

(c) In a private communication in 1976, J. D. Monk has informed us that he has shown that given a cardinal $m > \aleph_0$ there are exactly 2^m isomorphism types of rigid Boolean algebras of power m.

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