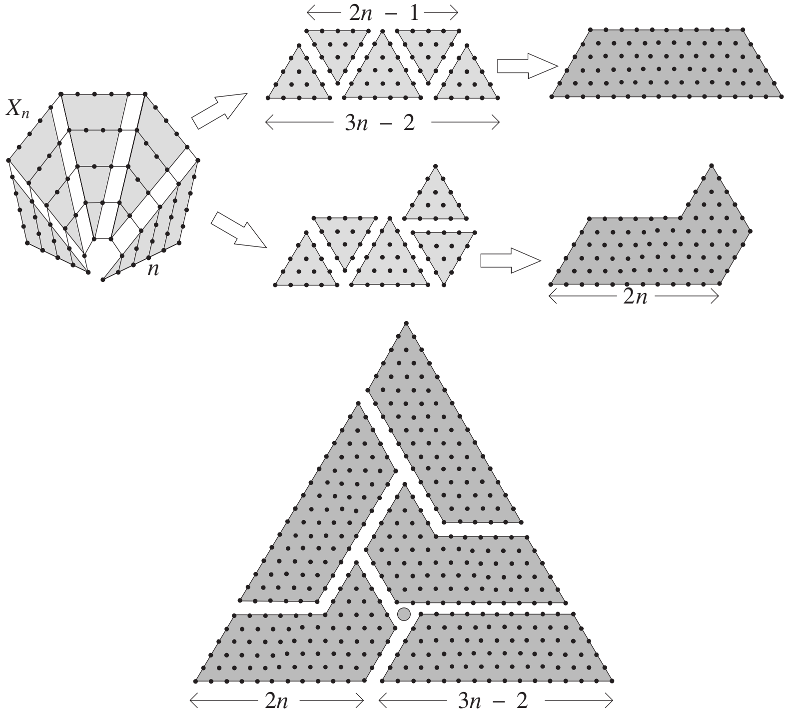
Proof: The proof is demonstrated for $n = 6$.



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108.25 A pair of interesting inequalities for e^x

A very interesting inequality for the number e was proved in [1] in three different ways. This inequality is: $e < \left(1 + \frac{1}{n}\right)^{n + \frac{1}{2}}$ for $n \geqslant 1$. The complete inequality can be stated as

$$\left(1 + \frac{1}{n}\right)^{n} < e < \left(1 + \frac{1}{n}\right)^{n + \frac{1}{2}}, n \geqslant 1. \quad (1)$$

The purpose of this Note is to provide similar inequalities for the function e^x. In fact, the intended inequality can be stated as follows.

For any real and positive x and y we have

$$\left(1 + \frac{x}{y}\right)^{y} < e^x < \left(1 + \frac{x}{y}\right)^{\frac{1}{2}x + y}, x > 0, y > 0. \quad (2)$$

Rewritten in the form of a double sequence, (2) becomes

$$\left(1 + \frac{n}{m}\right)^m < e^n < \left(1 + \frac{n}{m}\right)^{\frac{1}{2}n + m}, m > 0, n > 0. \tag{3}$$

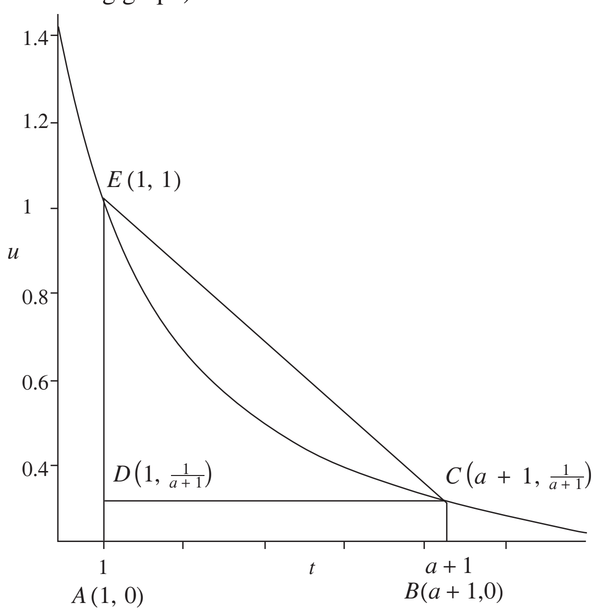
We first prove (2). Here is a simple proof.

