

QUANTISED AFFINE ALGEBRAS AND
PARAMETER-DEPENDENT R -MATRICES

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Let $U_q(\mathcal{G}^{(1)})$ be a quantised non-twisted affine Lie algebra with $U_q(\mathcal{G})$ the corresponding quantised simple Lie algebra. Using the previously obtained universal R -matrices for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$, explicitly spectral-dependent universal R -matrices for $U_q(A_1)$ and $U_q(A_2)$ are determined. These spectral-dependent universal R -matrices are evaluated in some concrete representations; well-known results for the fundamental representations are reproduced, and an explicit formula for the spectral-dependent R -matrix associated with the $V_{(3)} \otimes V_{(6)}$ module is derived, where $V_{(3)}$ and $V_{(6)}$ carry the 3- and 6-dimensional representations of $U_q(A_2)$, respectively.

1. INTRODUCTION

Quantum deformations of universal enveloping algebras, or for short, quantum algebras, are perhaps amongst the most important discoveries in recent years in mathematics and theoretical physics [8, 10]. The novelty of these algebras is that they have a quasitriangular Hopf algebra structure, that is to say each contains a canonical element R , called the universal R -matrix, which satisfies the spectral-independent quantum Yang-Baxter equation (QYBE). This equation plays a key role in applications to conformal field theories [1, 2] and knot theory [20, 23, 24]. Integrable models [9, 4, 22], on the other hand, involve spectral-dependent R -matrices which satisfy the spectral-dependent QYBE.

Since the works of Jimbo and Jones [11, 13], a central problem has been the construction of spectral parameter-dependent R -matrices using quantum group techniques [25, 7, 5]. There are two commonly used procedures to this end in the literature: one is the so-called “Yang-Baxterisation” process, the other is the “fusion” method. By “Yang-Baxterisation” we mean two seemingly different but essentially related methods: (i) one starts from a quantum simple Lie algebra, makes it affine, thus giving rise to Jimbo’s equations [11, 25], and then solves them; (ii) one begins with a braid group representation associated with a quantum simple Lie algebra and then tries to

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introduce a spectral parameter in such a way that the spectral-dependent QYBE is satisfied [13]. In practice, it is usually more convenient to use (i) combined with some features of (ii), as shown in [24, 7, 5]. Abelian Yang-Baxterisation, for cases where the decomposition of the tensor product of a representation with itself is multiplicity-free, has been extensively studied by many authors for small representations. The “fusion” method was invented to cope with larger representations, but it is not easy to apply in practice. Very few attempts have been made in the non-Abelian case, where the tensor product decomposition has finite multiplicities, because of the associated complexities. For these reasons, it is highly desirable to develop more effective and general methods of constructing solutions of the spectral-dependent QYBE.

We have presented recently a new way [26] of obtaining spectral-dependent R -matrices associated with simple quantum Lie algebras. The idea is essentially to reverse the above process. More precisely, we start from the universal R -matrix of a quantum affine algebra $U_q(\mathcal{G}^{(1)})$ and then evaluate it in the finite-dimensional loop representation $V(z)$ of $U_q(\mathcal{G}^{(1)})$ which is known to define also the representation $V \otimes \mathbb{C}(z, z^{-1})$ of the corresponding simple quantum Lie algebra $U_q(\mathcal{G})$. In this way, we introduce a spectral parameter z automatically and obtain a spectral parameter-dependent solution of the QYBE. Our approach is an extension of that initiated by Khoroshkin and Tolstoy [15], who considered the simplest case of the fundamental representation of $U_q(A_1)$ which has a classical analogue [3, 21]. A remarkable advantage of this approach is that it is totally irrelevant whether or not the tensor product decomposition is multiplicity-free, and whether or not the representations being tensored are the same or different.

Moreover, considering different gradations of the quantum affine Lie algebras, our approach leads to quantum R -matrices which obey the QYBE but not the intertwiner property and thus cannot be *directly* obtained by the Yang-Baxterisation and fusion methods. These results will be published elsewhere [6].

The present paper continues the investigation in the authors’ previous papers [26, 6] to other interesting cases. It is hoped that the explicit R -matrix formulae derived in this paper will turn out to be useful in physical applications.

The paper is structured in the following fashion. In Sections 1 and 2 we give an account of the necessary background. In Section 3, we present the universal R -matrix with explicit spectral dependence for $U_q(A_1)$ and $U_q(A_2)$. In Section 4, we evaluate the spectral-dependent universal R -matrix in some concrete representations and reproduce some well-known results. We also obtain a quite explicit formula for the spectral-dependent R -matrix on the module $V_{(3)} \times V_{(6)}$ of $U_q(A_2)$. Some concluding remarks are made in Section 5, while some technical details are relegated to the Appendix.

2. QUANTUM AFFINE LIE ALGEBRAS

We start with the definition of the non-twisted quantum affine Lie algebra $U_q(\mathcal{G}^{(1)})$. Let $A^0 = (a_{ij})_{1 \leq i, j \leq r}$ be a symmetrisable Cartan matrix. Let \mathcal{G} stand for the finite-dimensional simple Lie algebra associated with the symmetric Cartan matrix $A_{\text{sym}}^0 = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j)$, $i, j = 1, 2, \dots, r$, where r is the rank of \mathcal{G} . Let $A = (a_{ij})_{0 \leq i, j \leq r}$ be a symmetrisable, generalised Cartan matrix in the sense of Kac [14]. Let $\mathcal{G}^{(1)}$ denote the non-twisted affine Lie algebra associated with the corresponding symmetric Cartan matrix $A_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j)$, $i, j = 0, 1, \dots, r$. Then the quantum algebra $U_q(\mathcal{G}^{(1)})$ is defined to be a Hopf algebra with generators: $\{E_i, F_i, q^{h_i} \ (i = 0, 1, \dots, r), q^d\}$ and relations,

$$\begin{aligned}
 q^h \cdot q^{h'} &= q^{h+h'} \quad (h, h' = h_i \ (i = 0, 1, \dots, r), d) \\
 q^h E_i q^{-h} &= q^{(h, \alpha_i)} E_i, \quad q^h F_i q^{-h} = q^{-(h, \alpha_i)} F_i \\
 [E_i, F_j] &= \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}
 \end{aligned}$$

(1) $(\text{ad}_q E_i)^{1-\alpha_{ij}} E_j = 0, \quad (\text{ad}_{q^{-1}} F_i)^{1-\alpha_{ij}} F_j = 0 \quad (i \neq j)$

where

(2) $(\text{ad}_q x_\alpha) x_\beta = [x_\alpha, x_\beta]_q = x_\alpha x_\beta - q^{(\alpha, \beta)} x_\beta x_\alpha.$

The algebra $U_q(\mathcal{G}^{(1)})$ is a Hopf algebra with coproduct, counit and antipode, much as in the case of $U_q(\mathcal{G})$: explicitly, the coproduct is defined by

$$\begin{aligned}
 \Delta(q^h) &= q^h \otimes q^h, \quad h = h_i, d \\
 \Delta(E_i) &= E_i \otimes 1 + q^{-h_i} \otimes E_i \\
 \Delta(F_i) &= 1 \otimes F_i + F_i \otimes q^{h_i}, \quad i = 0, 1, \dots, r.
 \end{aligned}$$

(3)

Formulae for the co-unit and antipode may also be given, but are not required below.

