Can. J. Math. Vol. **43** (1), 1991 pp. 182–212

ON FUNCTIONAL PROPERTIES OF INCOMPLETE GAUSSIAN SUMS

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ABSTRACT. The following special function of two real variables x_2 and x_1 is considered:

$$H(x_2, x_1) = \text{p. v.} \sum_{n \neq 0} \frac{e^{2\pi i (n^2 x_2 + n x_1)}}{2\pi i n} = \lim_{N \to \infty} \sum_{0 < |n| \leq N} \cdots$$

and its connections with the incomplete Gaussian sums

$$W\left(\omega, \frac{a}{q}\right) = \sum_{\frac{n}{q} \in \omega} e^{2\pi i \frac{an^2}{q}}, \ q = 1, 2, \ldots, (a, q) = 1,$$

where ω are intervals of length $|\omega| \leq 1$. In particular, it is proved that for each fixed x_2 and uniformly in x_2 the function $H(x_2, x_1)$ is of weakly bounded 2-variation in the variable x_1 over the period $[0, 1]$. In terms of the sums W this means that for collections $\Omega = \{\omega_k\}$, consisting of nonoverlapping intervals $\omega_k \subset [0, 1)$ the following estimate is valid:

$$\sup_{\Omega} \text{card} \left\{ \omega_k : \left| W\left(\omega_k; \frac{a}{q}\right) \right| > \varepsilon \sqrt{q} \right\} \leq c\varepsilon^{-2} \quad (\varepsilon > 0),$$

where card denotes the number of elements, and c is an absolute positive constant. The exact value of the best absolute constant κ in the estimate

$$\left| W\left(\omega, \frac{a}{q}\right) \right| \leq \kappa \sqrt{q}$$

(which is due to G. H. Hardy and J. E. Littlewood) is discussed.

1. **Introduction.** We study here the special function of two real variables x_2, x_1

$$H(x_2, x_1) = \text{p. v.} \sum_{n \neq 0} \frac{e^{2\pi i (n^2 x_2 + n x_1)}}{2\pi i n} = \lim_{N \to \infty} \sum_{|n| \leq N} \cdots \tag{1.1}$$

and its connections with incomplete Gaussian sums, i.e.,

$$W\left(\omega; \frac{a}{q}\right) = \sum_{n \in \omega} e^{2\pi i \frac{an^2}{q}}; \ q = 1, 2, \ldots, ; a = 0, \pm 1, \ldots; (a, q) = 1, \tag{1.2}$$

where ω denotes closed intervals on the real axis, with the length $|\omega|$ satisfying $0 < |\omega| \leq q$. (If one or both endpoints of ω are integers, we take the corresponding summands with the factor $1/2$). For $|\omega| = q$, $W(\omega)$ turns into complete Gaussian sum for which we use the notation $S\left(\frac{a}{q}\right)$; moreover, let

$$S\left(\frac{a}{q}, \frac{b}{q}\right) = \sum_{n=1}^{q} e^{\frac{2\pi i (an^2 + bn)}{q}}; \quad a, b = 0, \pm 1, \ldots, (a, q) = 1.$$

Received by the editors May 31, 1989.

The sums S were introduced and computed by K. F. Gauss, and in particular their moduli are defined by the following classical relation

$$(1.3i)\qquad \left|S\left(\frac{a}{q},\frac{b}{q}\right)\right| = \sqrt{q} \quad \text{if} \quad q \equiv 1 \pmod 2;$$

$$(1.3ii)\qquad \left|S\left(\frac{a}{q},\frac{b}{q}\right)\right| = \sqrt{Q(1+(-1)^{aQ+b})} \quad \text{if} \quad q \equiv 0 \pmod 2,\ Q = \frac{q}{2}.$$

The following estimate is valid for incomplete Gaussian sums:

$$(1.4)\qquad W\left(\omega,\frac{a}{q}\right) << \sqrt{q}.$$

Here and in what follows relations such as $A << B$ mean, for a complex quantity A and nonnegative B, that $|A| \leq cB$ where c is an absolute constant; $\overline{A}$ denotes the conjugate of A. If $|A| \leq c_{\gamma;\delta\dots} \cdot B$, where the factor c depends only on $\gamma, \delta, \dots$, we use the notation $A <<_{\gamma,\delta,\dots} B$. We comment on (1.4) a little later, and now immediately illustrate relationships between incomplete Gaussian sums and the function $H(x_2,x_1)$ or its version

$$(1.1i)\qquad \begin{aligned} H^*(x_2,x_1) &= e^{-\frac{\pi i x_2}{2}}\left(H(\frac{x_2}{4},x_1) - \frac{1}{2}H(x_2,2x_1)\right) \\ &= \frac{e^{\frac{-\pi i x_2}{2}}}{2}\ \text{p.v.} \sum_{\nu:\nu\equiv 1/2 (\mathrm{mod}\, 1)} \frac{e^{2\pi i(\nu^2 x_2+\nu x_1)}}{2\pi i\nu}. \end{aligned}$$

(See Section 2 for the proof.)

LEMMA 1. *Let $0 < \alpha \leq 1$, q a positive integer, a an integer with $(a,q) = 1$, a' the solution of the congruence $aa' \equiv 1 \pmod q$ (unique $\bmod q$)*

$$W_\alpha\left(\frac{a}{q}\right) = W\left([0,q\alpha];\frac{a}{q}\right) = \sum_{n\in[0,q\alpha]} e^{2\pi i\frac{an^2}{q}}.$$

Then

$$(1.5i)\qquad W_\alpha\left(\frac{a}{q}\right) = S\left(\frac{a}{q}\right)\left(\alpha + \overline{H}\left(\frac{(4a)'}{q},\alpha\right)\right),\ \textit{if}\ q \equiv 1 \pmod 2;$$

$$(1.5ii)\qquad W_\alpha\left(\frac{a}{q}\right) = S\left(\frac{a}{q}\right)\left(\alpha + \frac{1}{2}\overline{H}\left(\frac{a'}{q},2\alpha\right)\right),\ \textit{if}\ q \equiv 0 \pmod 4;$$

$$(1.5iii)\qquad W_\alpha\left(\frac{a}{q}\right) = S\left(\frac{a}{q},\frac{1}{q}\right)\overline{H}^*\left(\frac{a'}{4q},\alpha\right),\ \textit{if}\ q \equiv 2 \pmod 4.$$

It is clear from the assertion that the general problem of distribution of values of incomplete Gaussian sums is equivalent to that of investigation of functional properties of $H(x_2,x_1)$. In particular, the *estimate* (1.4) *is equivalent to the boundedness of* $H(x_2,x_1)$ *on the whole real plane* $E^2 = \{x = (x_2,x_1), x_2,x_1 \in E^1 = (-\infty,\infty)\}$.

Estimate (1.4) is due to G. H. Hardy and J. E. Littlewood [5]. Although (1.4) has not been explicitly emphasized in [5], it is a corollary of iterative application of the approximate functional equation which was discovered by Hardy and Littlewood in [5] for the sums

$$W_n'(x_2,x_1)=\sum_{m=1}^{n} e^{\pi i(m^2x_2+2mx_1)},\ (x_2,x_1)\in E^2.$$

Those iterations were carried out in [5, pp.212–213]. E.C. Titchmarsh commented on [5] in [5i], and presented (cf. [5i, pp. 113–114]) a more detailed estimate of incomplete Gaussian sums in terms of denominators of continued fractions from the variable x_2. That comment implies (1.4) as a special case. We note also the joint paper of H. Fiedler, W. Jurkat and O. Koerner [4], devoted to the asymptotical formula of I. M. Vinogradov's type for the quadratic exponential sums W_n' with real variables x_2, x_1. [4] also contains the proof of (1.4), using iterations of the above-mentioned functional equation (cf.[4,pp. 138–139]), which is also derived in [4]. To a certain extent, the approach which we apply here to incomplete Gaussian sums is related with that of [4]. We also note that $W_n'(x_2,x_1)$ for real x_2, x_1, after the Hardy-Littlewood's paper [5], have been considered by several authors. For a survey of those results, the reader may be referred to [5i], [4].