Proof: Let $t = \frac{x}{y}, t > 0$, hence (2) becomes $1 + t < e^t < (1 + t)^{\frac{1}{2}t + 1}$. The left-hand inequality is well known. To see the right-hand inequality let $g(t) = \left(1 + \frac{1}{2}t\right)\ln(1 + t) - t$. Clearly $g(0) = 0$ and $g'(t) = \frac{1}{2}\left(\ln(1 + t) - \frac{t}{1+t}\right) > 0$, which implies that $g(t) > 0$ for $t > 0$, and (2) follows.

It is possible to give another interesting proof with a little generalisation. Here we follow the idea in [2]. Now let $t = \frac{y}{x} > 0$ so that (2) becomes

$$\left(1 + \frac{1}{t}\right)^t < e < \left(1 + \frac{1}{t}\right)^{\frac{1}{2} + t}, \tag{4}$$

Let $u = \frac{1}{t}$ and consider the graph of u on the t - u axes between the points $A(1,\ 0)$ and $B(a + 1,\ 0)$ for $a > 0$. Now consider three more points as $C\left(a + 1, \frac{1}{a+1}\right)$, $D\left(1, \frac{1}{a+1}\right)$ and $E(1,\ 1)$. Clearly, points C and E are on the graph. Now consider the rectangle A, B, C, D and the trapezoid A, B, C and E. (See the following graph):



Since the area under the curve, given by $\ln(1 + a) = \int_1^{1+a} \frac{1}{t}\,dt$, lies between that of the rectangle and the trapezoid, we have

$$\frac{a}{1 + a} \leqslant \ln(1 + a) \leqslant \frac{1}{2}a\left(1 + \frac{1}{a + 1}\right) = \frac{a(2 + a)}{2(1 + a)}. \tag{5}$$

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Now by abuse of notation we let again $u = \frac{1}{t}$ and write the right-hand expression in (4) as $R = (1 + u)^{\frac{1}{2} + \frac{1}{u}}$, and the left-hand expression as $L = (1 + u)^{\frac{1}{u}}$. Now we will use a little calculus to show (4). Since $u = \frac{1}{t}$, clearly, $L = \left(1 + \frac{1}{t}\right)^t$, and it is easily seen that

$$\frac{d}{dt}\left(\log L\right) = \log\left(1 + \frac{1}{t}\right) - \frac{1}{1 + t}.$$

Therefore, using the lower part of the inequality (5), $\log(1 + a) \geqslant \frac{a}{1 + a}$ and $a = \frac{1}{t}$, it follows that $\frac{d}{dt}(\log L) \geqslant 0$. Hence $L(u)$, $u = \frac{1}{t}$ is increasing in t, and increases to e. Now we will show the remaining part of (4). Recall that $u = \frac{1}{t}$ and $R(u) = R = \left(1 + \frac{1}{t}\right)^{\frac{1}{2} + t}$, so that one verifies

$$\frac{d\ \log R}{dt} = \log\left(t + \frac{1}{t}\right) - \frac{\frac{1}{2} + t}{t(1 + t)}. \tag{6}$$

Now using $a = \frac{1}{t}$ and the second part of (5) one checks that

$$\log\left(1 + \frac{1}{t}\right) \leqslant \frac{\frac{1}{2} + t}{t(1 + t)}.$$

Hence it follows from (6) that $\frac{d}{dt}\left(\log R\right) \leqslant 0$ implying $R' \leqslant 0$. Thus $R(u)$, $u = \frac{1}{t}$ is decreasing in t to e. Hence (4) is proved, and it verifies the truth of all the inequalities on the previous pages.

It is interesting to point out that (2) can be generalised to the effect that $\frac{1}{2}x$ in the right-hand side can be replaced by ηx where $\eta \geqslant \frac{1}{2}$. In fact, if $\eta > \frac{1}{2}$, and since (2) follows from (4), the assertion is obvious from

$$\left(1 + \frac{1}{t}\right)^{\eta t + 1} = \left(1 + \frac{1}{t}\right)^{(\eta - \frac{1}{2})t}\left(1 + \frac{1}{t}\right)^{\frac{1}{2}t + 1} > \left(1 + \frac{1}{t}\right)^{\frac{1}{2}t + 1}, t > 0, \eta > \tfrac{1}{2}.$$

However, the best value of η is $\frac{1}{2}$, and a value of $\eta < \frac{1}{2}$ does not satisfy the inequality.

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