Let Δ' be the opposite coproduct: $\Delta' = T \Delta$, $T(x \otimes y) = y \otimes x$ for all $x, y \in U_q(\mathcal{G}^{(1)})$. Then Δ and Δ' are related by the universal R -matrix R in $U_q(\mathcal{G}^{(1)}) \otimes U_q(\mathcal{G}^{(1)})$ satisfying

$$\begin{aligned}
 \Delta'(x)R &= R\Delta(x), \quad x \in U_q(\mathcal{G}^{(1)}) \\
 (\Delta \otimes id)R &= R^{13}R^{23}, \quad (id \otimes \Delta)R = R^{13}R^{12}.
 \end{aligned}$$

(4)

We define an anti-involution θ on $U_q(\mathcal{G}^{(1)})$ by

(5) $\theta(q^h) = q^{-h}, \theta(E_i) = F_i, \theta(F_i) = E_i, \theta(q) = q^{-1},$

which extends uniquely to an algebra anti-involution on all of $U_q(\mathcal{G}^{(1)})$ so that $\theta(ab) = \theta(b)\theta(a)$ for all $a, b \in U_q(\mathcal{G}^{(1)})$. Throughout the paper, we use the notations:

$$(6) \quad \begin{aligned} (n)_q &= \frac{1 - q^{-n}}{1 - q^{-1}}, [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, q_\alpha = q^{(\alpha, \alpha)} \\ \exp_q(x) &= \sum_{n \geq 0} \frac{x^n}{(n)_q!}, (n)_q! = (n)_q(n-1)_q \dots (1)_q. \end{aligned}$$

3. UNIVERSAL R -MATRICES FOR $U_q(A_1^{(1)})$ AND $U_q(A_2^{(1)})$

This section is devoted to a brief review of the construction of the universal R -matrix for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$ [16, 27, 28]. We start with the rank 2 case. Fix a normal ordering in the positive root system Δ_+ of $A_1^{(1)}$:

$$(7) \quad \alpha, \alpha + \delta, \dots, \alpha + n\delta, \dots, \delta, 2\delta, \dots, m\delta, \dots, \dots, (\delta - \alpha) + l\delta, \dots, \delta - \alpha,$$

where α and $\delta - \alpha$ are simple roots; δ is the minimal positive imaginary root. Construct Cartan-Weyl generators $E_\gamma, F_\gamma = \theta(E_\gamma), \gamma \in \Delta_+$ of $U_q(A_1^{(1)})$ as follows: define

$$(8) \quad \begin{aligned} \widetilde{E}_\delta &= [(\alpha, \alpha)]_q^{-1} [E_\alpha, E_{\delta - \alpha}]_q \\ E_{\alpha + n\delta} &= (-1)^n (\text{ad } \widetilde{E}_\delta)^n E_\alpha \\ E_{(\delta - \alpha) + n\delta} &= (\text{ad } \widetilde{E}_\delta)^n E_{\delta - \alpha}, \dots \\ \widetilde{E}_{n\delta} &= [(\alpha, \alpha)]_q^{-1} [E_{\alpha + (n-1)\delta}, E_{\delta - \alpha}]_q \end{aligned}$$

where $[\widetilde{E}_{n\delta}, \widetilde{E}_{m\delta}] = 0$ for any $n, m > 0$. Then

- (i) for any $n > 0$, there exists a unique element $E_{n\delta}$ [16] satisfying $[E_{n\delta}, E_{m\delta}] = 0$ for any $n, m > 0$ and the relation

$$(9) \quad \widetilde{E}_{n\delta} = \sum_{p_1 + 2p_2 + \dots + np_n = n} \frac{(q^{(\alpha, \alpha)} - q^{-(\alpha, \alpha)})^{\sum_i p_i - 1}}{p_1! \dots p_n!} E_\delta^{p_1} E_{2\delta}^{p_2} \dots E_{n\delta}^{p_n}.$$

- (ii) the vectors E_γ and $F_\gamma = \theta(E_\gamma), \gamma \in \Delta_+$ are the Cartan-Weyl generators for $U_q(A_1^{(1)})$. The universal R -matrix for $U_q(A_1^{(1)})$ may be written as [16]

$$(10) \quad \begin{aligned} R &= \left(\prod_{n \geq 0} \exp_{q_\alpha}((q - q^{-1})(E_{\alpha + n\delta} \otimes F_{\alpha + n\delta})) \right) \cdot \exp \left(\sum_{n > 0} n[n]_{q_\alpha}^{-1} (q_\alpha - q_\alpha^{-1})(E_{n\delta} \otimes F_{n\delta}) \right) \\ &\cdot \left(\prod_{n \geq 0} \exp_{q_\alpha}((q - q^{-1})(E_{(\delta - \alpha) + n\delta} \otimes F_{(\delta - \alpha) + n\delta})) \right) \cdot q^{1/2} h_\alpha \otimes h_\alpha + c \otimes d + d \otimes c \end{aligned}$$

where $c = h_\alpha + h_{\delta-\alpha}$. The order in the product (10) coincides with the chosen normal order (7).

Now consider the rank 3 case. Let $A_{\text{sym}}^0 = (a_{ij}^{\text{sym}})$, $i, j = 1, 2$ and Δ_+^0 respectively be the symmetrical Cartan matrix and positive root system of the rank 2 finite-dimensional simple Lie algebra A_2 . In what follows we use A_{sym}^0 in the form

$$(11) \quad A_{\text{sym}}^0 = (a_{ij}^{\text{sym}}) = \begin{pmatrix} (\alpha, \alpha) & (\alpha, \beta) \\ (\beta, \alpha) & (\beta, \beta) \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The simple roots are α, β and $\delta - \psi$ with $\psi = \alpha + \beta$ the highest root of A_2 :

One fixes the following order in Δ_+ of $A_2^{(1)}$:

$$(12) \quad \begin{aligned} &\alpha, \alpha + \beta, \alpha + \delta, \alpha + \beta + \delta, \dots, \dots, \alpha + m_1\delta, \alpha + \beta + m_1\delta, \dots, \dots, \\ &\beta, \beta + \delta, \dots, \beta + m_2\delta, \dots, \delta, 2\delta, \dots, k\delta, \dots, \dots, (\delta - \beta) + l_1\delta, \dots, \delta - \beta, \dots, \\ &\dots, (\delta - \alpha) + l_2\delta, (\delta - \alpha - \beta) + l_2\delta, \dots, \dots, \delta - \alpha, \delta - \alpha - \beta, \end{aligned}$$

where $m_i, k, l_i \geq 0, i = 1, 2$. We set

$$(13) \quad \begin{aligned} E_{\alpha+\beta} &= [E_\alpha, E_\beta]_q; & E_{\delta-\alpha} &= [E_\beta, E_{\delta-\alpha-\beta}]_q \\ E_{\delta-\beta} &= [E_\alpha, E_{\delta-\alpha-\beta}]_q \end{aligned}$$

and use the formula for $E_{\gamma+n\delta}$ and $E_{(\delta-\gamma)+n\delta}, \gamma \in \Delta_+^0$, given by

$$(14) \quad \begin{aligned} \tilde{E}_\delta^{(i)} &= [(\alpha_i, \alpha_i)]_q^{-1} [E_{\alpha_i}, E_{\delta-\alpha_i}]_q, & \alpha_i &= \alpha, \beta, \alpha + \beta \\ E_{\alpha_i+n\delta} &= (-1)^n (\text{ad } \tilde{E}_\delta^{(i)})^n E_{\alpha_i} \\ E_{\delta-\alpha_i+n\delta} &= (\text{ad } \tilde{E}_\delta^{(i)})^n E_{\delta-\alpha_i}, & \dots \\ \tilde{E}_{n\delta}^{(i)} &= [(\alpha_i, \alpha_i)]_q^{-1} [E_{\alpha_i+(n-1)\delta}, E_{\delta-\alpha_i}]_q, \end{aligned}$$

where $[\tilde{E}_{n\delta}^{(i)}, \tilde{E}_{m\delta}^{(j)}] = 0$ for any $n, m > 0$. One has the following statement similar to the case of $U_q(A_1^{(1)})$:

