J. Austral. Math. Soc. (*Series A*) **35** (1983), 338–348

A NOTE ON LINEAR FORMS IN A CLASS OF E-FUNCTIONS AND G-FUNCTIONS

XU GUANGSHAN

(Received 20 November 1981; revised 18 June 1982)

Communicated by J. H. Loxton

Abstract

This note establishes estimates for lower bounds of linear forms at algebraic points of E-functions and G-functions defined over a general algebraic number field.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 10 F 35, 10 F 10.

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In this note we consider linear forms in a class of E-functions and G-functions defined over an algebraic number field, obtaining lower bounds for the linear forms at algebraic points. Usually such lower bounds for linear forms depend on the maximum of the absolute values of the coefficients of the linear forms, as in Shidlovskii [3]; this note gives estimates that take into account the growth of each of the coefficients. There is similar work of Makarov [2], Galochkin [1], Väänänen [4], the author [6] and [7], but these papers consider E-functions and G-functions defined over the rational numbers or over an imaginary quadratic field. We provide a generalisation of [4], [6] and [7] to general algebraic number fields.

I should like to express my thanks to Professor A. J. van der Poorten for suggesting the problem dealt with here.

1. Notation, definition and results

As usual $\mathbf{C}$ is the field of complex numbers, $\mathbf{Z}$ the domain of rational integers and $\mathbf{Q}$ the field of rational numbers. We denote by $\mathbf{I}$ an imaginary quadratic field, $\mathbf{K}$ an algebraic number field over $\mathbf{Q}$ of degree h, $O_{\mathbf{K}}$ the domain of integers of $\mathbf{K}$,

https://doi.org/10.1017/S1446788700027038 Published online by Cambridge University Press

whilst $\mathbf{K}_i$ denotes a field conjugate to $\mathbf{K}$. If $\alpha \in \mathbf{K}$, we write $\alpha^* = \max_{1 \leqslant i \leqslant h} |\alpha_i|$, where $\alpha_i \in \mathbf{K}_i$ are the conjugates of α. Then $P(z)^*$ denotes the maximum of the absolute values of the coefficients of the polynomial $P(z)$ and its conjugates over $\mathbf{K}$.

We now define E-functions and G-functions. Let $C \geqslant 1$ be a constant, and $a_0, a_1, \ldots$ be a sequence of elements of $\mathbf{K}$ satisfying $a_l^* \leqslant C^l$, $l \geqslant 0$ and so that there is a sequence of natural numbers $q_0, q_1, \ldots$ with $q_l \leqslant C^l$, $l \geqslant 0$, such that $q_l a_i \in O_{\mathbf{K}}$, $0 \leqslant i \leqslant l$. Then the E-function

$$f(z) = \sum_{l=0}^{\infty} a_l z^l / l!$$

is said to belong to the class $\mathbf{K}E(C)$, and the G-function

$$g(z) = \sum_{l=0}^{\infty} a_l z^l$$

is said to belong to the class $\mathbf{K}G(C)$. We shall consider E-functions $\{f_{ij}(z)\}$ and G-functions $\{g_{ij}(z)\}$ as above, which satisfy the same defining system of differential equations

$$(1) \qquad y_{ij}' = Q_{ij0}(z) + \sum_{l=1}^{n_i} Q_{ijl}(z) y_{il}, \qquad i = 1, \ldots, k,\ j = 1, \ldots, n_i,$$

where $Q_{ijl}(z) \in \mathbf{K}(z)$. For $\kappa = 0, 1, \ldots$ we have

$$y_{ij}^{(\kappa)} = Q_{ij0\kappa}(z) + \sum_{l=1}^{n_i} Q_{ijl\kappa}(z) y_{il}, \qquad i = 1, \ldots, k, j = 1, \ldots, n_i,$$

with all the $Q_{ijl\kappa}(z) \in \mathbf{K}(z)$.

In the sequel $T(z) \in O_{\mathbf{K}}(z)$ will be supposed to satisfy the following conditions:

(i) In the case of E-functions, all the $T(z) Q_{ijl}(z) \in O_{\mathbf{K}}(z)$.