On the other hand, the total boundedness of the function $H(x_2,x_1)$ and hence also the estimate (1.4), (cf. (1.5)) is a consequence of a more general result, which was proved in [1] using I. M. Vinogradov's method of exponential sums (cf. [8]). Namely, *for each fixed positive integer r the following discrete Hilbert transforms*

$$H_N(x_r,\dots,x_2,x_1)=\sum_{1\le|n|\le N}\frac{e^{2\pi i(n^rx_r+\cdots+n^2x_2+nx_1)}}{2\pi in}$$

are uniformly bounded in $N=1,2,\dots$ and real coefficients $x_r,\dots,x_2,x_1$ of the polynomial in the imaginary exponent. Furthermore, as $N\longrightarrow\infty$, the limit

$$H(x_r,\dots,x_1)=\lim_{N\longrightarrow\infty}H_N(x_r,\dots,x_1)$$

exists at all points of the real space $E^r=\{(x_r,\dots,x_2,x_1)\}$.

Here we present a more detailed analysis of the function $H(x_2,x_1)$, and apply it to further considerations of incomplete Gaussian sums. Recall some necessary functional definitions.

Let $h(\xi)$ be some complex-valued and bounded function of the real variable ξ on a certain interval ω, and let

$$\operatorname{osc}(h,\omega)=\sup\{|h(\xi)-h(\eta)|:\xi,\eta\in\omega\}.$$

Let $\gamma\ge1$ and consider collections $\Omega=\{\omega_k\}$ of nonoverlapping subintervals ω_k of ω. If

$$\sup_{\Omega}\sum_k\operatorname{osc}^{\gamma}(h,\omega_k)<\infty,$$

the function h is said to be of *(strongly) bounded γ-variation* on ω; we use the notation $\operatorname{var}_\gamma(h,\omega)$ for the value of the sup at the right hand side. Furthermore, fix a collection

Ω and a positive ε, and count the number (denoted by card) of those $\omega_k \in \Omega$, for which $\mathrm{osc}(h, \omega_k) > \varepsilon$. If

$$\sup_{\Omega} \sup_{\varepsilon > 0} \varepsilon^{\gamma} \operatorname{card}\{\, \omega_k : \mathrm{osc}(h, \omega_k) > \varepsilon \} < \infty,$$

we shall say that h is of *weakly bounded* γ *-variation* on ω, and denote $\mathrm{war}_{\gamma}(h, \omega)$ the value of the sup above. The notion of (strong) 2 -variation was introduced by N. Wiener, and usefulness of γ-variations in Fourier analysis has been thoroughly studied, cf. e.g. [2, Ch. 4]. We can easily see that for $1 \le \gamma < \delta$

$$\mathrm{war}_{\gamma}(h, \omega) \le \mathrm{var}_{\gamma}(h, \omega); \qquad (\mathrm{var}_{\delta}(h, \omega))^{1/\delta} <<_{\gamma,\delta} (\mathrm{war}_{\gamma}(h, \omega))^{1/\gamma},$$

so that if for example h is of weakly bounded 2-variation, then it is of strongly bounded γ -variation for each $\gamma > 2$.

The following assertion is valid.

THEOREM 1. *1) For each fixed real x_2 the function $H(x_2, x_1)$ is of weakly bounded* 2 *-variation in the variable x_1 over the period* $[0, 1)$ *and this property is uniform in x_2 :*

$$\sup_{x_2} \mathrm{war}_2(H(x_2, \cdot), [0, 1)) < \infty \tag{1.6i}$$

2) Let $\Omega = \{\omega_k\}$ be an arbitrary set of nonoverlapping intervals on $[0, 1]$; $q = 1, 2, \ldots; a = 0, \pm 1, \ldots, (a, q) = 1$. *Then the following estimate holds true for the incomplete Gaussian sums which correspond to the intervals $q\omega_k$:*

$$\operatorname{card}\left\{\omega_k : \left|W\left(q\omega_k, \frac{a}{q}\right)\right| > \varepsilon\sqrt{q}\right\} << \varepsilon^{-2} (\varepsilon > 0). \tag{1.6ii}$$

It is clear that for sufficiently large ε the left hand side of (1.6ii) is less than 1 due to (1.6ii), i.e. card $= 0$, which in particular implies (1.4). Furthermore, by (1.5) and (1.3) the second assertion is a consequence of (1.6i). We also note that Theorem 1 is best possible, that is we cannot take strong instead of weak 2-variation in (1.6i), and it can be easily seen that

$$\sum_{k=1}^{q-1} \left| \sum_{n \in q\omega_k} e \frac{2\pi i n^2}{q} \right|^2 >> q \log q, \text{where } \omega_k = (2q)^{-1/2}(k^{1/2}, (k+1)^{1/2}). \tag{1.7}$$

For more details, see section 5 where the proof of Theorem 1 is carried out. In (1.6ii) and (1.7) $q\omega$ is used to denote the interval $\omega' = \{\, \xi : \xi = q\eta, \eta \in \omega \}$.

Consideration of the local properties of $H(x_2, x_1)$ (see Theorem 2, Section 5) enable also to estimate the exact value of the absolute constant times $\sqrt{q}$ in the right hand side of (1.4). The corresponding extremal problem, say in the class of odd denominators q, deals with the computation of the quantity

$$\kappa_0 = \sup \left\{ q^{-1/2} \left| W\left(\omega, \frac{a}{b}\right) \right| : |\omega| \le q, (a, q) = 1, q = 1, 3, 5, \ldots \right\}.$$

This problem was posed by A. A. Karatsuba in a personal conversation with the author. It can also be split into parts, corresponding to the cases of short, middle-length and long, i.e., almost complete sums. The corresponding asymptotic constants can be defined as follows. Let $q \longrightarrow \infty, q \equiv 1 \pmod 2$, and let $\beta > 0$ be such that $\beta q \longrightarrow 0$; furthermore, let

$$\alpha_1 = \beta, \quad \alpha_2 = \frac{1}{2} + \beta, \quad \alpha_3 = 1 - \beta,$$

$$\kappa_i = \limsup_{\substack{q \to \infty \\ \beta q \to 0}} \max_{a:(a,q)=1} q^{-1/2} W_{\alpha_i q}\left(\frac{a}{q}\right), \quad i = 1,2,3;$$

$$F_1 = \max_{\xi > 0} \left| \int_0^\xi e^{-\pi i \lambda^2} d\lambda \right|, \quad F_2 = \max_{\xi > 0} \left| \frac{1}{2} + e^{\frac{\pi i}{4}} \int_0^\xi e^{-\pi i \lambda^2} d\lambda \right|,$$

$$F_3 = \max_{\xi > 0} \left| 1 - \frac{i e^{\frac{\pi i}{4}}}{\sqrt{3}} \int_0^\xi e^{-\pi i \lambda^2} \right|.$$

Each of the latter three quantities is expressed in terms of incomplete Fresnel integral, i.e.,

$$g(\xi) = e^{\frac{\pi i}{4}} \int_0^\xi e^{-\pi i \lambda^2} d\lambda \quad (\xi \geq 0). \tag{1.8}$$

They provide lower bounds for the constants $\kappa_i, i = 1,2,3$ (see Proposition 1 below). The explicit value of κ_0 remains unknown. However, it also can be expressed in terms of incomplete Fresnel integral, taken over *periodic intervals* on the real axis. We call a set $\tilde{\omega}$ in $(-\infty, \infty)$ *a periodic interval* iff there are a positive number T and an ordinary interval ω in $(-\infty, \infty)$ such that

$$\tilde{\omega} = \{\xi : \xi = kT + \eta, k = 0, \pm 1, \dots, \eta \in \omega\}$$

and associate with each $\tilde{\omega}$ the corresponding Fresnel integral, understood in the Cauchy principle value sense:

$$\mathcal{G}(\tilde{\omega}) = e^{-\frac{x\pi i}{4}} \text{p.v.} \int_{-\infty}^{\infty} \chi(\tilde{\omega}, \lambda) e^{-\pi i \lambda^2} d\lambda = e^{-\frac{\pi i}{4}} \lim_{\xi \to +\infty} \int_{-\xi}^{\xi} \dots,$$

where $\chi(\tilde{\omega}, \lambda) = 1$, if $\lambda \in \tilde{\omega}$ and $\chi(\tilde{\omega}, \lambda) = 0$, if $\lambda \notin \tilde{\omega}$. (Note that convergence and boundedness of $\mathcal{G}(\tilde{\omega})$ is a serious matter. In fact it is equivalent to (1.4) or the boundedness of H, cf. Section 3, Lemma 3, and also relations (3.3)).

