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TWO-WEIGHTED INEQUALITIES FOR SINGULAR INTEGRALS

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ABSTRACT. We consider operators T of the form $Tf = \{T_jf_j\}$, where $T_jf_j(x) = (p.v) \int_{R^n} k_j(x-y)f_j(y)\,dy$. Under appropriate conditions on the k_j, two-weighted estimates for T are obtained, the weights being radial and suitably linked.

In this paper we prove two-weighted inequalities for vector-valued singular integrals. The description of the class of weight functions that provide the validity of a one-weighted inequality for Hilbert transforms was given in [6]. Subsequent generalizations for singular Calderon-Zygmund integrals can be found in [3], [7] and other papers. In [1], [9] similar questions are treated for vector-valued singular integrals.

The solution of a two-weighted problem for singular integrals has turned out to be more difficult. This problem is solved in [4] for the case of monotone weights. The present paper deals with a more general case.

A measurable function $w: R^n \longrightarrow R^1$ which is positive almost everywhere is called a *weight function*; w is called *radial* if it is of the form $w(x) = f(|x|)$ for some f, and in such cases we shall for convenience often write $w(|x|)$ instead of the more correct $w(x)$. By $L^p_w(R^n)$ we denote the space of measurable functions $f: R^n \longrightarrow R^1$ with finite norm

$$\|f\|_{L^p_w} = \Big(\int_{R^n} |f(x)|^p w(x)\,dx\Big)^{\frac{1}{p}}.$$

Let us recall the definition of the Muckenhoupt class A_p. We say that $w \in A_p(R^n)$ $(1 < p < \infty)$, if

$$\sup \frac{1}{|Q|}\int_Q w(x)\,dx\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}(x)\,dx\Big)^{p-1} < \infty,$$

where the supremum is taken with respect to all balls Q in R^n.

Let $\mathbf{M}f$ denote the maximal function of a locally summable function $f: R^n \longrightarrow R^1$ defined by

$$\mathbf{M}f(x) = \sup \frac{1}{|Q|}\int_Q |f(y)|\,dy$$

where the supremum is taken with respect to all balls Q containing the point x.

THEOREM A [11]. *The operator $\mathbf{M}: f \longmapsto \mathbf{M}f$ is continuous in $L^p_w(R^n)$, $1 < p < \infty$, if and only if $w \in A_p(R^n)$.*

Further, we shall consider the convolution kernel $k(x)$ satisfying the conditions:

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i) $|\hat{k}(x)| \leq L, x \in R^n$;
ii) $|k(x)| \leq L|x|^{-n}, x \in R^n$;
iii) $|k(x-y) - k(x)| \leq \omega(\frac{|y|}{|x|})|x|^{-n}$ for $|y| \leq \frac{|x|}{2}$.

Here $\hat{k}$ denotes the Fourier transform of k, L is a constant and $\omega(t)$ is a nondecreasing function on $(0, \infty)$ such that $\omega(2t) \leq c\omega(t)$ and

$$\int_0^1 \frac{\omega(t)}{t}\,dt < \infty.$$

Suppose $\{k_j(x)\}$ is a sequence of convolution kernels satisfying the conditions i), ii), iii) with a uniform constant L and a fixed ω independent of j.

For $f = \{f_j\}$ let $Tf = \{T_j f_j\}$ where

$$T_j f_j(x) = (p.\,v)\int_{R^n} k_j(x-y) f_j(y)\,dy.$$

Next, for θ, $1 < \theta < \infty$, and a vector $\varphi = \{\varphi_j\}$ we put

$$|\varphi|_\theta = \Big(\sum_{j=1}^{\infty} |\varphi_j(x)|^\theta\Big)^{1/\theta}.$$

The following vector-valued one-weighted inequality for the operator T was proved in [1].