(ii) In the case of G-functions, all the $d_n/\kappa! [T(z)]^\kappa Q_{ijl\kappa}(z) \in O_{\mathbf{K}}(z)$, $\kappa = 1, \ldots, n$, $n = 0, 1, \ldots,$ where $\{d_n\}$ is a sequence of natural numbers satisfying $d_n \leqslant D^n$, $n = 0, 1, \ldots,$ with $D \geqslant 1$.

(iii) $T = \max_{i,j,l}(T(z)^*, (T(z) Q_{ijl}(z))^*)$.

(iv) $g = \max_{i,j,l}(\deg T(z), \deg T(z) Q_{ijl}(z))$.

Let $L = \sum_{i=1}^{k} n_i$, $n_0 = 1$, and let $r_1, \ldots, r_k$ be positive integers satisfying

$$r = r_0 = \max(r_1, \ldots, r_k) \geqslant 3.$$

Let x_{ij} $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i)$ be an arbitrary set of algebraic integers, not all zero, which belong to $O_{\mathbf{K}}$. Put

$$\bar{x}_i = \max_{1 \leqslant j \leqslant n_i} (1, x_{ij}^*), \qquad x = \max_{1 \leqslant i \leqslant k} (\bar{x}_i, x_{01}^*),$$

where x_{01} is any algebraic integer of $O_{\mathbf{K}}$.

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We denote linear forms in E-functions by

$$l(z) = x_{01} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} f_{ij}(z),$$

and linear forms in G-functions by

$$L(z) = x_{01} + \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} g_{ij}(z).$$

The linear forms we call $l_i(\xi_i)$ and $L_i(\xi_i)$ are obtained from the linear forms $l(\xi)$ and $L(\xi)$ by replacing ξ, x_{ij} and all the coefficients of $f_{ij}(z)$, $g_{ij}(z)$ with their conjugates (from $\mathbf{K}_i$).

We now summarise our additional notational conventions.

(i) In the case of E-functions, ξ is an algebraic number in $\mathbf{K}$, such that $\xi T(\xi) \neq 0$, and there is a natural number a such that $a\xi \in O_{\mathbf{K}}$. We write

$$H = \max(a, a\xi^*), \qquad B = 4C^2HT.$$

Let p denote the minimum of the orders of the zeros at $z = 0$ of all the functions $f_{ij}(z)$. Finally, we define two functions

$$c(r) = h(L+1)^2(g+\sigma+1)(\log B)^{1/2} r(\log r)^{1/2},$$
$$\mu(r) = e^{-2c(r)} r!$$

where σ is a constant depending only on the $f_{ij}(z)$. (It is defined explicitly in [7].)

(ii) In the case of G-functions, ω and τ will be positive numbers satisfying

$$(2) \qquad \omega > (g+2)L, \qquad \tau > \frac{(g+2)}{\omega - L(g+2)},$$

and the constants E and F are given by

$$(3) \qquad E = \gamma_0^{2\omega h}(3T\, De\, L\omega^2)^{1/\omega} C^{4(2\omega-1)Lh},$$

$$(4) \qquad F = E(1 + 2(1 - 1/\omega))^{1/\omega} C^{5(2\omega-1)}$$

where the constant γ_0 depends only on $g_{ij}(z)$, the system of differential equations and L.