PROPOSITION 1. *The following relations are valid:*

$$\kappa_0 = \sup_{\tilde{\omega}} |\mathcal{G}(\tilde{\omega})|, \tag{1.9i}$$

where the sup *is taken over the set of all periodic intervals* $\tilde{\omega}$ *;*

$$\kappa_0 \geq 2F_1 = 1.341\dots; \tag{1.9ii}$$

$$\kappa_1 = F_1 > \frac{1}{2}; \tag{1.9iii}$$

$$\kappa_2 \geq F_2 > 1; \tag{1.9iv}$$

$$\kappa_3 \geq F_3 > \sqrt{\frac{13}{12}}. \tag{1.9v}$$

https://doi.org/10.4153/CJM-1991-010-0 Published online by Cambridge University Press

It can be seen from (1.9) that incomplete Gaussian sums exhibit, with respect to the corresponding complete ones, a kind of the well-known Gibbs phenomena in the theory of Fourier series, cf. [10, Ch. 2, Section 9].

We also note that Theorem 1 can be equivalently formulated in terms of functional properties of generalized solutions to the Cauchy initial value problem for Schroedinger equation

$$\frac{\partial \Psi}{\partial t} = \frac{1}{2\pi i}\frac{\partial^2 \Psi}{\partial x^2}, \quad \Psi(x,t)\big|_{t=0} = f(x), \tag{1.10}$$

with periodical (of period 1) initial data $f(x)$. (For more details concerning this relation cf. [6], [6i]). Namely, let $\mathrm{var}_1(f,[0,1)) < \infty$, i.e., $f(x)$ is of ordinary bounded variation on the period. Then there is a solution to (1.10) in the class of regular $\Psi(x,t)$, which is bounded everywhere on $E^2 = \{(x,t)\}$, and moreover, for each fixed t, $\Psi(x,t)$ is of weakly bounded 2-variation in x over $[0,1)$, and

$$\sup_t \mathrm{war}_2(\Psi(\cdot,t),[0,1)) << \mathrm{var}_1^2(f,[0,1)).$$

ACKNOWLEDGEMENTS. The author is indebted to G. I. Arkhipov, A. A. Karatsuba and V. N. Chubarikov for consultations regarding exponential sums.

The present paper has been completed during the author's visit to Canada in frames of the Exchange Agreement between the Steklov Math. Institute–Queens University, May 1989. The author expresses his sincere gratitude to S. Riemenschneider and the staff of the University of Alberta, Edmonton, for their hospitality and friendly cooperation. In particular, the author is indebted to A. Sharma and N. Sivakumar, who checked the language of the manuscript. Special thanks are due to Christine Fischer for beautiful and prompt typesetting.

2. **Identities for Gaussian sums and Vinogradov's continuations.** Let $f(\lambda)$ be a complex-valued, periodic (of period 1) function of the real variable λ, integrable over the period, and let $\hat{f}(n)$ stand for its Fourier coefficients

$$\hat{f}(n) = \int_0^1 f(\lambda)e^{-2\pi i n\lambda}\,d\lambda, \quad n = 0, \pm 1, \dots .$$

We define the *Vinogradov's series (V-series)* of f in two real variable (x_2, x_1) (cf. [6],[6i]) as follows:

$$V(f;x_2,x_1) \sim \sum_n \hat{f}(n)e^{2\pi i(n^2x_2+nx_1)}, \tag{2.1i}$$

where $n = 0, \pm 1, \dots$. The sum of the series (2.1i), whenever it exists is called the (two dimensional) *V-continuation* of f. In our case, we understand the sum of (2.1i) as the limit of the sequence of symmetrical partial sums of the series:

$$\begin{aligned} V(f;x_2,x_1) &= \lim_{N\to\infty} V_N(f;x_2,x_1), \\ V_N(f;x_2,x_1) &= \sum_{|n|\le N} \hat{f}(n)e^{2\pi i(n^2x_2+nx_1)} \end{aligned} \tag{2.1ii}$$

If $\mathrm{var}_1(f,[0,1)) < \infty$, then $V(f;s_2,x_1)$ exists on the whole real plane $E^2 = \{(x_2,x_1)\}$ and is a bounded function of (x_2,x_1) (cf. [6],[6i]; the main point here is the existence and boundedness of $H(x_2,x_1)$).

From this point of view $H(x_2,x_1)$ coincides with the V-continuation of the Bernoulli kernel of the first order:

$$H(x_2,x_1) = V(B;x_2,x_1), \quad B(\lambda) = \text{p.v.} \sum_{n\neq 0} \frac{e^{2\pi in\lambda}}{2\pi in} = \frac{1}{2} - \{\lambda\} \tag{2.2}$$

where $\{\lambda\}$ denotes the fractional part of $\lambda \in (-\infty,\infty), \lambda \neq 0, \pm 1, \dots$. At the rational points on the plane E^2 of the kind $(x_2,x_1) = (\frac{a}{q},\frac{b}{q})$, where $(a,q) = 1$, it is not difficult to compute a representation for H as a finite sum:

$$H\left(\frac{a}{q},\frac{b}{q}\right) = \frac{1}{2qi}\sum_{n=1}^{q-1} e^{2\pi i\frac{an^2+bn}{q}} \cot\frac{\pi n}{q}. \tag{2.3}$$

V-continuations of general functions possess the following periodicity properties in the variables x_2, x_1:

$$V(f;x_2+1,x_1) \equiv V(f;x_2,x_1+1) \equiv V(f;x_2+\frac{1}{2},x_1+\frac{1}{2}) \equiv V(f;x_2,x_1) \tag{2.4}$$

and the symmetry properties of $H(x_2,x_1)$ are expressed by

$$H(\theta_2x_2,\theta_1x_1) = \theta_1\Big(\frac{1+\theta_2}{2}H(x_2,x_1) + \frac{1-\theta_2}{2}\overline{H}(x_2,x_1)\Big) \tag{2.5}$$
$$(\theta_2 = \pm 1, \theta_1 = \pm 1).$$

Next we note some necessary elementary properties of the complete Gaussian sums (shift-formulae in the linear variable). Let q be a positive integer, a,b,ν-integers, $(a,q) = 1$, and denote by a' the (unique in mod$q$) solution of the congruence

$$aa' \equiv 1 \pmod q.$$

Then

$$S\left(\frac{a}{q},\frac{b+2\nu}{q}\right) = e^{-2\pi i\frac{a'(\nu^2+b\nu)}{q}} S\left(\frac{a}{q},\frac{b}{q}\right); \tag{2.6i}$$

$$S\left(\frac{a}{q},\frac{b+\nu}{q}\right) = e^{\frac{-2\pi ia'(4'\nu^2+2'b\nu)}{q}} S\left(\frac{a}{q},\frac{b}{q}\right), \text{if } q\equiv 1 \pmod 2. \tag{2.6ii}$$

Note that if $q \equiv 1 \pmod 2$ and $Q = \frac{q+1}{2}$, then

$$2' = Q, 4' = Q^2, \dots, (2^k)' = Q^k \quad (k = 0,1,\dots). \tag{2.7}$$

Furthermore, given a function $f(\lambda)$, we let

$$f_+(\lambda) = \frac{1}{2}\left(f\left(\frac{\lambda}{2}\right) + f\left(\frac{\lambda+1}{2}\right)\right) \sim \sum_n \hat{f}(2n)e^{2\pi in\lambda},$$

$$f_-(\lambda) = \frac{e^{\pi i\lambda}}{2}\left(f\left(\frac{\lambda}{2}\right) - f\left(\frac{\lambda+1}{2}\right)\right) \sim \sum_n \hat{f}(2n-1)e^{2\pi in\lambda}.$$