- (i) there exists a unique element $E_{n\delta}^{(i)}, n > 0$ satisfying $[E_{n\delta}^{(i)}, E_{m\delta}^{(j)}] = 0$ for any $n, m > 0$ and the relation $(\alpha_i = \alpha, \beta)$
- (15)

$$\tilde{E}_{n\delta}^{(i)} = \sum_{p_1+2p_2+\dots+np_n} \frac{(q^{(\alpha_i, \alpha_i)} - q^{-(\alpha_i, \alpha_i)})^{\sum_i p_i - 1}}{p_1! \dots p_n!} (E_\delta^{(i)})^{p_1} (E_{2\delta}^{(i)})^{p_2} \dots (E_{n\delta}^{(i)})^{p_n}.$$

- (ii) the vectors E_γ and $F_\gamma = \theta(E_\gamma), \gamma \in \Delta_+$ are the Cartan-Weyl generators for $U_q(A_2^{(1)})$.

One can show [16, 27] (see, in particular, [27]) that the universal R -matrix for $U_q(A_2^{(1)})$ takes the explicit form

$$\begin{aligned}
 R = & \prod_{\gamma < \delta} \exp_{q_\gamma} ((q - q^{-1})(E_\gamma \otimes F_\gamma)) \\
 & \cdot \exp \left(\sum_{n > 0} \sum_{i,j=1}^2 C_{ij}^n(q)(q - q^{-1}) (E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)}) \right) \\
 & \cdot \prod_{\gamma > \delta} \exp_{q_\gamma} ((q - q^{-1})(E_\gamma \otimes F_\gamma)) \\
 (16) \quad & \cdot q^{\sum_{i,j=1}^2 (a_{sym}^{-1})^{ij} h_i \otimes h_j + c \otimes d + d \otimes c}
 \end{aligned}$$

where $c = h_0 + h_\psi$, the order in the product of (16) is defined by (12), and the constants $C_{ij}^n(q)$ are given by

$$(C_{ij}^n(q)) = (C_{ji}^n(q)) = \frac{n}{[n]_q} \frac{[2]_q^2}{q^{2n} + 1 + q^{-2n}} \begin{pmatrix} q^n + q^{-n} & (-1)^n \\ (-1)^n & q^n + q^{-n} \end{pmatrix}.$$

4. UNIVERSAL R -MATRICES WITH SPECTRAL PARAMETERS

We turn now to the main object of the paper. We shall determine explicitly spectral-dependent universal R -matrices for $U_q(A_1)$ and $U_q(A_2)$ by using the universal R -matrices (10) and (16) for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$, respectively.

From the Appendix it follows that for any $z \in \mathbb{C}^\times$, there is a homomorphism of algebras $ev_z: U_q(A_1^{(1)}) \rightarrow U_q(A_1)$ given by

$$\begin{aligned}
 (18) \quad ev_z(E_\alpha) &= E_\alpha, \quad ev_z(F_\alpha) = F_\alpha, \quad ev_z(h_\alpha) = h_\alpha, \quad ev_z(c) = 0 \\
 ev_z(E_{\delta-\alpha}) &= zF_\alpha, \quad ev_z(F_{\delta-\alpha}) = z^{-1}E_\alpha, \quad ev_z(h_{\delta-\alpha}) = -h_\alpha.
 \end{aligned}$$

Then suppressing “ ev_z ” for notational convenience, and through straightforward calculations and induction in n , using (18) and the defining relations (8), we obtain

$$\begin{aligned}
 E_{\alpha+n\delta} &= (-1)^n z^n q^{-nh_\alpha} E_\alpha \\
 F_{\alpha+n\delta} &= (-1)^n z^{-n} F_\alpha q^{nh_\alpha} \\
 E_{(\delta-\alpha)+n\delta} &= (-1)^n z^{n+1} F_\alpha q^{-nh_\alpha} \\
 F_{(\delta-\alpha)+n\delta} &= (-1)^n z^{-n-1} q^{nh_\alpha} E_\alpha \\
 \tilde{E}_{n\delta} &= [2]_q^{-1} (-1)^{n-1} z^n q^{-(n-1)h_\alpha} (E_\alpha F_\alpha - q^{-2n} F_\alpha E_\alpha) \\
 \tilde{F}_{n\delta} &= [2]_q^{-1} (-1)^{n-1} z^{-n} q^{(n-1)h_\alpha} (F_\alpha E_\alpha - q^{2n} E_\alpha F_\alpha).
 \end{aligned}
 (19)$$

We introduce the primed and z -independent quantities, $\tilde{E}'_{n\delta}$ and $\tilde{F}'_{n\delta}$ which are obtained from the corresponding unprimed quantities in (19) by removing the parameter z , and determine $E'_{n\delta}$ and $F'_{n\delta}$ using the following equalities of formal power series:

$$(20) \quad \begin{aligned} 1 + (q_\alpha - q_\alpha^{-1}) \sum_{k=1}^\infty \tilde{E}'_{k\delta} u^k &= \exp \left((q_\alpha - q_\alpha^{-1}) \sum_{l=1}^\infty E'_{l\delta} u^l \right) \\ 1 - (q_\alpha - q_\alpha^{-1}) \sum_{k=1}^\infty \tilde{F}'_{k\delta} u^{-k} &= \exp \left(-(q_\alpha - q_\alpha^{-1}) \sum_{l=1}^\infty F'_{l\delta} u^{-l} \right), \end{aligned}$$

which are variants of (9). Then from (10) we deduce the universal R -matrix of $U_q(A_1)$ with the explicit dependence of spectral parameter, $R(x, y) \equiv (ev_x \otimes ev_y)R$,

$$(21) \quad \begin{aligned} R(x, y) &= \prod_{n \geq 0} \exp_{q_\alpha} \left((q - q^{-1}) \left(\frac{x}{y} \right)^n (q^{-n h_\alpha} E_\alpha \otimes F_\alpha q^{n h_\alpha}) \right) \\ &\quad \cdot \exp \left(\sum_{n > 0} n [n]_{q_\alpha}^{-1} (q_\alpha - q_\alpha^{-1}) \left(\frac{x}{y} \right)^n (E'_{n\delta} \otimes F'_{n\delta}) \right) \\ &\quad \cdot \prod_{n \geq 0} \exp_{q_\alpha} \left((q - q^{-1}) \left(\frac{x}{y} \right)^{n+1} (F_\alpha q^{-n h_\alpha} \otimes q^{n h_\alpha} E_\alpha) \right) \cdot q^{(h_\alpha \otimes h_\alpha)/2}. \end{aligned}$$