THEOREM 1. *Let $\{f_{ij}(z)\}$ belong to the class $\mathbf{K}E(C)$. Suppose that the $f_{ij}(z)$ and* 1, *are linearly independent over $\mathbf{C}(z)$ and satisfy the system of differential equations* (1). *Put $h \geqslant 2$, noting that the theorem has been proved in* [7] *in the case $h = 1$. If r is the positive integer satisfying the inequality*

$$(5) \qquad \mu(r-1) \leqslant x < \mu(r),$$

then we have

$$(6) \qquad r \geqslant B^{4h^2(L+1)^4(g+\sigma+1)^2} + 1,$$

and

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \cdot \max_{1 \leqslant i \leqslant h} |l_i(\xi_i)| > e^{-2(L+1)c(r)}. \tag{7}$$

From Theorem 1, we obtain that

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \max_{1 \leqslant i \leqslant h} |l_i(\xi_i)| > x^{-12(L+1)^3 h(g+\sigma+1)(\log B/\log\log x)^{1/2}},$$

provided that $x \geqslant X^X$ where $X = B^{16h^2(L+1)^4(g+\sigma+1)^2}$ and when $\mathbf{K} = \mathbf{I}$, we obtain the results of [2] and [7].

THEOREM 2. *Let* $\{g_{ij}(z)\}$ *belong to the class* $\mathbf{K}G(C)$. *Suppose that the* $g_{ij}(z)$ *and* 1 *are linearly independent over* $\mathbf{C}(z)$ *and satisfy the system of differential equations* (1). *Let* q *be a natural number satisfying* $T(1/q) \neq 0$ *and also satisfying the following conditions*:

$$q > \max\{F^{1/(\tau/(1+L\tau)-(g+2)\omega)}, 4C\}. \tag{8}$$

Then we have

$$\max_{1 \leqslant i \leqslant h} |L_i(q^{-1})| > q^{-\lambda}\left(\prod_{i=1}^{k} \bar{x}_i^{n_i}\right)^{-(1+L\tau)(1+(g+1)\omega^{-1}+\log E/\log q)} \tag{9}$$

where λ *is an effectively computable constant.*

From Theorem 2, when $\mathbf{K} = \mathbf{I}$, we obtain a result similar to theorems of [4] and [6]; when $\mathbf{K} = \mathbf{I}$ and $n_i = 1$ $(i = 1, \ldots, k)$ we obtain a result similar to the theorem of [1].

2. Proof of Theorem 1

LEMMA 1. *Let* a_{ij} $(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, n > m)$ *be integers of* $\mathbf{K}$, *and let* A *be a positive number with* $\max_{i,j} a_{ij}^* \leqslant A$. *Then there are algebraic integers* $z_1, \ldots, z_n \in O_{\mathbf{K}}$ *with*

$$0 < \max_i z_i^* \leqslant A^{(h-1)n/(n-m)} h (2^{1/h} nhA)^{m/(n-m)} \tag{10}$$

such that

$$\sum_{i=1}^{n} a_{ij} z_i = 0 \qquad (1 \leqslant j \leqslant m).$$

This lemma is easily obtained from Lemma 1.22 of [5].

LEMMA 2. *Let $\omega = \omega(r)$ be a non-decreasing function of r, $2 \leqslant \omega(r) \leqslant r$ and let*

$$m = \sum_{i=0}^{k} n_i r_i + L - [r\omega^{-1}], \qquad n = \sum_{i=0}^{k} n_i r_i + L + 1,$$

$$M = (2C^2)^{h(L+1)^2\omega(r+1)^2/r}(2h)^{2(L+1)\omega+1}((L+1)(r+1))^{2(L+1)\omega}.$$

Then there are $L+1$ polynomials $P_{ij}(z) \in O_{\mathbf{K}}(z)$ $(0 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i)$ which do not all vanish identically and which have the following properties:

(i)

$$\begin{aligned} &\deg P_{ij}(z) \leqslant r, &&\operatorname{ord} P_{ij}(z) \geqslant r - r_i, \\ &P_{ij}(z)^* \leqslant r_i! 2^r M &&(0 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i). \end{aligned} \tag{11}$$

(ii) *Let*

$$\begin{aligned} F(z) &= \sum_{i=0}^{k} \sum_{j=1f}^{n_i} P_{ij}(z) f_{ij}(z) \qquad (\textit{where } f_{01}(z) \equiv 1) \\ &= \sum_{\nu=m}^{\infty} r!\sigma_\nu(\nu!)^{-1} z^\nu = \sum_{\nu=m}^{\infty} \rho_\nu z^\nu; \end{aligned}$$

then we have

$$\rho_\nu^* \leqslant (L+1) r! (\nu!)^{-1} (2C)^\nu M, \qquad \nu \geqslant m. \tag{12}$$

The proof of this lemma is similar to Lemma 2 of [7].