The following lemma collects identities that are valid for V-continuations. We assume that the values of $f(\lambda)$ everywhere equal to the sum of its Fourier series, which converges everywhere in the sense of limit of symmetrical partial sums. In particular,

$$f(\lambda)=\frac{f(\lambda+0)-f(\lambda-0)}{2}$$

whenever both of one-sided limits exist. In particular

$$\{0\}=\frac{1}{2}. \tag{2.8}$$

LEMMA 2. *1) For each* $\lambda\in(-\infty,\infty)$

$$V(f;\frac{a}{q},\lambda)=\frac{1}{q}\sum_{\nu=0}^{q-1}S\left(\frac{a}{q},\frac{\nu}{q}\right)f(\lambda-\frac{\nu}{q}). \tag{2.8i}$$

In particular, for odd q and $Q=\frac{q+1}{2}$

$$V(f;\frac{a}{q},\lambda)=\frac{1}{q}S\left(\frac{a}{q}\right)\sum_{\nu=0}^{q-1}e^{\frac{2\pi i a'(Q\nu)^2}{q}}f(\lambda-\frac{\nu}{q}). \tag{2.8ii}$$

For even q and $Q=\frac{q}{2}$

$$V(f;\frac{a}{q},\lambda)=\frac{1}{q}S\left(\frac{a}{q}\right)\sum_{\nu=0}^{Q-1}e^{-\frac{2\pi i a'\nu^2}{q}}f(\lambda-\frac{2\nu}{q}) \tag{2.8iii}$$

$$\textit{if } q\equiv 0 \pmod 4.$$

$$V(f;\frac{a}{q},\lambda)=\frac{1}{q}S\left(\frac{a}{q},\frac{1}{q}\right)\sum_{\nu=0}^{Q-1}e^{-\frac{2\pi i a'(\nu^2+\nu)}{q}}f(\lambda-\frac{2\nu+1}{q}) \tag{2.8iv}$$

$$\textit{if } q\equiv 2 \pmod 4.$$

2) *Define the following rational points on the plane E^2:*

$$\mathbf{y}=\left(\frac{a}{q},\frac{b}{q}\right),\quad \mathbf{y}'=\left(\frac{a}{q},\frac{b-1}{q}\right),\quad \mathbf{y}_*=-\left(\frac{a'}{q},\frac{a'b}{q}\right)$$

$$\mathbf{y}'_*=\left(\frac{a'}{q},\frac{a'(b-1)}{q}\right);\quad \mathbf{y}_{**}=-\left(\frac{(4a)'}{q},\frac{(2a)'b}{q}\right)\quad \textit{if } q\equiv 1 \pmod 2$$

Then if $q\equiv 1 \pmod 2$, we have

$$\sum_{n=1}^{q}f\left(\frac{n}{q}\right)e^{2\pi i\frac{an^2+bn}{q}}=S(y)V(f;y_{**}); \tag{2.9i}$$

if $q\equiv 0 \pmod 2$, then

$$\sum_{n=1}^{q}f\left(\frac{n}{q}\right)e^{2\pi i\frac{(an^2+bn)}{q}}=S(y)V(f_+;y_{**})+S(y')V(f_-;y'_*), \tag{2.9ii}$$

and in the latter case only one summand at the righthand side can be nonzero, i.e. $S(y)S(y') = 0$

The proof of all the assertions is the same. Its background is the following relation

$$\frac{1}{q}\sum_{\nu=0}^{q-1} e^{2\pi i \frac{n\nu}{q}} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod q, \\ 0, & \text{if } n \not\equiv 0 \pmod q, \end{cases}$$

and it suffices to check the identities for functions f of the form $f_m(\lambda) = e^{2\pi i m\lambda}, m = 0, \pm 1, \dots$ (cf. also [6]). Relations (2.8ii)–(2.8iv) follow from (2.8i) and (2.6), where one should take $b = 0$ in the cases (2.8ii) and (2.8iii), and $b = 1$ in the case (2.8iv) cf. also (1.3ii). Relation (2.9i) follows from (2.6ii), (2.9ii) is a corollary of (2.6i), and here the cases of $f_m(\lambda)$ with even and odd m should be treated separately, cf. also (1.3ii). We omit the details.

Next we apply (2.9i), (2.9ii) to the case, when $f(\lambda)$ is the characteristic function of some interval $\omega = [\alpha, \beta]$, whose length $|\omega|$ satisfies $0 < |\omega| \leq 1$. We continue that characteristic function periodically, with period 1, over $(-\infty, \infty)$. Then

$$\text{(2. 10)} \qquad \hat{f}(0) = |\omega|, \quad \hat{f}(n) = \frac{e^{-2\pi i n\alpha} - e^{-2\pi i n\beta}}{2\pi i n} \quad (n = \pm 1, \pm 2, \dots).$$

Given a function $h(\lambda)$, denote

$$h(\lambda)|_{\lambda\in\omega} = h(\beta) - h(\alpha) \quad (\omega = [\alpha, \beta]).$$

Using (2. 10), we see that (here again we omit the details of simple calculations)

$$V(f; x_2, x_1) = (\lambda - H(x_2, x_1 - \lambda))|_{\lambda\in\omega},$$

$$V(f_+; x_2, x_1) = (\lambda - \frac{1}{2}H(x_2, x_1 - 2\lambda))|_{\lambda\in\omega},$$

$$V(f_-; x_2, x_1) = e^{\pi i(\frac{x_2}{2}+x_1)}(\frac{1}{2}H(x_2, x_1 + x_2 - 2\lambda) - H(\frac{x_2}{4}, \frac{x_1 + x_2}{2} - \lambda))|_{\lambda\in\omega}.$$

Thus if we define the vector $\mathbf{x} = (x_2, x_1)$ in three different ways corresponding to Assertion 2) of Lemma $\mathbf{x} = \mathbf{y}_{**}, \mathbf{x} = \mathbf{y}_*, \mathbf{x} = \mathbf{y}'_*$, then keeping in mind that $H(-x) = -\bar{H}(x)$, we arrive at the following relations (cf. also (1.1i))

$$\text{(2. 11}i\text{)} \qquad V(f; \mathbf{y}_{**}) = (\lambda + \bar{H}(\frac{(4a)'}{q}, \frac{(2a)'b}{q} + \lambda))|_{\lambda\in\omega},$$

$$\text{(2. 11}ii\text{)} \qquad V(f_+; \mathbf{y}_*) = (\lambda + \frac{1}{2}\bar{H}(\frac{a'}{q}, \frac{a'b}{q} + 2\lambda))|_{\lambda\in\omega},$$

$$\text{(2. 11}iii\text{)} \qquad V(f_-; \mathbf{y}_*) = e^{-\frac{\pi i a' b}{q}}\bar{H}^*(\frac{a'}{q}, \frac{a'b}{2q} + \lambda)|_{\lambda\in\omega}.$$

In particular, for $b = 0, \omega = [0, \alpha]$, (2.11) and (2.9) (see also (1.3i)) imply (1.5), which proves Lemma 1.

3. **Incomplete Fresnel integrals.** The following assertion is of equiconvergence type and connects the values of V-continuations with the corresponding Fresnel integrals. It may also be considered as a version of the Poisson summation formula.