THEOREM B. *Let p, $\theta \in (1, \infty)$, $w \in A_p$. Then there exists a positive constant c such that the inequality*

$$\int_{R^n} |Tf|_\theta^p w(x)\,dx \leq c \int_{R^n} |f(x)|_\theta^p w(x)\,dx$$

holds for every f for which $|f|_\theta \in L_w^p$.

Our further discussion will deal with two-weighted estimates for the operator T with radial weights.

We introduce

DEFINITION 1. Let $1 < p < \infty$ and let $p' = p/(p-1)$. Denote by $a_p(n)$ the family of all pairs (h_1, h) of nonnegative measurable functions on $(0, \infty)$ which satisfy the condition

$$\sup_{t>0}\Big(\int_t^\infty h_1(\tau)\tau^{n-np-1}\,d\tau\Big)\Big(\int_0^{\frac{t}{2}} h^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1} < \infty;$$

$b_p(n)$ will denote the family of all pairs of functions (h_1, h) satisfying the condition

$$\sup_{t>0}\Big(\int_0^{\frac{t}{2}} h_1(\tau)\tau^{n-1}\,d\tau\Big)\Big(\int_t^\infty h^{1-p'}(\tau)\tau^{-1-\frac{n}{p-1}}\,d\tau\Big)^{p-1} < \infty.$$

We have

THEOREM 1. *Let $p, \theta \in (1,\infty)$, let σ and u be positive monotone functions defined on $(0,\infty)$, and suppose that the radial function $\rho(|x|) \in A_p$. We put $v = \sigma\rho$, $w = u\rho$; that is, $v(|x|) = \sigma(|x|)\rho(|x|)$ and $w(|x|) = u(|x|)\rho(|x|)$. If either σ and u are increasing and $(v,w) \in a_p(n)$, or σ and u are decreasing and $(v,w) \in b_p(n)$, then there exists a constant $c > 0$ such that the inequality*

$$\int_{R^n} |Tf(x)|_\theta^p v(|x|)\,dx \le c \int_{R^n} |f(x)|_\theta^p w(|x|)\,dx$$

holds whenever $|f|_\theta \in L^p_{w(|x|)}$.

To prove the theorem we shall use the following analogue of the well-known Hardy inequality and some simple lemmas.

THEOREM C. *Let $1 \le p \le q < \infty$ and let $\alpha(t)$, $\beta(t)$ be positive functions on $(0,\infty)$.*

i) The inequality

$$\Big(\int_0^\infty \alpha(t)\Big|\int_0^t F(\tau)\,d\tau\Big|^q dt\Big)^{q/p} \le c_1\Big(\int_0^\infty |F(t)|^p\beta(t)\,dt\Big)^{1/p}$$

with a constant c_1 independent of F holds if and only if the condition

$$\sup_{t>0}\Big(\int_t^\infty \alpha(\tau)\,d\tau\Big)^{p/q}\Big(\int_0^t \beta^{1-p'}(\tau)\,d\tau\Big)^{p-1} < \infty$$

is fulfilled.

ii) The inequality

$$\Big(\int_0^\infty \alpha(t)\Big|\int_t^\infty F(\tau)\,d\tau\Big|^q dt\Big)^{1/q} \le c_2\Big(\int_0^\infty |F(t)|^p\beta(t)\,dt\Big)^{1/p}$$

with a constant c_2 independent of F holds if and only if

$$\sup_{t>0}\Big(\int_0^t \alpha(\tau)\,d\tau\Big)^{p/q}\Big(\int_t^\infty \beta^{1-p'}(\tau)\,d\tau\Big)^{p-1} < \infty.$$

For $1 \le p = q < \infty$ the above proposition is proved in [12], which also contains information on previous work in this direction. For subsequent generalizations see [2], [8], [10].

LEMMA 1. *Let $w = u\rho$ where $\rho(|x|) \in A_p$ for some p, $1 < p < \infty$, $u(t)$ increases on $(0,\infty)$ and*

$$\int_0^t w^{1-p'}(\tau)\tau^{n-1}\,d\tau < \infty$$

for each $t > 0$. Suppose the kernel k satisfies the conditions i), ii) and iii).