Let

$$F_0(z) = F(z), \quad F_\tau(z) = T(z)\frac{d}{dz}F_{\tau-1}(z), \qquad \tau = 1, 2, \ldots. \tag{13}$$

Then the system of differential equations (1) implies that

$$F_\tau(z) = \sum_{i=0}^{k} \sum_{j=1}^{n_i} P_{ij\tau}(z) f_{ij}(z),$$

where $P_{ij\tau}(z) \in O_{\mathbf{K}}(z)$. Further write

$$\begin{aligned} \tilde{f}_0(z) &= f_{01}(z), &\tilde{f}_\nu(z) &= f_{ij}(z), \\ \tilde{P}_{\tau 0}(z) &= P_{01\tau}(z), &\tilde{P}_{\tau\nu}(z) &= P_{ij\tau}(z), \end{aligned}$$

where $\nu = \sum_{l=0}^{i-1} n_l + j - 1$ $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n_i)$, and $\nu = 0$ if $i = 0$, $j = 1$.

https://doi.org/10.1017/S1446788700027038 Published online by Cambridge University Press

LEMMA 3. *Under the hypotheses of Theorem* 1, *let* $N_0 = L(L+1)(g+1)/2 + [r\omega^{-1}] + \sigma + 1$, $t = N_0 + p - \sigma$ *and suppose* $r^* = \min(r_i) > N_0$. *Then there exist* $(L+1)^2$ *algebraic integers* $q_{\tau\nu}(\xi)$ $(0 \leqslant \tau, \nu \leqslant L) \in O_{\mathbf{K}}$ *with the following properties*:

(i)

$$\det(q_{\tau\nu}(\xi))_{0 \leqslant \tau,\nu \leqslant L} \neq 0. \tag{14}$$

(ii) *For each pair* (τ, ν) *we have*

$$q^*_{\tau\nu}(\xi) \leqslant C_1 r_{i(\nu)}! \tag{15}$$

where $i = i(\nu)$ *satisfies* $\sum_{l=0}^{i-1} n_l \leqslant \nu \leqslant \sum_{l=0}^{i} n_l - 1$, *and*

$$C_1 = 2^r((L+t)g + r + L)^{L+t} T^{L+t} (2H)^{2r+2(L+t)g} M.$$

(iii) *For* $\tau = 0, 1, \ldots, L$, *we have*

$$\left| \sum_{\nu=0}^{L} q_{\tau\nu}(\xi) \tilde{f}_\nu(\xi) \right| \leqslant C_2 \prod_{i=1}^{k} (r_i!)^{-n_i}, \tag{16}$$

where

$$C_2 = (L+1)((L+t)g)^{L+t}(2H)^{2(L+t)g} H^{(L+2)r}(2T)^{L+t} \times ((L+1)r)^{2t}(1+C)^{(L+1)r} e^{2CH} M.$$

Of course the inequality (16) remains valid if we change ξ, $q_{\tau\nu}(\xi)$ and the coefficients of $\tilde{f}_\nu(z)$ to their conjugates in any of the fields $\mathbf{K}_i$.

The proof of this lemma is similar to Lemma 5 of [7]. Indeed we need only notice that it suffices to take

$$q_{\tau\nu}(\xi) = a^{r+(L+t)g} \tilde{P}_{J(\tau)\nu}(\xi)$$

where $0 \leqslant J(1) < \cdots < J(L) \leqslant L + t$.