LEMMA 3. *Let $f(\lambda)$ be square-integrable over the period* $[0,1)$ *and let* ξ, η *be real numbers,* $\xi \neq 0$. *Then*

$$(3.1i) \qquad e^{\frac{\pi i}{4}} \int_{|\lambda| \leq \Lambda} e^{-\pi i \lambda^2} f(\lambda\xi + \eta)\, d\lambda - \sum_{n:|n\xi| \leq \Lambda} \hat{f}(n) e^{\pi i(n^2\xi^2 + 2n\eta)} \longrightarrow 0 \quad (\Lambda \longrightarrow \infty).$$

In particular, if $\mathrm{var}_1(f,[0,1)) < \infty$, *then both the integral and the series converge in Cauchy principal value sense, and*

$$(3.1ii) \qquad \text{p.v. } e^{\frac{\pi i}{4}} \int_{-\infty}^{\infty} e^{-\pi i \lambda^2} f(\lambda\xi + \eta)\, d\lambda = \text{p.v. } \sum_n \hat{f}(n) e^{\pi i(n^2\xi^2 + 2n\eta)} = V(f; \frac{\xi^2}{2}, \eta).$$

PROOF. For fixed ξ, η and $\Lambda \geq 2\xi$ denote $\Delta(\Lambda)$ the expression on the left hand side of (3.1i). Then using the estimates

$$e^{\frac{\pi i}{4}} \int_{|\lambda - \xi n| \leq \Lambda} e^{-\pi i \lambda^2} d\lambda - 1 << \min(1, (\Lambda - |\xi n|)^{-1}) \quad \text{if } |\xi n| \leq \Lambda,$$
$$e^{\frac{\pi i}{4}} \int_{|\lambda - \xi n| \leq \Lambda} e^{-\pi i \lambda^2} d\lambda << \min(1, (|\xi n| - \Lambda)^{-1}) \quad \text{if } |\xi n| > \Lambda,$$

and observing the Fourier expansion of f, we obtain the following estimate:

$$(3.2) \qquad \Delta(\Lambda) << \sum_n |\hat{f}(n)| \min(1, |\Lambda - |\xi n||^{-1}).$$

Here, the right hand side tends to 0 as $\Lambda \longrightarrow +\infty$. To see this, fix a real number M with $|\xi| \leq M \leq \Lambda$, and let

$$\omega_1 = \{n : |\Lambda - |\xi n|| \leq M\}, \omega_2 = \{n : |\Lambda - |\xi n|| > M\},$$
$$\Delta_1(\Lambda, M) = \sum_{n\in\omega_1} |\hat{f}(n)|, \quad \Delta_2(\Lambda, M) = \sum_{n\in\omega_2} |f(n) \|\ \Lambda - |\xi n\ \||^{-1}.$$

We have $\Delta_1(\Lambda, M) \longrightarrow 0$ as $\Lambda \longrightarrow +\infty$, since $|\hat{f}(n)| \longrightarrow 0$ and ω_1 consists of two intervals whose lengths are fixed and equal to $2M|\xi|^{-1}$ and the centres $\pm\Lambda|\xi|^{-1}$ tend to $\pm\infty$. Furthermore,

$$\sum_{n\in\omega_2} (\Lambda - |\xi n|)^{-2} = \xi^{-2} \sum_{n\in\omega_2} (|n| - \Lambda|\xi|^{-1})^{-2} << (M|\xi|)^{-1}$$

which, after application of the Cauchy inequality implies that $\Delta_2(\Lambda, M) <<_f (M|\xi|)^{-1/2}$, and thus by (3.2) $\limsup_{\Delta\to\infty} \Delta(\Lambda) <<_f (M|\xi|)^{-1/2}$. Since M can be arbitrarily large, it follows that $\Delta(\Lambda) \longrightarrow 0$ as $\Lambda \longrightarrow +\infty$, which completes the proof of (3.1i). Finally, if $\mathrm{var}_1(f,[0,1)) < \infty$, then cf. (2.1ii) the limit V of symmetrical partial sums V_N exists by the results in [6],[6i], and (3.1ii) follows.

A comparison of (3.1ii) with (2.9i) shows that if $q \equiv 1 \pmod 2$, then

$$(3.3i) \qquad \sum_{n=1}^{q} f\left(\frac{n}{q}\right) e^{2\pi i \frac{an^2}{q}} = S\left(\frac{a}{q}\right) \text{p.v. } e^{\frac{\pi i}{4}} \int_{-\infty}^{+\infty} e^{-\pi i \lambda^2} f(\lambda\xi)\, d\lambda,$$

https://doi.org/10.4153/CJM-1991-010-0 Published online by Cambridge University Press

where ξ is an arbitrary real solution of the congruence

$$\frac{\xi^2}{2}+\frac{Q^2a'}{q}\equiv 0 \pmod 1 \quad (Q=\frac{q+1}{2}) \tag{3.3ii}$$

In particular, for odd q the incomplete Gaussian sums (1.2) can be expressed as product of the corresponding complete sum and the Fresnel integral $\mathcal{G}(\tilde{\omega})$ over appropriate periodic interval:

$$\sum_{n\in\omega} e^{2\pi i\frac{an^2}{q}} = S\left(\frac{a}{q}\right)e^{\frac{\pi i}{4}}\,\text{p. v.}\int_{\xi^{-1}\tilde{\omega}} e^{-\pi i\lambda^2}\,d\lambda. \tag{3.3iii}$$

Here

$$\begin{aligned}\tilde{\omega}&=\{\lambda:\lambda=k+\eta, k=0,\pm1,\dots;\eta\in\omega\};\\ \xi^{-1}\tilde{\omega}&=\{\lambda:\lambda=\xi^{-1}\eta,\eta\in\tilde{\omega}\}.\end{aligned} \tag{3.3iv}$$

and ξ satisfies (3.3ii). Thus, relation (1.9i), which provides the implicit value of the best posssible constant times $\sqrt{q}$ for $q\equiv 1 \pmod 2$ in (1.4) is a consequence of (3.3).

Next we consider some properties of the following integral analogue of the function $H(x_2,x_1)$:

$$G(x_2,x_1)=\text{p. v.}\int_{-\infty}^{+\infty}\frac{e^{2\pi(\lambda^2x_2+\lambda x_1)}}{2\pi i\lambda}d\lambda = \lim_{\substack{\delta\to 0\\ \Lambda\to\infty}}\int_{\delta<|\lambda|<\Lambda}\cdots. \tag{3.4}$$

PROPOSITION 2. *1) Let $x_2\neq 0, 0<\delta\leq\Lambda, \omega=(\delta,\Lambda)$. Then*

$$\int_{|\lambda|\in\omega}\frac{e^{2\pi(\lambda^2x_2+\lambda x_1)}}{2\pi i\lambda}d\lambda << \min(1,(\delta^2|x_2|+\delta|x_1|)^{-1/2}). \tag{3.5}$$

In particular, the integral defining G exists in the Cauchy principal value sense. 2) Let $g(\xi)(\xi>0)$ denote the incomplete Fresnel integral

$$g(\xi)=e^{\frac{\pi i}{4}}\int_0^{\xi} e^{-\pi i\lambda^2}d\lambda,$$

and let t be a real parameter. Then the values of G satisfy the following relations:

$$G(x_2t^2,x_1t)=\operatorname{sign}t\cdot G(x_2,x_1); \tag{3.6i}$$

$$G(x_2,x_1)=\operatorname{sign}x_1\int_0^{\xi} e^{\pi i(\frac14-\lambda^2)\operatorname{sign}x_2}d\lambda,\ \xi=|x_1|(2|x_2|)^{-1/2}, \tag{3.6ii}$$

or, with the same value of ξ and $\theta_2=\pm1,\theta_1=\pm1$,

$$G(\theta_2x_2,\theta_1x_1)=\theta_1\left(\frac{1+\theta_2}{2}g(\xi)+\frac{1-\theta_2}{2}\bar{g}(\xi)\right); \tag{3.6iii}$$

$$\max_{x_2,x_1}|G(x_2,x_1)| = F_1=\max_{\xi>0}|g(\xi)|; \tag{3.6iv}$$

3) For each fixed x_2 the function $G(x_2,x_1)$ is of weakly bounded 2-variation in the variable x_1 over the whole real axis $(-\infty,\infty)$, and this property is uniform in x_2:

$$\sup_{x_2}\operatorname{war}_2(G(x_2,\cdot),(-\infty,\infty))<\infty; \tag{3.7}$$

4) if $x_2 \neq 0$, then $G(x_2, x_1)$ is not of strongly bounded 2-variation in the variable x_1 over $(-\infty, \infty)$.