Then the singular integral

$$T\varphi(x) = \int_{R^n} k(x-y)\varphi(y)\,dy$$

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exists almost everywhere in R^n for any $\varphi \in L^p_{w(|x|)}(R^n)$.

PROOF. Fix arbitrarily a number $\alpha > 0$. Suppose $S_\alpha = \{x : |x| > \frac{\alpha}{2}\}$, $\varphi_1(x) = \varphi(x) \cdot \chi_{S_\alpha}$ and $\varphi_2(x) = \varphi(x) - \varphi_1(x)$. Since $u(t)$ is increasing on $(0, \infty)$, we have

$$\int_{R^n} |\varphi_1(x)|^p \rho(|x|)\, dx = \int_{S_\alpha} |\varphi(x)|^p \rho(|x|)\, dx \leq c \int_{R^n} |\varphi(x)|^p w(|x|)\, dx.$$

Since $\rho \in A_p$, by virtue of Theorem B (in the scalar case) we conclude that $T\varphi_1$ exists almost everywhere on R^n. Now we shall show that $T\varphi_2(x)$ converges absolutely for all x provided that $|x| > \alpha$. Note that for $|x| > \alpha$ and $|y| \leq \frac{\alpha}{2}$ we have $|x - y| \geq |x| - |y| \geq \frac{\alpha}{2}$. Further, application of Hölder's inequality gives

$$|T\varphi_2(x)| \leq c \int_{R^n \setminus S_\alpha} \frac{|\varphi(y)|}{|x-y|^n}\, dy \leq \left(\frac{2}{\alpha}\right)^n \int_{R^n \setminus S_\alpha} \frac{|\varphi(y)| w^{\frac{1}{p}}(|y|)}{w^{\frac{1}{p}}(|y|)}\, dy$$

$$\leq \left(\frac{2}{\alpha}\right)^n \left(\int_{R^n} |\varphi(y)|^p w(|y|)\, dy\right)^{\frac{1}{p}} \left(\int_{R^n \setminus S_\alpha} w^{1-p'}(|y|)\, dy\right)^{\frac{1}{p'}} < \infty.$$

Since α may be chosen arbitrarily small we conclude that $T\varphi_2$ converges absolutely almost everywhere on R^n.

Therefore $T\varphi$ exists almost everywhere in R^n. ∎

LEMMA 2. *Let the radial function $\rho \in A_p$ for some p, $1 < p < \infty$ and suppose that $0 \leq c_1 < c_2 \leq c_3 < c_4$. Then we have the inequality*

$$\int_{c_3 t}^{c_4 t} \rho(\tau)\tau^{n-1}\, d\tau \leq c_0 \int_{c_1 t}^{c_2 t} \rho(\tau)\tau^{n-1}\, d\tau$$

with some constant c_0 independent of $t \in (0, \infty)$.

PROOF. We introduce the notation:

$$\Gamma = \{x : c_1 t < |x| < c_2 t\},$$
$$\Gamma_1 = \{x : c_3 t < |x| < c_4 t\}$$

and

$$B = \{x : |x| < c_4 t\}.$$

By virtue of the definition of a maximal function for an arbitrary $x \in \Gamma_1$ and of the function $\varphi \in L^p_\rho$,

$$\mathbf{M}\varphi(x) > \frac{1}{|B|} \int_B |\varphi(y)|\, dy\, \chi_{\Gamma_1}(x) \geq \frac{c}{|\Gamma|} \int_\Gamma |\varphi(y)|\, dy\, \chi_{\Gamma_1}(x). \tag{1}$$

Due to Theorem A we have

$$\int_{R^n} \left(\mathbf{M}\varphi(x)\right)^p \rho(x)\, dx \leq c \int_{R^n} |\varphi(x)|^p \rho(x)\, dx.$$