We now prove Theorem 1. As in the proof of Theorem 1 of [7], r satisfying (5) must satisfy $r \geqslant B^{4h^2(L+1)^4(g+\sigma+1)^2} + 1$. Similarly, if we define the integers $r_1, \ldots, r_k$ by

$$(r_i - 1)! \leqslant e^{2c(r)} \bar{x}_i < r_i!, \qquad i = 1, \ldots, k,$$

then we may define the function $\omega(r)$ by the relation

$$\omega^{-1} = \omega(r)^{-1} = (L+1)(\log B / \log r)^{1/2}.$$

Clearly $2 \leqslant \omega \leqslant r$. As in the proof of Theorem 1 of [7] we can verify that $r^* > N_0$. According to Lemma 3, there is no loss of generality in assuming that, say

$$D(\xi) = \begin{vmatrix} x_{01} & x_{11} & x_{kn_k} \\ q_{10}(\xi) & q_{11}(\xi) & q_{1L}(\xi) \\ \cdots & \cdots & \cdots \\ q_{L0}(\xi) & q_{L1}(\xi) & q_{LL}(\xi) \end{vmatrix} \neq 0.$$

Let

$$R_\tau(\xi) = \sum_{\nu=0}^{L} q_{\tau\nu}(\xi)\tilde{f}_\nu(\xi), \qquad \tau = 1,\dots,L.$$

Thus $D(\xi)$ can be rewritten as

$$D(\xi) = \begin{vmatrix} l(\xi) & x_{11} & x_{kn_k} \\ R_1(\xi) & q_{11}(\xi) & q_{1L}(\xi) \\ \cdots & \cdots & \cdots \\ R_L(\xi) & q_{L1}(\xi) & q_{LL}(\xi) \end{vmatrix}.$$

Clearly $D(\xi) \in O_{\mathbf{K}}$. We denote by $D_i(\xi)$ the conjugates of $D(\xi)$ in $\mathbf{K}$, noting that the $D(\xi)$ can be obtained by simultaneously replacing all the elements of the determinant $D(\xi)$ by their conjugates from the field $\mathbf{K}_i$. Because $D(\xi) \neq 0$, we have $\prod_{i=1}^{h}|D_i(\xi_i)| \geqslant 1$, and thus, for some i, $|D_i(\xi_i)| \geqslant 1$.

Decomposing the determinant $D_i(\xi_i)$ according to its first column and by (15) and (16) of Lemma 3, we have

$$1 \leqslant |D_i(\xi_i)| \leqslant L!C_1^L \prod_{i=1}^{k} (r_i!)^{n_i}|l_i(\xi_i)| + L!C_1^{l-1}C_2 \sum_{i=1}^{k} \frac{n_i\bar{x}_i}{r_i!} = U + V,$$

and by the definitions of the r_i we have

$$\prod_{i=1}^{k} (r_i!)^{n_i} \leqslant r^L e^{2Lc(r)} \prod_{i=1}^{k} \bar{x}_i^{n_i}, \qquad \sum_{i=1}^{k} \frac{n_i\bar{x}_i}{r_i!} \leqslant Le^{-2c(r)}.$$

Then

$$2U \leqslant 2L!C_1^L r^L e^{2Lc(r)}|l_i(\xi_i)| \prod_{i=1}^{k} \bar{x}_i^{n_i},$$

$$2V \leqslant 2(L+1)!C_1^{L-1}C_2 e^{-2c(r)},$$

and in a manner similar to the computation in Theorem 1 of [7] we may verify that

$$2L!C_1^L r^L < e^{2c(r)}, \qquad 2(L+1)!C_1^{L-1}C_2 e^{-2c(r)} < 1,$$

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then

$$\prod_{i=1}^{k} \bar{x}_i^{n_i} \cdot \max_{1 \leqslant i \leqslant h} |l_i(\xi_i)| > e^{-2(L+1)c(r)},$$

proving Theorem 1.