PROOF. The estimate (3.5) is a special case of the following more general one, which is true for real polynomials of higher degree in the imaginary exponent:

$$\text{(3.8)} \qquad \int_{\delta<|\lambda|<\Lambda} \frac{e^{2\pi i(\lambda^r x_r+\cdots+\lambda^2 x_2+\lambda x_1)}}{\lambda} d\lambda <<_r \min(1, (\delta^r|x_r| + \cdots + \delta|x_1|)^{-1/r}).$$

For the proof of (3.8) see [6i]; we note that (3.8) follows by integration by parts from I. M. Vinogradov's estimate of oscillatory integral (cf. [8, Ch. 2, Lemma 4]):

$$\text{(3.9)} \qquad \int_0^1 e^{2\pi i(\lambda^r x_r+\cdots+\lambda x_1)} d\lambda \leq \min(1, 32 \parallel x \parallel^{-1/r}), \quad \parallel x \parallel = \max_{1\leq x\leq r} |x_s|.$$

(3.6i) immediately follows by change of variables in the integral, defining G. For $x_2 > 0, x_1 > 0$, (3.6i) implies:

$$\text{(3.10)} \qquad G(x_2, x_1) = G\left(\frac{1}{2}, x_1(2x_2)^{-1/2}\right) = h(\xi), \quad \xi = x_1(2x_2)^{-1/2}$$

where

$$h(\xi) = \text{p.v.} \int_{-\infty}^{+\infty} \frac{e^{\pi i(\lambda^2+2\lambda\xi)}}{2\pi i\lambda} d\lambda \quad (\xi > 0).$$

We have $h(0) = 0 = g(0)$ and

$$\begin{aligned} h'(\xi) &= \text{p.v.} \int_{-\infty}^{+\infty} e^{\pi i(\lambda^2+2\lambda\xi)}\lambda = e^{-\pi i\xi^2}\text{p.v.} \int_{-\infty}^{\infty} e^{\pi i\lambda^2} d\lambda \\ &= e^{\frac{\pi i}{4}} e^{-\pi i\xi^2} = g'(\xi) \quad (\xi > 0) \end{aligned}$$

(it is easy to see that in the above differentiation under the sign of integral is in fact justified). Thus $h(\xi) \equiv g(\xi)$, and (3.10) proves (3.6ii) for $x_2 > 0, x_1 > 0$. Relations (3.6ii)–(3.6iv) then follow, since the symmetry properties of $G(x_2, x_1)$ are the same as those of $H(x_2, x_1)$, cf. (2.5).

To prove assertion 3) according to (3.6ii) it suffices to show that

$$\text{(3.11)} \qquad \text{war}_2(g, [0, \infty)) < \infty.$$

Fix a $\xi > 0$. Since $|g'(\lambda)| \equiv 1$, we have

$$\text{var}_1(g, [0, \xi]) = \int_0^{\xi} |g'(\lambda)| \, d\lambda = \xi.$$

Thus, given a collection $\Omega = \{\omega_k\}$ of nonoverlapping intervals ω_k on $[0, \infty)$ and $\varepsilon > 0$, Chebyshev's inequality yields

$$\text{(3.12)} \qquad \text{card}\{\omega_k : \text{osc}(g, \omega_k) > \varepsilon, \ \omega_k \cap [0, \xi] \neq \emptyset\} \leq \varepsilon^{-1}\xi + 1.$$

Furthermore,

$$g(\infty) = \frac{1}{2}, \quad g(\infty) - g(\xi) = e^{\frac{\pi i}{4}} \int_{\xi}^{\infty} e^{-\pi i\lambda^2} d\lambda << \xi^{-1} \quad (\xi > 0),$$

which implies, that for $\xi^{-1} << \varepsilon$

$$\operatorname{card}\{\,\omega_k : \operatorname{osc}(g,\omega_k) > \varepsilon,\ \omega_k \cup [\xi,\infty) \neq \phi\,\} \leq 1. \tag{3.13}$$

Choose some ξ with $\varepsilon^{-1} << \xi << \varepsilon^{-1}$ so that (3.13) is valid; then (3.11) follows from (3.12).

We note that assertion 3) remains valid for traces of the function $G(x_2, x_1)$ on a much wider class of curves on the plane E^2 than just lines parallel to x_1-axis. In particular, representation (3.6ii) implies that $G(x_2, x_1)$ is of weakly bounded 2-variation on every straight line in E^2.

To prove assertion 4), it again suffices in view of (3.6ii) to show, that

$$\operatorname{var}_2(g,[0,\infty)) = \infty. \tag{3.14}$$

We make use of the following estimate of the tails of the Fresnel integrals:

$$e^{-\frac{\pi i}{4}}(g(\infty) - g(\xi)) - \frac{e^{-\pi i \xi^2}}{2\pi i \xi} << \xi^{-3} \quad (\xi > 0). \tag{3.15}$$

If we take $\xi_k = k^{1/2}, k = 1, 2, \ldots,$ we see from (3.15) that

$$|g(\xi_{k+1}) - g(\xi_k)|^2 - \frac{1}{\pi k} << \frac{1}{k^{3/2}},$$

and thus for $\Lambda > 0$

$$\begin{aligned}\operatorname{var}_2(g,[0,\Lambda]) &\geq \sum_{k:\xi_{k+1}\leq\Lambda} |g(\xi_{k+1}) - g(\xi_k)|^2 \\ &\geq \frac{1}{\pi^2} \sum_{1\leq k\leq \Lambda^2-1} \frac{1}{k} - \text{const} \sim \frac{2}{\pi^2}\log\Lambda \quad (\Lambda \longrightarrow \infty),\end{aligned}$$

which proves (3.14).

REMARK 1. Let $r \geq 3$, and consider the following improper oscillatory integral:

$$G(x_r, \ldots, x_1) = \text{p. v.} \int_{-\infty}^{\infty} \frac{e^{2\pi i(\lambda^r x_r + \cdots + \lambda x_1}}{2\pi i \lambda}\, d\lambda \;=\; \lim_{\substack{\delta\to 0\\ \Lambda\to\infty}} G_{\delta,\Lambda}(x_r, \ldots, x_1),$$

where $x_r, \ldots, x_1$ are real variables and (cf. (3.8))

$$G_{\delta,\Lambda}(x_r, \ldots, x_1) = \int_{\delta<|\lambda|<\Lambda} \frac{e^{2\pi i(\lambda^2 x_r + \cdots + \lambda x_1)}}{\lambda}\, d\lambda.$$

Then it follows from (3.8) that G converges and represents a function which is bounded in the whole r-dimensional real space $E^r = \{(x_r, \ldots, x_1)\}$. The boundedness result can be easily derived from the corresponding one for the discrete sums H_N, mentioned in the introduction. (In fact, Darboux sums for the integral $G_{\delta,\Lambda}$ taken with respect to equidistant partitions of $[\delta, \Lambda]$ are exactly of the form H_N).

However, the assertions concerning the boundedness of $G_{\delta,\Lambda}$ and G are not new. They are due to E. M. Stein and S. Wainger [7] and the author learnt about it recently from the book [3] where the integrals G are discussed in [3i] and [3ii], particularly, the case $r = 2$. For more details on $G(x_r, \ldots, x_1)$ and $H(x_r, \ldots, x_1)$ as functions of real variables $x_r, \ldots, x_1$ and for applications to Cauchy initial value problems for a class of Schroedinger type equations see [6i].