Then (1) implies the estimate

$$\int_{\Gamma_1} \left(\frac{1}{|\Gamma|} \int_\Gamma |\varphi(y)|\, dy\right)^p \rho(x)\, dx \leq c \int_{R^n} |\varphi(y)|^p \rho(y)\, dy.$$

The choice $\varphi(y) = \chi_\Gamma(y)$ in the above inequality shows that

$$\int_{\Gamma_1} \rho(x)\, dx \leq c \int_\Gamma \rho(x)\, dx,$$

which implies the validity of the desired inequality. ∎

LEMMA 3. *Let the pair of radial functions* $(v, w) \in a_p(n)$ *where* $v = \sigma\rho$, $w = u\rho$ *and* σ *and* v *increase on* $(0, \infty)$, $\rho(|x|) \in A_p$, $1 < p < \infty$. *Then there exists a constant* $c > 0$ *such that for all* $t > 0$,

$$\sigma(2t) \le cu(t).$$

PROOF. Obviously, due to the increase of the functions σ and u, we obtain

$$(2) \qquad \int_t^\infty \sigma(\tau)\rho(\tau)\tau^{n-np-1}\,d\tau \ge \sigma(t)\int_t^\infty \rho(\tau)\tau^{n-np-1}\,d\tau \ge \sigma(t)\int_t^{2t} \rho(\tau)\tau^{n-np-1}\,d\tau$$

and

$$(3) \qquad \Big(\int_0^{\frac{t}{2}} u^{1-p'}(\tau)\rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1} \ge \frac{1}{u(\frac{t}{2})}\Big(\int_0^{\frac{t}{2}} \rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1}.$$

By Hölder's inequality we have

$$(4) \qquad 1 \le ct^{-np}\int_t^{2t} \rho(\tau)\tau^{n-1}\,d\tau\Big(\int_t^{2t} \rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1}.$$

Further note that the definition of the class A_p shows that $\rho \in A_p$ implies $\rho^{1-p'} \in A_{p'}$. By virtue of (4), Lemma 2, inequalities (2), (3) and the condition $(v, w) \in a_p(n)$ we now conclude that the inequalities

$$\begin{aligned}\frac{\sigma(t)}{u(\frac{t}{2})} &\le c\frac{\sigma(t)}{u(\frac{t}{2})}t^{-np}\int_t^{2t} \rho(\tau)\tau^{n-1}\,d\tau\Big(\int_t^{2t} \rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1}\\ &\le c\frac{\sigma(t)}{u(\frac{t}{2})}\int_t^{2t} \rho(\tau)\tau^{n-np-1}\,d\tau\Big(\int_t^{2t} \rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1}\\ &\le c\frac{\sigma(t)}{u(\frac{t}{2})}\int_t^{2t} \rho(\tau)\tau^{n-np-1}\,d\tau\Big(\int_0^{\frac{t}{2}} \rho^{1-p'}(\tau)\tau^{n-1}\,d\tau\Big)^{p-1} \le c_1\end{aligned}$$

hold. ∎

PROOF OF THEOREM 1. First let σ and u be increasing. Suppose without loss of generality that the function $\sigma(t)$ can be represented as

$$\sigma(t) = \sigma(0) + \int_0^t \varphi(u)\,du,$$

where φ is a positive function. Then we shall have

(5)

$$\begin{aligned}\int_{R^n} |Tf(x)|_\theta^p v(|x|)\,dx &= \sigma(0)\int_{R^n} |Tf(x)|_\theta^p \rho(|x|)\,dx + \int_{R^n} |Tf(x)|_\theta^p \rho(|x|)\,dx\Big(\int_0^{|x|} \varphi(t)\,dt\Big)\,dx\\ &= I_1 + I_2.\end{aligned}$$