We now consider the case of $U_q(A_2^{(1)})$. We have (see Appendix) that for any $z \in \mathbb{C}^\times$, there is a homomorphism of algebras $ev_z: U_q(A_2^{(1)}) \rightarrow U_q(A_2)$ given by

$$(22) \quad \begin{aligned} ev_z(E_\alpha) &= E_\alpha, \quad ev_z(F_\alpha) = F_\alpha, \quad ev_z(h_\alpha) = h_\alpha \\ ev_z(E_\beta) &= E_\beta, \quad ev_z(F_\beta) = F_\beta, \quad ev_z(h_\beta) = h_\beta \\ ev_z(E_{\delta-\alpha-\beta}) &= z F_{\alpha+\beta} q^{(h_\beta-h_\alpha)/3}, \quad ev_z(F_{\delta-\alpha-\beta}) = z^{-1} q^{(h_\alpha-h_\beta)/3} E_{\alpha+\beta} \\ ev_z(h_{\delta-\alpha-\beta}) &= -h_{\alpha+\beta}, \quad ev_z(c) = 0. \end{aligned}$$

Again suppressing “ ev_z ” and using (22) and the defining relations of the generators, equations (14) and (15), we obtain

$$\begin{aligned} E_{\alpha+n\delta} &= (-1)^n z^n q^{-n h_\alpha} E_\alpha q^{-n(h_\alpha+2h_\beta)/3} \\ F_{\alpha+n\delta} &= (-1)^n z^{-n} q^{n(h_\alpha+2h_\beta)/3} F_\alpha q^{n h_\alpha} \\ E_{\alpha+\beta+n\delta} &= (-1)^n z^n q^{-n h_{\alpha+\beta}} E_{\alpha+\beta} q^{n(h_\beta-h_\alpha)/3} \\ F_{\alpha+\beta+n\delta} &= (-1)^n z^{-n} q^{n(h_\alpha-h_\beta)/3} F_{\alpha+\beta} q^{n h_{\alpha+\beta}} \\ E_{\beta+n\delta} &= (-1)^n [2]_q^{-n} z^n q^n \{ (ad'_{q^{-1}} \mathcal{E})^n E_\beta \} q^{n(h_\beta-h_\alpha)/3} \end{aligned}$$

$$\begin{aligned}
 F_{\beta+n\delta} &= [2]_q^{-n} z^{-n} q^{n(h_\alpha-h_\beta)/3} (\text{ad}'_{q^{-1}} \mathcal{F})^n F_\beta \\
 E_{(\delta-\beta)+n\delta} &= [2]_q^{-n} z^{n+1} q^{-n} \{ (\text{ad}'_q \mathcal{E})^n (\text{ad}'_{q^{-2}} E_\alpha) F_{\alpha+\beta} \} q^{(n+1)(h_\beta-h_\alpha)/3} \\
 F_{(\delta-\beta)+n\delta} &= (-1)^n [2]_q^{-n} z^{-n-1} q^{(n+1)(h_\alpha-h_\beta)/3} (\text{ad}'_q \mathcal{F})^n (\text{ad}'_{q^2} E_{\alpha+\beta}) F_\alpha \\
 E_{(\delta-\alpha)+n\delta} &= (-1)^n z^{n+1} q^{-(n+1)(h_\alpha+2h_\beta)/3} F_\alpha q^{-nh_\alpha} \\
 F_{(\delta-\alpha)+n\delta} &= (-1)^n z^{-n-1} q^{nh_\alpha} E_\alpha q^{(n+1)(h_\alpha+2h_\beta)/3} \\
 E_{(\delta-\alpha-\beta)+n\delta} &= (-1)^n z^{n+1} q^{(n+1)(h_\beta-h_\alpha)/3} F_{\alpha+\beta} q^{-nh_{\alpha+\beta}} \\
 F_{(\delta-\alpha-\beta)+n\delta} &= (-1)^n z^{-n-1} q^{nh_{\alpha+\beta}} E_{\alpha+\beta} q^{(n+1)(h_\alpha-h_\beta)/3} \\
 \tilde{E}'_{n\delta}(\alpha) &= (-1)^{n-1} [2]_q^{-1} z^n (E_\alpha F_\alpha - q^{-2n} F_\alpha E_\alpha) q^{-(n-1)h_\alpha} q^{-n(h_\alpha+2h_\beta)/3} \\
 \tilde{F}'_{n\delta}(\alpha) &= (-1)^{n-1} [2]_q^{-1} z^{-n} q^{(n-1)h_\alpha} q^{n(h_\alpha+2h_\beta)/3} (F_\alpha E_\alpha - q^{2n} E_\alpha F_\alpha) \\
 \tilde{E}'_{n\delta}(\beta) &= (-1)^n [2]_q^{-n} z^n q^{n-2} \{ (\text{ad}'_{q^{-n+2}} \mathcal{F}') \cdot (\text{ad}'_{q^{-1}} \mathcal{E})^{n-1} E_\beta \} q^{n(h_\beta-h_\alpha)/3} \\
 \tilde{F}'_{n\delta}(\beta) &= [2]_q^{-n} z^{-n} q^{n-1} q^{n(h_\alpha-h_\beta)/3} (\text{ad}'_{q^{-n+2}} \mathcal{E}') \cdot (\text{ad}'_{q^{-1}} \mathcal{F})^{n-1} F_\beta,
 \end{aligned}
 \tag{23}$$

where $(\text{ad}'_Q \mathcal{A}) \cdot \mathcal{B} \equiv \mathcal{A}\mathcal{B} - \mathcal{Q}\mathcal{B}\mathcal{A}$, and

$$\begin{aligned}
 \mathcal{E} &= (\text{ad}'_{q^{-1}} E_\beta) (\text{ad}'_{q^{-2}} E_\alpha) F_{\alpha+\beta}, \quad \mathcal{F} = (\text{ad}'_q (\text{ad}'_{q^2} E_{\alpha+\beta}) F_\alpha) F_\beta \\
 \mathcal{E}' &= E_{\alpha+\beta} F_\alpha - q^2 F_\alpha E_{\alpha+\beta}, \quad \mathcal{F}' = E_\alpha F_{\alpha+\beta} - q^{-2} F_{\alpha+\beta} E_\alpha.
 \end{aligned}
 \tag{24}$$