4. **Local properties of** H. Due to (3.6) the origin (0,0) is the only point on the plane E^2 at which $G(x_2, x_1)$ is discontinuous. The set of points discontinuity of $H(x_2, x_1)$ is much more rich, cf. [6i], in fact, it is everywhere dense on the plane. For $q = 1, 2, \ldots$ let $R(q)$ denote the set of rational points on the plane E^2, defined as follows:

$$R(q) = \{\mathbf{y} = (y_2, y_1) = \left(\frac{a}{q}, \frac{b}{q}\right);\ q = 1, 2, \ldots;\ a, b = 0, \pm 1, \ldots, (a, q) = 1\}.$$

and let (cf.(1.3))

$$R = \bigcup_{q=1}^{\infty} R(q); \quad R' = \{\mathbf{y} \in R, S(\mathbf{y}) \neq 0\}$$

Then R' is precisely the set of all points, where $H(x_2, x_1)$ is discontinuous. Moreover, if we fix a vector $\mathbf{y} \in R'$, and consider increments of H in small neighbourhoods of $\mathbf{y}$ in the variable x_1, we can easily compute the exact value of the jump of H at the point $\mathbf{y} = \left(\frac{a}{q}, \frac{b}{q}\right)$:

$$\lim_{\xi,\eta \to +0} (H(y_2, y_1 + \xi) - H(y_2, y_1 - \eta)) = s(\mathbf{y}). \tag{4.1}$$

Here and in what follows $s(\mathbf{y})$ for $\mathbf{y} = (y_2, y_1) \in R(q)$ denotes the normalized Gaussian sum:

$$s(\mathbf{y}) = q^{-1} S(\mathbf{y}).$$

Relation (4.1) is an easy consequence of (2.8i), if we recall that $H(x_2, x_1)$ is the V-continuation of the Bernoulli kernel of the first order (cf. (2.2) and (2.8)), and thus

$$H\left(\frac{a}{q}, \lambda\right) = \frac{1}{q} \sum_{\nu=0}^{q-1} S\left(\frac{a}{q}, \frac{\nu}{q}\right) \left(\frac{1}{2} - \left\{\lambda - \frac{\nu}{q}\right\}\right).$$

Furthermore, it was shown in [6i] that in small neighbourhoods of each rational point $\mathbf{y}$ the increments of H (with respect to $H(\mathbf{y})$) approximately reproduce those of G, the coefficient of reduction being exactly equal to $s(\mathbf{y})$:

$$(H(\mathbf{y} + \mathbf{z}) - H(\mathbf{y})) - s(\mathbf{y}) G(\mathbf{z}) \longrightarrow 0 \quad (\mathbf{z} \longrightarrow 0). \tag{4.2}$$

in particular we have

$$H(\mathbf{x}) - G(\mathbf{x}) \longrightarrow 0 \quad (\mathbf{x} \longrightarrow 0). \tag{4.3}$$

These relations are valid for all $r \geq 2$ (cf. [6i]). Here we specify the case $r = 2$ and provide a more detailed description of increments of H in rectangular neighbourhoods of each $y \in R$. These neighbourhoods are of the form

$$\square(\mathbf{y}) = \{\mathbf{x} = (x_2, x_1), |x_2 - y_2| << q^{-2}, |x_1 - y_1| \leq 0.5 q^{-1}\} \tag{4.4}$$

if $\mathbf{y} = (y_2, y_1) \in R(q)$.

THEOREM 2. *Let $q = 1, 2, \dots, \mathbf{y} = (y_2, y_1) \in R(q)$, and let $\mathbf{z} = (z_2, z_1), \zeta = (\zeta_2, \zeta_1) \in E^2$ be such that the points $x = y + z$ and $x' = y + \zeta$ belong to $\sqsubset\!\!\sqsupset(y)$. Then the following asymptotic formula holds true:*

$$(4.5)\qquad \begin{aligned}&(H(y+z) - H(y+\zeta)) - s(y)(G(z) - G(\zeta))\\ &\qquad << q^{1/2}(|z_2|^{1/2} + |\zeta_2|^{1/2} + |z_1 - \zeta_1|).\end{aligned}$$

PROOF. The main point of the proof is the following auxiliary statement which provides asymptotic formulae of Vinogradov's type (cf. also [4]) for finite quadratic exponential sums.

LEMMA 4. *Let ω be an interval on $[1, \infty)$, κ a real number and q a positive integer, $y = (y_2, y_1) \in R(q), x = y + z$, where $z = (z_2, z_1) \in E^2$ and let*

$$(4.6)\qquad 8nq|z_2| \le 1 \quad (n \in \omega); \; 2q|z_1| \le 1.$$

Then the following estimates hold true:

$$(4.7i)\qquad \sum_{n \in \omega} n^\kappa e(x, n) - s(y) \int_\omega \lambda^\kappa e(z, \lambda)\, d\lambda <<_x q^{1/2} \max_{\lambda \in \omega} \lambda^\kappa,$$

$$(4.7ii)\qquad \sum_{|n| \in \omega} n^\kappa e(x, n) - s(y) \int_{|\lambda| \in \omega} \lambda^\kappa e(z, \lambda)\, d\lambda <<_\kappa q^{1/2} \max_{\lambda \in \omega} \lambda^\kappa,$$

where

$$e(x, \lambda) = e^{2\pi i(\lambda^2 x_2 + \lambda x_1)} \textit{ for } x = (x_2, x_1) \in E^2, \; \lambda \in (-\infty, \infty),$$

$(-1)^\kappa = e^{\pi i \kappa}$, *and in (4.7ii) the summation is taken over all positive and negative integers n with $|n| \in \omega$.*

We postpone the proof of Lemma 4 to Section 6, and now make use of this proposition.

Given a point $x = (x_2, x_1) \in E^2$, consider the sequence of continued fractions (cf. [5], or [9, Ch. 1]) of the leading coordinate x_2:

$$y_{j2} = y_{j2}(x_2) = \frac{a_j}{Q_j} \quad (j = 1, 2, \dots).$$

We add $Q_0 = 1$ to the sequence of denominators $Q_j = Q_j(x_2)$, and if x_2 is a rational number, then the sequence of y_{j2} is finite, and we add $Q = +\infty$ as the final denominator. Furthermore, fix $j = 0, 1, \dots$ and define the rational number y_{j1} as the closest to the coordinate x_1 among the fractions (not necessarily irreducible) with denominators $Q = Q_j(x_2)$, i.e.,

$$y_{j1} = y_{j1}(x_1, x_2) = \frac{b}{Q}, \quad -\frac{1}{2Q} \le x_1 - \frac{b}{Q} < \frac{1}{2Q}.$$

Let $y_j = (y_{j2}, y_{j1}), z_j = x - y_j, j = 0, 1, \dots$. We make use of the following three well-known properties of continued fractions (cf. [9, Ch. 1]) and best rational approximation to real numbers:

(4.8i) the denominators Q_j increase at least as the Fibonacci numbers do:

$$Q_{j+2} \geq Q_{j+1} + Q_j;$$

(4.8ii) the errors z_{2j} satisfy relations

$$\frac{1}{2Q_jQ_{j+1}} \leq |z_{2j}| \leq \frac{1}{Q_jQ_{j+1}};$$

(4.8iii) y_{j2} is the closest rational number to x_2 among all those whose denominators do not exceed Q_j;

(4.8iv) if ξ is a real number, $Q = 1, 2, \ldots,$ A an integer $(A, Q) = 1$ and

$$\left|\xi - \frac{A}{Q}\right| \leq \frac{1}{2Q^2},$$

then $\frac{A}{Q}$ is the closest rational number to ξ among all those rationals whose denominators do not exceed Q.

The latter statement may be complemented as follows:

(4.8v) If ξ is real, $Q = 1, 2, \ldots,$ and for some constant $c \geq 1$ and integer A we have

$$\left|\xi - \frac{A}{Q}\right| \leq \frac{c}{2Q^2},$$

then each rational number of the form B/P with $P \leq c^{-1}Q$ is not closer to ξ than A/Q. Since otherwise we would have had

$$\frac{1}{PQ} \leq \left|\frac{B}{P} - \frac{A}{Q}\right| \leq \left|\xi - \frac{B}{P}\right| + \left|\xi - \frac{A}{Q}\right| < \frac{1}{cQ^2}$$

which obviously contradicts the assumption $P \leq c^{-1}Q$.