If $\sigma(0) = 0$, then $I_1 = 0$. If $\sigma(0) > 0$, then by Theorem B we obtain

$$(6) \qquad \begin{aligned}\sigma(0)\int_{R^n} |Tf(x)|_\theta^p \rho(|x|)\,dx &\le c\sigma(0)\int_{R^n} |f(x)|_\theta^p \rho(|x|)\,dx\\ &\le c_1\int_{R^n} |f(x)|_\theta^p \rho(|x|)\sigma(|x|)\,dx\\ &\le c_2\int_{R^n} |f(x)|^p \rho(|x|)u(|x|)\,dx.\end{aligned}$$

Next, change of the order of integration and use of Minkowski's inequality give

$$
\begin{aligned}
I_2 &= \int_{R^n} |Tf(x)|_\theta^p \rho(|x|)\Big(\int_0^{|x|} \varphi(t)\,dt\Big)\,dx = \int_0^\infty \varphi(\tau)\Big(\int_{|x|>\tau} |Tf(x)|_\theta^p \rho(|x|)\,dx\Big)\,d\tau \\
&\le c_3 \int_0^\infty \varphi(\tau)\Big(\int_{|x|>\tau}\Big|\int_{|y|>\frac{\tau}{2}} k_j(x-y)f_j(y)\,dy\Big|_\theta^p \rho(|x|)\,dx\Big)\,d\tau \qquad (7)\\
&\quad + c_3 \int_0^\infty \varphi(\tau)\Big(\int_{|x|>\tau}\Big|\int_{|y|<\frac{\tau}{2}} k_j(x-y)f_j(y)\,dy\Big|^p \rho(|x|)\,dx\Big)\,d\tau = I_{21} + I_{22}.
\end{aligned}
$$

Again application of Theorem B and Lemma 3 gives

$$
\begin{aligned}
I_{21} &= \int_0^\infty \varphi(\tau)\Big(\int_{|x|>\tau}\Big|\int_{R^n} k_j(x-y)f_j(y)\chi_{\{y:|y|>\tau/2\}}(y)\,dy\Big|_\theta^p \rho(|x|)\,dx\Big)\,d\tau \\
&\le c_\theta \int_0^\infty \varphi(\tau)\Big(\int_{|y|>\frac{\tau}{2}} |f(y)|_\theta^p \rho(|y|)\,dy\Big)\,d\tau = c_\theta \int_{R^n} |f(y)|_\theta^p \rho(|y|)\Big(\int_0^{2|y|} \varphi(\tau)\,d\tau\Big)\,dy \\
&\le c_\theta \int_{R^n} |f(y)|_\theta^p \rho(|y|)\sigma(2|y|)\,dy \le c_4 \int_{R^n} |f(y)|_\theta^p \rho(|y|)u(|y|)\,dy.
\end{aligned}
$$

Therefore

$$
I_{21} \le c_4 \int_{R^n} |f(y)|_\theta^p w(|y|)\,dy. \qquad (8)
$$

Further, the property ii) of the kernels k_j enables us to obtain the estimate

$$
\begin{aligned}
I_{22} &\le c_5 \int_0^\infty \varphi(t)\Big(\int_{|x|>\tau} \frac{\rho(|x|)}{|x|^{np}}\,dx\Big)\Big(\int_{|y|\le\frac{\tau}{2}} |f(y)|_\theta\,dy\Big)^p\,d\tau \qquad (9)\\
&= c_5 \int_0^\infty \varphi(2s)\Big(\int_{\gamma>2s} \frac{\rho(\gamma)}{\gamma^{np-n+1}}\,d\gamma\Big)\Big(\int_{|y|\le s} |f(y)|_\theta\,dy\Big)^p\,ds.
\end{aligned}
$$

By the hypotheses of the theorem we have

$$
\Big(\int_t^\infty \frac{\sigma(\tau)\rho(\tau)}{\tau^{1+n(p-1)}}\,d\tau\Big)\Big(\int_0^{\frac{t}{2}} \frac{\tau^{n-1}}{w^{p'-1}(\tau)}\,d\tau\Big)^{p-1} < c_6. \qquad (10)
$$