We introduce the primed and z -independent quantities $\tilde{E}'_{n\delta}(\alpha)$, $\tilde{F}'_{n\delta}(\alpha)$, $\tilde{E}'_{n\delta}(\beta)$, $\tilde{F}'_{n\delta}(\beta)$, $E'_{\beta+n\delta}$, $F'_{\beta+n\delta}$, $E'_{(\delta-\beta)+n\delta}$ and $F'_{(\delta-\beta)+n\delta}$, which are obtained from the corresponding unprimed quantities in (23) by suppressing the parameter z , and determine $E'^{(i)}_{n\delta}$, $F'^{(i)}_{n\delta}$ by the following equalities of formal series: ($\alpha_i = \alpha, \beta$)

$$\begin{aligned}
 k\delta u^k &= \exp \left((q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{l=1}^{\infty} E'_{l\delta}(\alpha) u^l \right) \\
 1 - (q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{k=1}^{\infty} \tilde{F}'_{k\delta}(\alpha) u^{-k} &= \exp \left(-(q_{\alpha_i} - q_{\alpha_i}^{-1}) \sum_{l=1}^{\infty} F'_{l\delta}(\alpha) u^{-l} \right),
 \end{aligned}$$

which are variants of (15). Then from (16) we may deduce the universal R -matrix of $U_q(A_2)$ with the explicit dependence on a spectral parameter, $R(x, y) \equiv (\text{ev}_x \otimes \text{ev}_y) R$, given by

$$\begin{aligned}
 R(x, y) &= \prod_{n \geq 0} \left\{ \exp_{q_\alpha} \left((q - q^{-1}) \left(\frac{x}{y} \right)^n \left(q^{-nh_\alpha} E_\alpha q^{-n(h_\alpha+2h_\beta)/3} \right. \right. \right. \\
 &\quad \left. \left. \left. \otimes q^{n(h_\alpha+2\beta)/3} F_\alpha q^{nh_\alpha} \right) \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \exp_{q_{\alpha+\beta}} \left((q - q^{-1}) \left(\frac{x}{y} \right)^n \left(q^{-nh_{\alpha+\beta}} E_{\alpha+\beta} q^{n(h_{\beta}-h_{\alpha})/3} \right. \right. \\
 & \quad \left. \left. \otimes q^{n(h_{\alpha}-\beta)/3} F_{\alpha+\beta} q^{nh_{\alpha+\beta}} \right) \right) \Big\} \\
 & \cdot \prod_{n \geq 0} \exp_{q_{\beta}} \left((q - q^{-1}) \left(\frac{x}{y} \right)^n (E'_{\beta+n\delta} \otimes F'_{\beta+n\delta}) \right) \\
 & \cdot \exp \left(\sum_{n > 0} \sum_{i,j=1}^2 C_{ij}^n(q) (q - q^{-1}) \left(\frac{x}{y} \right)^n (E'_{n\delta}^{(i)} \otimes F'_{n\delta}^{(j)}) \right) \\
 & \cdot \prod_{n \geq 0} \exp_{q_{\beta}} \left((q - q^{-1}) \left(\frac{x}{y} \right)^{n+1} (E'_{(\delta-\beta)+n\delta} \otimes F'_{(\delta-\beta)+n\delta}) \right) \\
 & \cdot \prod_{n \geq 0} \left\{ \exp_{q_{\alpha}} \left((q - q^{-1}) \left(\frac{x}{y} \right)^{n+1} \left(q^{-(n+1)(h_{\alpha}+2h_{\beta})/3} F_{\alpha} q^{-nh_{\alpha}} \right. \right. \right. \\
 & \quad \left. \left. \otimes q^{nh_{\alpha}} E_{\alpha} q^{(n+1)(h_{\alpha}+2h_{\beta})/3} \right) \right) \\
 & \cdot \exp_{q_{\alpha+\beta}} \left((q - q^{-1}) \left(\frac{x}{y} \right)^{n+1} \left(q^{(n+1)(h_{\beta}-h_{\alpha})/3} F_{\alpha+\beta} q^{-nh_{\alpha+\beta}} \right. \right. \\
 & \quad \left. \left. \otimes q^{nh_{\alpha+\beta}} E_{\alpha+\beta} q^{(n+1)(h_{\alpha}-h_{\beta})/3} \right) \right) \Big\} \cdot q \sum_{i,j=1}^2 (a_{\beta\gamma m}^{-1})^{ij} h_i \otimes h_j.
 \end{aligned}
 \tag{25}$$

5. APPLICATIONS

To illustrate the general theory developed in the previous section, we present a detailed study of the spectral-dependent R -matrix for some concrete and interesting representations.

First consider the $U_q(A_1^{(1)})$ case. Let $V_{l/2}$, $l \in \mathbb{Z}_+$ denote the $(l + 1)$ -dimensional module of $U_q(A_1)$ (spin $l/2$ representation) with basis $\{v_m^{(l)} \mid 0 \leq m \leq l\}$. We have for this representation,

$$\begin{aligned}
 h_{\alpha} v_m^{(l)} &= (l - 2m) v_m^{(l)} \\
 E_{\alpha} v_m^{(l)} &= [l - m + 1]_q v_{m-1}^{(l)} \\
 F_{\alpha} v_m^{(l)} &= [m + 1]_q v_{m+1}^{(l)} \\
 E'_{n\delta} v_m^{(l)} &= [2]_q^{-1} \frac{(-1)^{n-1}}{n} q^{nm} \left([n(l - m)]_q - q^{-n(l+2)} [nm]_q \right) v_m^{(l)} \\
 F'_{n\delta} v_m^{(l)} &= [2]_q^{-1} \frac{(-1)^{n-1}}{n} q^{-nm} \left([n(l - m)]_q - q^{n(l+2)} [nm]_q \right) v_m^{(l)},
 \end{aligned}
 \tag{26}$$

where it is understood that $v_m^{(l)}$ is identically zero if $m > l$ or $m < 0$. These results

are obtained by straightforward computation plus induction in n .

(i) for the spin 1/2 representation, we have from (26)

$$(27) \quad h_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$E'_{n\delta} = [2]_q^{-1} \frac{(-1)^{n-1}}{n} [n]_q \begin{pmatrix} 1 & 0 \\ 0 & -q^{-2n} \end{pmatrix}, \quad F'_{n\delta} = [2]_q^{-1} \frac{(-1)^{n-1}}{n} [n]_q \begin{pmatrix} 1 & 0 \\ 0 & -q^{2n} \end{pmatrix}.$$

We apply (21) to $V_{1/2} \otimes V_{1/2}$, where $V_{1/2}$ carries the spin-1/2 representation of $U_q(A_1)$. Using (27), it follows from (21) that

$$(28) \quad R_{1/2,1/2}(x,y) = f_q(x,y) \cdot \begin{pmatrix} 1 & & & \\ & \frac{q^{-1}(y-x)}{y-q^{-2}x} & \frac{y(1-q^{-2})}{y-q^{-2}x} & \\ & \frac{x(1-q^{-2})}{y-q^{-2}x} & \frac{q^{-1}(y-x)}{y-q^{-2}x} & \\ & & & 1 \end{pmatrix}$$

where

$$(29) \quad f_q(x,y) = q^{1/2} \cdot \exp\left(\sum_{n>0} \frac{q^n - q^{-n}}{q^n + q^{-n}} \frac{(x/y)^n}{n}\right)$$

and use has been made of the notation:

$$(30) \quad (A \otimes B) = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1N}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1}B & A_{M2}B & \cdots & A_{MN}B \end{pmatrix}$$

Equation (28) reproduces a well-known result [12], up to the scalar factor $f_q(x,y)$. Khoroshkin and Tolstoy [16] obtained (29) directly from (10).