3. Proof of Theorem 2

LEMMA 4. *Let* $N = \sum_{i=1}^{k} n_i r_i$, $m = r + N - [N\omega^{-1}] - 1$ *and* $n = r + N$ *where* ω *satisfies* (2). *Then there exist* $L + 1$ *polynomials* $p_{ij}(z) \in O_{\mathbf{K}}(z)$ $(i = 0, \ldots, k,$ $j = 1, \ldots, n_i)$ *not all identically zero with the following properties*:

(i)

$$\text{(17)} \qquad \deg p_{ij}(z) \leqslant r - 1, \qquad \operatorname{ord} p_{ij}(z) \geqslant r - r_i, \qquad p_{ij}(z)^* \leqslant C^{4(2\omega-1)hN}\gamma_1^{2\omega \log hN}$$

where $\gamma_1, \gamma_2, \ldots$ *are positive constants not depending on* N *or* ω.

(ii) *We have*

$$\text{(18)} \qquad F(z) = \sum_{i=0}^{k} \sum_{j=1}^{n_i} p_{ij}(z) g_{ij}(z) = \sum_{h=m}^{\infty} \rho_h z^h,$$

and for all $|z| < (2C)^{-1}$,

$$\text{(19)} \qquad |F(z)| \leqslant C^{5(2\omega-1)hN}\gamma_2^{2\omega \log hN}|z|^m.$$

The inequality (19) remains valid if we change all the coefficients of $F(z)$ to their conjugates in any of the fields $\mathbf{K}_i$.

Let

$$F_\kappa(z) = d_\kappa(\kappa!)^{-1}(T(z))^\kappa F^{(\kappa)}(z).$$

It follows from (1) that

$$F_\kappa(z) = P_{01\kappa}(z) + \sum_{i=1}^{k} \sum_{j=1}^{n_i} P_{ij\kappa}(z) g_{ij}(z).$$

Put

$$r_\kappa(z) = q^{r+\kappa g} F_\kappa(z) = C_{01\kappa}(z) + \sum_{i=1f}^{k} \sum_{j=1}^{n_i} C_{ij\kappa}(z) g_{ij}(z).$$

Clearly $P_{ij\kappa}(z)$ and $C_{ij\kappa}(z) \in O_{\mathbf{K}}(z)$.

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LEMMA 5. *Let* $r_i > [N\omega^{-1}] + \gamma_3$, $\theta = q^{-1}$, *where* q *is a natural number satisfying* $q > 2C$ *and* $T(q^{-1}) \neq 0$. *Then there exist* $L+1$ *numbers* $\kappa_0, \kappa_1, \dots, \kappa_L$, *such that*

$$\kappa_0 + \kappa_1 + \cdots + \kappa_L \leqslant [N\omega^{-1}] + \gamma_4$$

and such that the linear forms

$$r_{\kappa_0}(\theta), r_{\kappa_1}(\theta), \dots, r_{\kappa_L}(\theta)$$

are linearly independent. Further we have

$$\max_j C^*_{ij\kappa}(\theta) \leqslant (1-\theta)^{-2\kappa}(3DT)^{\kappa}\gamma_5^{2\omega\log hN} C^{4(2\omega-1)hN} q^{r_i+(g+1)\kappa}\left(\frac{er}{\nu(\kappa)}\right)^{\nu(\kappa)} \tag{20}$$

where $\nu(\kappa) = \max(1, \min(r, \kappa))$. *If* $N > 4\gamma_4$ *and* $q > 4C$, *then*

$$|r_{\kappa_i}(\theta)| \leqslant (1-\theta)^{-\kappa}(DT)^{\kappa}\gamma_6^{2\omega\log hN} C^{5(2\omega-1)hN} q^{-N+[N\omega^{-1}]+(g+1)\kappa+1} \tag{21}$$
$$\times \left(\frac{er}{\nu(\kappa)}\right)^{\nu(\kappa)}(1 + N/r(1-1/\omega))^{\kappa}$$

and (21) *remains valid if we change all the coefficients of* $g_{ij}(\theta)$ *and* $C_{ij\kappa}(\theta)$ *in* $r_{\kappa_i}(\theta)$ *to their conjugates in any of the fields* $\mathbf{K}_i$.

The proof of Lemmas 4 and 5 is similar to that of Lemmas 1 and 3 of [4].