After change of order of integration we obtain

$$
\begin{aligned}
\int_{2t}^\infty \varphi(s)\Big(\int_{2s}^\infty \frac{\rho(\gamma)}{\gamma^{1+n(p-1)}}\,d\gamma\Big)\,ds &\le c_7 \int_{2t}^\infty \varphi(s)\Big(\int_{2s}^\infty \frac{\rho(\gamma)}{\gamma^{1+n(p-1)}}\,d\gamma\Big)\,ds \\
&= c_7 \int_{2t}^\infty \frac{\rho(\gamma)}{\gamma^{1+n(p-1)}}\Big(\int_{2t}^\gamma \varphi(s)\,ds\Big)\,d\gamma \\
&\le c_7 \int_{2t}^\infty \frac{\rho(\gamma)\sigma(\gamma)}{\gamma^{1+n(p-1)}}\,d\gamma.
\end{aligned}
$$

Therefore by (10) we have

$$
\int_{2t}^\infty \varphi(s)\Big(\int_{2s}^\infty \frac{\rho(\gamma)}{\gamma^{1+n(p-1)}}\,d\gamma\Big)\,ds\Big(\int_0^t \frac{\tau^{n-1}}{w^{p'-1}(\tau)}\,d\tau\Big)^{p-1} < c_8.
$$

Now, applying Theorem C to the right-hand side of (9) we find that

$$
I_{22} \le c_9 \int_{R^n} |f(x)|_\theta^p w(|x|)\,dx. \qquad (11)
$$

Finally, from (5), (6), (7), (8) and (11) we conclude that the theorem is valid.

When σ and u are decreasing functions, the proof is conducted in a similar manner; one should use only condition ii) of Theorem C. ∎

Let us consider a concrete singular integral, namely the Hilbert transform

$$Hf(x) = \int_{-\infty}^{\infty} \frac{f(y)}{x-y}\,dy.$$

In that case the conditions $a_p(1)$ and $b_p(1)$ are also necessary for the boundedness of the operator H from L^p_w to L^p_v. To be more precise, the following theorem is valid:

THEOREM 2. *Let $1 < p < \infty$. If the pair (v, w) satisfies the conditions of Theorem 1 for $n = 1$, then we have the inequality*

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(|x|)\,dx \le c\int_{-\infty}^{\infty} |f(x)|^p w(|x|)\,dx, \quad f \in L^p_w. \tag{12}$$

Conversely, if (12) is fulfilled, then $(v, w) \in a_p(1) \cap b_p(1)$.

PROOF. The first part of the theorem is a corollary of Theorem 1. Now let (12) be fulfilled: then by [13], $w^{1-p'} \in L\big((\alpha, \beta)\big)$ for arbitrary α and β, $0 < \alpha < \beta < \infty$. Fix arbitrarily α and t, $0 < \alpha < \frac{t}{2}$, and in (12) substitute the function

$$f(y) = \begin{cases} w^{1-p'}(y) & \text{for } \alpha < y < \frac{t}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$\int_{-\infty}^{\infty} |Hf(x)|^p v(|x|)\,dx \le c\int_{\alpha}^{\frac{t}{2}} w^{1-p'}(\tau)\,d\tau, \tag{13}$$

where the constant c does not depend on α and t.

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} |Hf(x)|^p v(|x|)\,dx &\ge \int_t^{\infty} \left| \int_{\alpha}^{\frac{t}{2}} \frac{w^{1-p'}(y)}{x-y}\,dy \right|^p v(|x|)\,dx \\ &\ge \int_t^{\infty} \frac{v(x)}{x^p}\,dx \Big(\int_{\alpha}^{\frac{t}{2}} w^{1-p'}(y)\,dy \Big)^p. \end{aligned} \tag{14}$$

Further, from (13) and (14) we obtain

$$\int_t^{\infty} \frac{v(x)}{x^p}\,dx \Big(\int_{\alpha}^{\frac{t}{2}} w^{1-p'}(y)\,dy \Big)^{p-1} \le c_1.$$

Making α tend to zero, we conclude that $(v, w) \in a_p(1)$.