(ii) for the spin 1 representation, (26) gives

$$(31) \quad h_\alpha = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & [2]_q & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & [2]_q & 0 \end{pmatrix}$$

$$F'_{n\delta} = [2]_q^{-1} \frac{(-1)^{n-1}}{n} [n]_q \begin{pmatrix} q^n + q^{-n} & 0 & 0 \\ 0 & -q^n(q^{2n} - q^{-2n}) & 0 \\ 0 & 0 & -q^{2n}(q^n + q^{-n}) \end{pmatrix}.$$

We now apply (21) to $V_{1/2} \otimes V_1$ with V_1 carrying the spin-1 representation of $U_q(A_1)$. Using (31), we obtain from (21)

$$(32) \quad R_{1/2,1}(x,y) = \frac{q^2(y - q^{-1}x)}{y - qx} \cdot \left(e_{11} + e_{66} + \frac{q^{-2}(y - qx)}{y - q^{-3}x}(e_{33} + e_{44}) + \frac{yq^{-1}(1 - q^{-2})}{y - q^{-3}x}e_{24} + \frac{xq^{-1}(1 - q^{-2})}{y - q^{-3}x}e_{53} \right),$$

where (and below) e_{ij} is the matrix satisfying $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

We now turn to the $U_q(A_2^{(1)})$ case. Hence we find that the explicit form of the generators on the fundamental representation of $U_q(A_2)$ is given by

$$(33) \quad \begin{aligned} h_\alpha &= \text{diag}(1, -1, 0), & h_\beta &= \text{diag}(0, 1, -1) \\ E_\alpha &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & F_\alpha &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ E_{\alpha+\beta} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & F_{\alpha+\beta} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ E'_{\beta+n\delta} &= q^{-2n-n/3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & F'_{\beta+n\delta} &= q^{2n+n/3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ E'_{(\delta-\beta)+n\delta} &= -q^{-2n-1-(n+1)/3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ F'_{(\delta-\beta)+n\delta} &= -q^{2n+1+(n+1)/3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ E'_{n\delta}^{(\alpha)} &= [2]_q^{-1}(-1)^{n-1} \frac{[n]_q}{n} q^{-n/3} \text{diag}(1, -q^{-2n}, 0) \\ F'_{n\delta}^{(\alpha)} &= [2]_q^{-1}(-1)^{n-1} \frac{[n]_q}{n} q^{n/3} \text{diag}(1, -q^{2n}, 0) \\ E'_{n\delta}^{(\beta)} &= [2]_q^{-1} \frac{[n]_q}{n} q^{-n-n/3} \text{diag}(0, -1, q^{-2n}) \\ F'_{n\delta}^{(\beta)} &= [2]_q^{-1} \frac{[n]_q}{n} q^{n+n/3} \text{diag}(0, -1, q^{2n}). \end{aligned}$$

We apply the results in Section 4 to derive the spectral-dependent R -matrix associated with $V_{(3)} \otimes V_{(3)}$, where $V_{(3)}$ stands for the fundamental representation of $U_q(A_2)$.

Using (33) we get

$$\begin{aligned}
 R_{(3),(3)}(x,y) = f_q(x,y) \cdot & \left(e_{11} + e_{55} + e_{99} + \frac{q^{-1}(y-x)}{y-q^{-2}x}(e_{22} + e_{33} + e_{44} + e_{66} + e_{77} + e_{88}) \right. \\
 (34) \quad & \left. + \frac{y(1-q^{-2})}{y-q^{-2}x}(e_{24} + e_{37} + e_{68}) + \frac{x(1-q^{-2})}{y-q^{-2}x}(e_{42} + e_{73} + e_{86}) \right),
 \end{aligned}$$

where

$$f_q(x,y) = q^{2/3} \cdot \exp \left(\sum_{n>0} \frac{q^{2n} - q^{-2n}}{q^{2n} + 1 + q^{-2n}} \frac{(x/y)^n}{n} \right).$$

Again this reproduces a well-known result [12], up to a scalar factor $f_q(x,y)$.

We now consider a very interesting case: to extract the spectral dependent R -matrix associated with the module $V_{(3)} \otimes V_{(6)}$ of $U_q(A_2)$. As indicated in the introduction, this case could not be treated by means of the usual Yang-Baxterisation procedures, simply because the representations being tensored are different.

We introduce the so-called Gelfand-Tsetlin basis vector $|(m)\rangle$ given by

$$|(m)\rangle = \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} & m_{22} \\ & & m_{11} \end{pmatrix} \right\rangle.$$

It can be shown that the action of the generators on these basis vectors is

$$\begin{aligned}
 h_\alpha |(m)\rangle &= (2m_{11} - m_{12} - m_{22}) |(m)\rangle \\
 h_\beta |(m)\rangle &= (2m_{12} + 2m_{22} - m_{11} - m_{13} - m_{23} - m_{33}) |(m)\rangle
 \end{aligned}$$

$$\begin{aligned}
 F_\alpha \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} & m_{22} \\ & & m_{11} \end{pmatrix} \right\rangle &= \{[m_{11} - m_{22}]_q [m_{12} - m_{11} + 1]_q\}^{1/2} \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} & m_{22} \\ & & m_{11} - 1 \end{pmatrix} \right\rangle \\
 F_\beta \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} & m_{22} \\ & & m_{11} \end{pmatrix} \right\rangle &= \left\{ \frac{[m_{12} - m_{11}]_q [m_{13} - m_{12} + 1]_q [m_{23} - m_{12}]_q [m_{33} - m_{12} - 1]_q}{[m_{12} - m_{22} + 1]_q [m_{12} - m_{22}]_q} \right\}^{1/2} \\
 &\times \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ & m_{12} - 1 & m_{22} \\ & & m_{11} \end{pmatrix} \right\rangle
 \end{aligned}$$

$$(37) \quad + \left\{ \frac{[m_{22} - m_{11} - 1]_q [m_{13} - m_{22} + 2]_q [m_{23} - m_{22} + 1]_q [m_{33} - m_{22}]_q}{[m_{12} - m_{22} + 2]_q [m_{12} - m_{22} + 1]_q} \right\}^{1/2} \\ \times \left| \begin{pmatrix} m_{13} & m_{23} & m_{33} \\ m_{12} & m_{22} - 1 & \\ & & m_{11} \end{pmatrix} \right\rangle.$$

The matrices of E_α and E_β are given by the transposes of F_α and F_β , respectively.

For the 6-dimensional representation, we have the following basis vectors:

$$(38) \quad \phi_1 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & 4/3 & -2/3 \\ & & 4/3 \end{pmatrix} \right\rangle, \quad \phi_2 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & 4/3 & -2/3 \\ & & 1/3 \end{pmatrix} \right\rangle \\ \phi_3 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & 4/3 & -2/3 \\ & & -2/3 \end{pmatrix} \right\rangle, \quad \phi_4 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & 1/3 & -2/3 \\ & & 1/3 \end{pmatrix} \right\rangle \\ \phi_5 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & 1/3 & -2/3 \\ & & -2/3 \end{pmatrix} \right\rangle, \quad \phi_6 = \left| \begin{pmatrix} 4/3 & -2/3 & -2/3 \\ & -2/3 & -2/3 \\ & & -2/3 \end{pmatrix} \right\rangle.$$