We now prove Theorem 2. Let $w = \prod_{i=1}^{k} \bar{x}_i^{n_i}$, and

$$r_i = \left[\frac{\log \bar{x}_i w^{\tau}}{\log q}\right], \qquad i = 1, \dots, k.$$

First we assume $w \geqslant q^{\alpha}$ where

$$\alpha > \frac{2(g+2)(\gamma_4+2)}{\varepsilon} \tag{22}$$

and ε is a sufficiently small positive number. We can easily verify that r_i $(1 \leqslant i \leqslant k)$ and N satisfy the conditions of Lemma 5. According to Lemma 5 we may suppose that, say,

$$\Delta(\theta) = \begin{vmatrix} x_{01} & x_{11} & x_{kn_k} \\ C_{01\kappa_1}(\theta) & C_{11\kappa_1}(\theta) & C_{kn_k\kappa_1}(\theta) \\ \cdots & \cdots & \cdots \\ C_{01\kappa_L}(\theta) & C_{11\kappa_L}(\theta) & C_{kn_k\kappa_L}(\theta) \end{vmatrix} \neq 0.$$

But $\Delta(\theta)$ can be rewritten as

$$\Delta(\theta) = \begin{vmatrix} L(\theta) & x_{11} & x_{kn_k} \\ r_{\kappa_1}(\theta) & C_{11\kappa_1}(\theta) & C_{kn_k\kappa_L}(\theta) \\ \cdots & \cdots & \cdots \\ r_{\kappa_L}(\theta) & C_{11\kappa_L}(\theta) & C_{kn_k\kappa_L}(\theta) \end{vmatrix}.$$

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Clearly $\Delta(\theta) \in O_{\mathbf{K}}$. We denote by $\Delta_i(\theta)$ the conjugates to $\Delta(\theta)$, whence

$$\prod_{i=1}^{h} |\Delta_i(\theta)| \geqslant 1.$$

So, for some i, $|\Delta_i(\theta)| \geqslant 1$. On decomposing the determinant Δ_i according to its first column, (20) and (21) of Lemma 5 yield

$$(23)\qquad 1 \leqslant |\Delta_i(\theta)| \leqslant |L_i(\theta)| E^N q^{N+(g+1)(N\omega^{-1}+\gamma_4)} + F^N q^{(g+2)(N\omega^{-1}+\gamma_4+1)} \max_i (\bar{x}_i q^{-r_i}).$$

But by the definitions of r_i we have

$$(24)\qquad \max_i (\bar{x}_i q^{-r_i}) \leqslant q w^{-\tau}$$

and from (2), (8) we have

$$\tau > (g+2)(1+L\tau)/\omega - \varepsilon,$$

and

$$(25)\qquad q^{\tau-(g+2)(1+L\tau)\omega^{-1}-\varepsilon} > F^{1+L\tau}.$$

Thus if follows from (22) and (24) that

$$(26)\qquad F^N q^{(g+2)(N\omega^{-1}+\gamma_4+2)} w^{-\tau} \leqslant \tfrac{1}{2}.$$

Further from (23) and (25) we obtain

$$|L_i(\theta)| \geqslant \tfrac{1}{2} q^{-(g+1)\gamma_4} w^{-(1+L\tau)(1+(g+1)\omega^{-1}+\log E/\log q)}$$

and

$$\max_{1\leqslant i\leqslant h} |L_i(\theta)| \geqslant q^{-\lambda} \left(\prod_{i=1}^{k} \bar{x}_i^{n_i} \right)^{-(1+L\tau)(1+(g+1)\omega^{-1}+\log E/\log q)}.$$

If $w < q^{\alpha}$, the proof of Theorem 2 can be completed by applying the above assertion to the linear form $I(z) = q^{\alpha} L(z)$. This proves Theorem 2.

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Institute of Mathematics
Academia Sinica
Beijing
People's Republic of China

Author's current address:
School of Mathematics and Physics
Macquarie University
North Ryde, N.S.W. 2113
Australia