Now fix arbitrarily t and β, $0 < t < \frac{\beta}{2}$, and in (12) substitute the function

$$f(y) = \begin{cases} \big(w(y)y\big)^{1-p'} & \text{for } 2t < y < \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Obtaining the estimates in the manner discussed above and making β tend to infinity, we find that $(v, w) \in b_p(1)$.

In what follows given any natural number m, Λ_m will denote the set of all measurable functions f for which

$$\int_{-\infty}^{\infty} |f(x)|(1+|x|)^m\,dx < \infty$$

and

$$\int_{-\infty}^{\infty} f(x)x^k\,dx = 0, \quad k = 0, 1, 2, \ldots, m.$$

We have

THEOREM 3. *Let $1 < p < \infty$. If the pair of functions (v, w) satisfies the condition of Theorem 1, then for arbitrary functions $f \in \Lambda_m$ for which $fP_m \in L^p_{w(|x|)}$ we have the inequality*

$$\int_{-\infty}^{\infty} |Hf(x)|^p |P_m(x)|^p v(|x|)\,dx \leq c\int_{-\infty}^{\infty} |f(x)|^p |P_m(x)|^p w(|x|)\,dx,$$

where $P_m(x)$ is an arbitrary polynomial with complex-valued coefficients of degree $m+1$ and the positive constant c is independent of f.

PROOF. The proof follows from Theorem 1 and the identity

$$P_m(x)Hf(x) = H(P_m f)(x), \quad f \in \Lambda_m. \qquad \blacksquare$$

We illustrate Theorems 1 and 2 by giving examples of distinct weights v and w for which these theorems hold.

EXAMPLE 1. Let $0 < \alpha \leq \beta < p-1$; define real-valued functions h_1 and h on $(0, \infty)$ by

$$h_1(t) = \begin{cases} t^{p-1} & \text{if } 0 < t \leq 1/2 \\ 2^{\alpha-p+1}t^\alpha & \text{if } 1/2 < t < \infty \end{cases}$$

and

$$h(t) = \begin{cases} t^{p-1}\log^p(1/t) & \text{if } 0 < t \leq 1/2 \\ 2^{\beta-p+1}t^\beta \log^p 2 & \text{if } 1/2 < t < \infty, \end{cases}$$

and define radial weights v, w by $v(|x|) = h_1(|x|)$, $w(|x|) = h(|x|)$. Routine calculations show that the pair (h_1, h) of increasing functions belongs to $a_p(1)\cap b_p(1)$. Thus Theorem 1 and the first part of Theorem 2 hold for the pair (v, w).

EXAMPLE 2. Here we let $0 < \beta \leq \alpha < p-1$, define h_1 and h by

$$h_1(t) = \begin{cases} 1/\big(t\log^p(1/t)\big) & \text{if } 0 < t \leq 1/2 \\ (2^{1-\alpha}/\log^p 2)t^{-\alpha} & \text{if } 1/2 < t < \infty, \end{cases}$$

and

$$h(t) = \begin{cases} t^{-1} & \text{if } 0 < t \leq 1/2 \\ 2^{1-\beta}t^{-\beta} & \text{if } 1/2 < t < \infty, \end{cases}$$

and define the radial weights v, w by $v(|x|) = h_1(|x|)$, $w(|x|) = h(|x|)$. Again it is easy to verify that the pair (h_1, h) of decreasing functions belongs to $a_p(1) \cap b_p(1)$, and that consequently Theorem 1 and the first part of Theorem 2 hold for the pair (v, w).

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