Then long computations show that the matrix form of the generators in the 6-dimensional representation of $U_q(A_2)$ is given by

$$h_\alpha = \text{diag}(2, 0, -2, 1, -1, 0), \quad h_\beta = \text{diag}(0, 1, 2, -1, 0, -2) \\ E_\alpha = [2]_q^{1/2} e_{12} + [2]_q^{1/2} e_{23} + e_{45}, \quad F_\alpha = [2]_q^{1/2} e_{21} + [2]_q^{1/2} e_{32} + e_{54} \\ E_\beta = e_{24} + [2]_q^{1/2} e_{35} + [2]_q^{1/2} e_{56}, \quad F_\beta = e_{42} + [2]_q^{1/2} e_{53} + [2]_q^{1/2} e_{65} \\ E_{\alpha+\beta} = [2]_q^{1/2} e_{14} + q e_{25} + [2]_q^{1/2} e_{46}, \quad F_{\alpha+\beta} = [2]_q^{1/2} e_{41} + q e_{52} + [2]_q^{1/2} e_{64} \\ E'_{\beta+n\delta} = q^{-2n/3} \{ q^{-3n} e_{24} + [2]_q^{1/2} q^{-3n} e_{35} + [2]_q^{1/2} q^{-n} e_{56} \} \\ F'_{\beta+n\delta} = q^{2n/3} \{ q^{3n} e_{42} + [2]_q^{1/2} q^{3n} e_{53} + [2]_q^{1/2} q^n e_{65} \} \\ E'_{(\delta-\beta)+n\delta} = -q^{-2(n+1)/3} \{ q^{-3n-2} e_{42} + [2]_q^{1/2} q^{-3n-1} e_{53} + [2]_q^{1/2} q^{-n-1} e_{65} \} \\ F'_{(\delta-\beta)+n\delta} = -q^{2(n+1)/3} \{ q^{3n+2} e_{24} + [2]_q^{1/2} q^{3n+1} e_{35} + [2]_q^{1/2} q^{n+1} e_{56} \} \\ E'_{n\delta} = [2]_q^{-1} (-1)^{n-1} \frac{[n]_q}{n} q^{-2n/3} \{ (q^n + q^{-n}) e_{11} + q^{-n} (q^{2n} - q^{-2n}) e_{22} \\ - q^{-2n} (q^n + q^{-n}) e_{33} + q^n e_{44} - q^{-n} e_{55} \} \\ F'_{n\delta} = [2]_q^{-1} (-1)^{n-1} \frac{[n]_q}{n} q^{2n/3} \{ (q^n + q^{-n}) e_{11} - q^n (q^{2n} - q^{-2n}) e_{22} \\ - q^{2n} (q^n + q^{-n}) e_{33} + q^{-n} e_{44} - q^n e_{55} \}$$

$$\begin{aligned}
 E'_{n\delta}^{(\beta)} &= [2]_q^{-1} \frac{[n]_q}{n} q^{-2n/3} \{ -q^{-2n} e_{22} - q^{-n} (q^n + q^{-n}) e_{33} + q^{-4n} e_{44} \\
 &\quad - q^{-2n} (q^{2n} - q^{-2n}) e_{55} + q^{-3n} (q^n + q^{-n}) e_{66} \} \\
 F'_{n\delta}^{(\beta)} &= -[2]_q^{-1} \frac{[n]_q}{n} q^{2n/3} \{ -q^{2n} e_{22} - q^n (q^n + q^{-n}) e_{33} + q^{4n} e_{44} \\
 &\quad + q^{2n} (q^{2n} - q^{-2n}) e_{55} + q^{3n} (q^n + q^{-n}) e_{66} \} .
 \end{aligned}
 \tag{39}$$

We have the following properties for the generators in (39),

$$\begin{aligned}
 (E_\alpha)^2 &= [2]_q e_{13}, (E_\alpha)^3 = 0, (F_\alpha)^2 = [2]_q e_{31}, (F_\alpha)^3 = 0 \\
 (E_{\alpha+\beta})^2 &= -[2]_q e_{16}, (E_{\alpha+\beta})^3 = 0, (F_{\alpha+\beta})^2 = -[2]_q e_{61}, (F_{\alpha+\beta})^3 = 0 \\
 (E'_{\beta+n\delta})^2 &= [2]_q q^{-5n-n/3} e_{36}, (E'_{\beta+n\delta})^3 = 0, (F'_{\beta+n\delta})^2 = [2]_q q^{5n+n/3} e_{63}, (F'_{\beta+n\delta})^3 = 0 \\
 (E'_{(\delta-\beta)+n\delta})^2 &= [2]_q q^{-5n-3-(n+1)/3} e_{63}, (E'_{(\delta-\beta)+n\delta})^3 = 0 \\
 (F'_{(\delta-\beta)+n\delta})^2 &= [2]_q q^{5n+3+(n+1)/3} e_{36}, (F'_{(\delta-\beta)+n\delta})^3 = 0
 \end{aligned}
 \tag{40}$$

as can easily be checked.

Now we are in the position to derive the quantum R -matrix, $R_{(3),(6)}(x, y)$, on the tensor product module $V_{(3)} \otimes V_{(6)}$. With the help of (33) and (39) and through long calculations similar to those that led to (34), we find

$$R_{(3),(6)}(x, y) = f'_q(x, y) \cdot R_+ R_0 R_- ,
 \tag{41}$$

where

$$\begin{aligned}
 f'_q(x, y) &= q^{1/3} \cdot \exp \left(\sum_{n>0} \frac{q^n - q^{-n}}{q^{2n} + 1 + q^{-2n}} \frac{(q^{1/3} x/y)^n}{n} \right) \\
 R_+ &= I + \frac{y(q - q^{-1})}{y - q^{4/3}x} \{ [2]_q^{1/2} (e_{12} \otimes e_{21} + e_{13} \otimes e_{41} + e_{23} \otimes e_{53}) + e_{23} \otimes e_{42} \} \\
 &\quad + \frac{y(q - q^{-1})}{y - q^{-2/3}x} \{ e_{12} \otimes e_{54} + [2]_q^{1/2} (e_{12} \otimes e_{32} + e_{13} \otimes e_{64}) + q^{-1} e_{13} \otimes e_{52} \\
 &\quad + [2]_q^{1/2} e_{23} \otimes e_{65} \} + \frac{y^2 (q - q^{-1})^2}{(y - q^{4/3}x)(y - q^{-2/3}x)} e_{13} \otimes e_{52} \\
 R_0 &= \frac{y - q^{-2/3}x}{y - q^{4/3}x} \{ e_{11} \otimes e_{11} + e_{22} \otimes e_{33} + e_{33} \otimes e_{66} \} \\
 &\quad + \frac{(y - q^{10/3}x)(y - q^{-2/3}x)}{(y - q^{4/3}x)^2} \{ e_{11} \otimes e_{22} + e_{11} \otimes e_{44} + e_{22} \otimes e_{55} \}
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{y - q^{10/3}x}{y - q^{4/3}x} \{e_{11} \otimes e_{33} + e_{11} \otimes e_{55} + e_{11} \otimes e_{66} + e_{22} \otimes e_{66}\} \\
 &+ \frac{y - q^{-2/3}x}{y - q^{-8/3}x} \{e_{22} \otimes e_{11} + e_{33} \otimes e_{11} + e_{33} \otimes e_{22} + e_{33} \otimes e_{33}\} \\
 &+ \frac{(y - q^{-2/3}x)^2}{(y - q^{4/3}x)(y - q^{-8/3}x)} \{e_{22} \otimes e_{22} + e_{33} \otimes e_{44} + e_{33} \otimes e_{55}\} \\
 &+ \frac{(y - q^{10/3}x)(y - q^{-2/3}x)^2}{(y - q^{4/3}x)^2(y - q^{-8/3}x)} e_{22} \otimes e_{44}
 \end{aligned}$$

$$\begin{aligned}
 R_- = &q\{e_{11} \otimes e_{11} + e_{22} \otimes e_{33} + e_{33} \otimes e_{66}\} + e_{11} \otimes e_{22} + e_{11} \otimes e_{44} \\
 &+ e_{22} \otimes e_{22} + e_{22} \otimes e_{55} + e_{33} \otimes e_{44} + e_{33} \otimes e_{55} + q^{-1}\{e_{11} \otimes e_{33} \\
 &+ e_{11} \otimes e_{55} + e_{11} \otimes e_{66} + e_{22} \otimes e_{11} + e_{22} \otimes e_{44} \\
 &+ e_{22} \otimes e_{66} + e_{33} \otimes e_{11} + e_{33} \otimes e_{22} + e_{33} \otimes e_{33}\} \\
 &+ \frac{xq^{1/3}(q - q^{-1})}{y - q^{4/3}x} \{[2]_q^{1/2}(e_{21} \otimes e_{12} + e_{31} \otimes e_{14} + e_{32} \otimes e_{35}) + e_{32} \otimes e_{24}\} \\
 &+ \frac{xq^{-2/3}(q - q^{-1})}{y - q^{-2/3}x} \{[2]_q^{1/2}(e_{21} \otimes e_{23} + e_{31} \otimes e_{46}) + e_{21} \otimes e_{45} + e_{31} \otimes e_{25} + \\
 (42) &+ [2]_q^{1/2}e_{32} \otimes e_{56}\} + \frac{x^2q^{2/3}(q - q^{-1})^2}{(y - q^{4/3}x)(y - q^{-2/3}x)} e_{31} \otimes e_{25}.
 \end{aligned}$$

It is understood that I in R_+ is the 18×18 unit matrix.

6. CONCLUDING REMARKS

To summarise: we have obtained explicit and compact formulae for the quantum R -matrices (or the inter-twiners) associated with some interesting representations of $U_q(A_2)$. For the case of $V_{(3)} \otimes V_{(6)}$, it is interesting to note in particular the appearance of fractional powers of q . It is hoped that these explicit formulae may be useful in physical applications, with associated interesting new features.

Up to now, we have been working in the homogeneous gradation of $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$. Similar calculations can be carried out for other gradations, in particular for the principal gradation. In this way, we could get a host of R -matrices with different spectral parameter-dependences. The simplest cases corresponding to the fundamental representation of $U_q(A_1)$ and $U_q(A_2)$ have been worked out very recently [6].

7. APPENDIX

We consider here finite-dimensional irreducible representations of $U_q(gl(n)^{(1)})$

with the Chevalley generators $\{E_i, F_i, q^{h_i}, 0 \leq i < n; q^d\}$ in which

$$(43) \quad E_i \equiv E_{i+1}, \quad F_i \equiv E_{i+1}i, \quad q^{h_i}, \quad h_i \equiv E_{ii} - E_{i+1}i+1, \quad 1 \leq i < n, \quad q^{E_{nn}}$$

are the usual Chevalley generators of $U_q(gl(n))$. We define

$$(44) \quad \begin{aligned} E_{ij} &= E_{ik}E_{kj} - q^{-1}E_{kj}E_{ik}, \quad i < k < j \\ E_{ij} &= E_{ik}E_{kj} - qE_{kj}E_{ik}, \quad i > k > j \end{aligned}$$

and put

$$(45) \quad E_\psi \equiv q^{E_{11}+E_{nn}}E_{1n}, \quad F_\psi \equiv E_{n1}q^{-E_{11}-E_{nn}}, \quad h_\psi \equiv E_{11} - E_{nn}.$$

Then we have

PROPOSITION. For any given $z \in \mathbb{C}^\times$, there is a homomorphism of algebras $ev_z: U_q(gl(n)^{(1)}) \rightarrow U_q(gl(n))$ where, in terms of the Chevalley generators,

$$(46) \quad \begin{aligned} ev_z(E_i) &= E_i, \quad ev_z(F_i) = F_i, \quad ev_z(h_i) = h_i \\ ev_z(E_0) &= zF_\psi, \quad ev_z(F_0) = z^{-1}E_\psi, \quad ev_z(h_0) = -h_\psi, \quad ev_z(c) = 0. \end{aligned}$$

PROOF: To show that a homomorphism $U_q(gl(n)^{(1)}) \rightarrow U_q(gl(n))$ is defined, one needs to check that the relations in (1) are satisfied. This is immediate except for the last two, which reduce to

$$(47) \quad (ad_{q^{-1}}F_i)^{1+(\psi, \alpha_i)}F_0 = (ad_{q^{-1}}F_i)^{1+(\psi, \alpha_i)}E_\psi = 0$$

$$(48) \quad (ad_{q^{-1}}F_0)^{1+(\psi, \alpha_i)}F_i = (ad_{q^{-1}}E_\psi)^{1+(\psi, \alpha_i)}F_i = 0,$$

and two similar relations with the interchanges $F_i \leftrightarrow E_i, F_0 \leftrightarrow E_0, q^{-1} \leftrightarrow q$. We now prove the result (47). First we consider the case: $1 < i < i + 1 < n$. In this case $(\psi, \alpha_i) = 0$, and the left hand side of (47) becomes

$$(49) \quad (ad_{q^{-1}}F_i)E_\psi = [F_i, E_\psi] = q^{E_{11}+E_{nn}}[E_{i+1}i, E_{1n}],$$

which can easily be seen to vanish. We then consider the $i = 1$ case. Then the left hand side of (47) reads

$$(50) \quad (ad_{q^{-1}}F_1)^2E_\psi = (ad_{q^{-1}}E_{21})(ad_{q^{-1}}E_{21})E_\psi.$$

One can easily show that $(ad_{q^{-1}}E_{21})E_\psi = q^{E_{22}+E_{nn}}E_{2n}$. Inserting this into (50), one sees that this equation reduces to

$$(51) \quad (ad_{q^{-1}}E_{21})q^{E_{22}+E_{nn}}E_{2n} = q^{E_{22}+E_{nn}-1}[E_{21}, E_{2n}] = 0$$

as required. Finally for $i = n$, we see that the left hand side of (47) reduces to

$$(52) \quad (\text{ad}_{q^{-1}} F_n)^2 E_\psi = (\text{ad}_{q^{-1}} E_{n\ n-1})(\text{ad}_{q^{-1}} E_{n\ n-1}) E_\psi.$$

Direct computations show that the right hand side of this equation reduces to

$$(53) \quad q^{E_{11} + E_{nn} - 1} \{ q^{-1} E_{n\ n-1} [E_{n\ n-1}, E_{1n}] - q [E_{n\ n-1}, E_{1n} E_{n\ n-1}] \}$$

which, using the formula

$$(54) \quad [E_{n\ n-1}, E_{1n}] = -q^{E_{n-1\ n-1} - E_{nn}} E_{1\ n-1},$$

is easily seen to vanish. We may prove (48) similarly. \square

REMARK. Since $N \equiv \sum_{i=1}^n E_{ii}$ commutes with all generators, it follows that if we set, instead of (45),

$$(55) \quad E_\psi \equiv q^{E_{11} + E_{nn} - \frac{2}{n}N} E_{1n}, \quad F_\psi \equiv E_{n1} q^{-E_{11} - E_{nn} + \frac{2}{n}N} E_{1n},$$

then the proposition of this Appendix still holds. It turns out that it is more convenient to use (55) as we did in the previous sections.

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