Furthermore, we also make use of the following estimate of the tails of the integral G (cf. (3.5))

$$\sup_{L \geq \Lambda}\left|\int_{\Lambda \leq |\lambda| \leq L} \frac{e(z, \lambda)}{\lambda}\, d\lambda\right| << \min(1, \Lambda^{-1}|z_2|^{-1/2}) \quad (\Lambda > 0; z = (z_2, z_1) \in E^2),$$

and we obviously have (cf. (1.3))

$$s(y_j) = Q_j^{-1}S(y_j) << Q_j^{-1/2}. \tag{4.10}$$

Next we define the following intervals on $[1/8, \infty)$, which depend on the denominators $Q_j = Q_j(x_2)$ of continued fractions to x_2 of $x = (x_2, x_1)$;

$$\omega_j = \{\lambda : Q_j(x_2) \leq 8\lambda < Q_{j+1}(x_2)\}, j = 0, 1, \ldots \tag{4.11}$$

(Note that ω_0 may be empty.) If $N \in \omega_j(x_2)$, then on account of (4.8ii) and the definition of y_{1j} we have

$$8nQ_j|z_{2j}| \leq 8nQ_{j+1}^{-1} \leq 1 \quad (n \in \omega_j); \quad 2Q_j|z_{1j}| \leq 1.$$

Thus by Lemma 5 with $\kappa = -1$ and by (4.9), (4.10) and (4.8ii) we obtain

$$\sum_{|n|\in\omega_j, |n|\geq N} \frac{e(x,n)}{n} - s(y_j)\int_{\substack{|\lambda|\in\omega_j\\|\lambda|\geq N}} \frac{e(z_j,\lambda)}{\lambda}\,d\lambda << N^{-1}Q_j^{1/2}; \tag{4.12i}$$

$$\sum_{\substack{|n|\in\omega_j\\|n|\geq N}} \frac{e(x,n)}{n} << N^{-1}Q_j^{1/2} + Q_j^{-1/2}\min(1, N^{-1}Q_j^{1/2}Q_{j+1}^{1/2}) \tag{4.12ii}$$

and in particular also

$$\sum_{\substack{|n|\in\omega_j\\|n|\geq N}} \frac{e(x,n)}{n} << Q_j^{-1/2}. \tag{4.12iii}$$

(Note that it may so happen that there are no integers n satisfying $|n| \in \omega_j$ and $|n| \geq N$, and in that case the corresponding sums obviously equal 0.)

Estimate (4.12iii) along with (4.8i) imply in particular the everywhere convergence and uniform boundedness of the sequence of symmetric partial sums

$$H_N(x) = \sum_{1\leq|n|\leq N} \frac{e(x,n)}{2\pi in}.$$

Obviously, for that purpose less accurate estimates than (4.12iii) would also be sufficient.

Next, for sake of definitiveness, we assume that $|z_2| \geq |\zeta_2|$ and consider separately the following two cases:

$$2q^2|z_2| \leq 1; \tag{4.13i}$$

$$1 < 2q^2|z_2| \leq c, \tag{4.13ii}$$

where c denotes the value of the absolute constant in the sign $<<$, occurring in the definition (4.4) of $\sqsubset(y)$. In the case (4.13i) we let

$$N = |32z_2|^{-1/2}, \quad M = |8qz_2|^{-1}; \quad N' = |32\zeta_2|^{-1/2}, \quad M' = |8q\zeta_2|^{-1}.$$

Then $N \leq M \leq M', N \leq N'$. Let $\omega = [1, N]$,

$$\Delta(\omega) = \sum_{|n|\in\omega} \frac{e(x',n)}{2\pi in} - \sum_{|n|\in\omega} \frac{e(x,n)}{2\pi in},$$

and $z(t) = (1-t)z + t\zeta$ for $0 \leq t \leq 1$. Then

$$\Delta(\omega) = \int_0^1 \left(\sum_{|n|\in\omega} \frac{e(y,n)}{2\pi in}\frac{\partial e(z(t),n)}{\partial t}\right) dt,$$

$$\frac{\partial e(z(t),n)}{\partial t} = 2\pi in((z_2 - z_2)n + (\zeta_1 - z_1)), \tag{4.14}$$

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and consequently

$$\text{(4.15)} \qquad \Delta(\omega) = \int_0^1 \left(\sum_{|n| \in \omega} ((\zeta_2 - z_2)n + (\zeta_1 - z_1))e(x(t), n) \right) dt$$

with $x(t) = y + z(t), 0 \le t \le 1$. For each fixed $t \in [0, 1]$

$$|z_2(t)| = |z_2(1-t) + \zeta_2 t| \le \max(|z_2|, |\zeta_2|) = |z_2|,$$

and thus for $|n| \in \omega, t \in [0, 1]$ we have (cf. also (4.4))

$$8|nqz_2(t)| \le 8Nq|z_2| = NM^{-1} \le 1,$$
$$2|qz_1(t)| \le 2q \max(|z_1|, |\zeta_1|) \le 1.$$

From this we see that for each fixed $t \in [0, 1]$ the conditions of Lemma 5 are satisfied. We apply that lemma, taking $\kappa = 1$ and $\kappa = 0$. Then using (4.7*ii*) we see that for $t \in [0, 1]$

$$\sum_{|n| \in \omega} ((\zeta_2 - z_2)n + (\zeta_1 - z_1))e(x(t), n)$$
$$- s(y) \int_{|\lambda| \in \omega} ((\zeta_2 - z_2)\lambda + (\zeta_1 - z_1))e(z(t), \lambda)\, d\lambda$$
$$<< q^{-1/2}(|\zeta_2 - z_2|N + |\zeta_1 - z_1|).$$

Next we integrate this relation over $t \in [0, 1]$. Interchanging the orders of integration and keeping in mind the definition of N and (4.14), we then arrive at the following approximate representation for $\Delta(\omega)$(cf. also (4.15)):

$$\text{(4.16}i\text{)} \quad \Delta(\omega) - s(y) \int_{|\lambda| \in \Omega} \frac{e(\zeta, \lambda) - e(z, \lambda)}{2\pi i \lambda}\, d\lambda << q^{1/2}(|\zeta_2|^{1/2} + |z_2|^{1/2} + |\zeta_1 - z_1|).$$

If we also apply the estimate $|\sin u| \le |u| (\Im(u) = 0)$ and

$$e(\zeta, \lambda) - e(z, \lambda) << |\zeta_2 - z_2|\lambda^2 + |(\zeta_1 - z_1)\lambda|,$$

we immediately can substitute (4.16i) by
(4.16*ii*)

$$\Delta(\omega) - s(\mathbf{y})\ \text{p. v.} \int_{-N}^{N} \frac{e(\zeta, \lambda) - e(z, \lambda)}{2\pi i \lambda}\, d\lambda << q^{1/2}(|\zeta_2|^{1/2} + |z_2|^{1/2} + |\zeta_1 - z_1|).$$

Furthermore, let $\omega_1 = (N, M), \omega_1' = (N, M')$. Then Lemma 5 is still applicable, and using (4.7ii) with $\kappa = 1$ we get:

$$\text{(4.17}i\text{)} \quad \sum_{|n| \in \Omega_1} \frac{e(x, n)}{2\pi i n} - s(y) \int_{|\lambda| \in \Omega_1} \frac{e(z, \lambda)}{2\pi i n}\, d\lambda << q^{1/2} N^{-1} << q^{1/2} |z_2|^{1/2},$$

$$\text{(4.17}ii\text{)} \quad \sum_{|n| \in \Omega_1'} \frac{e(x', n)}{2\pi i n} - s(y) \int_{|\lambda| \in \Omega_1'} \frac{e(\zeta, \lambda)}{2\pi i n}\, d\lambda << q^{1/2} |z_2|^{1/2}.$$

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It follows from (4.9) that

$$(4.18i)\qquad s(y)\int_{|\lambda|\geq M}\frac{e(z,\lambda)}{2\pi in}\,d\lambda << q^{1/2}M^{-1}|z_2|^{-1/2} << q^{1/2}|z_2|^{1/2},$$

$$(4.18ii)\qquad s(y)\int_{|\lambda|\geq M'}\frac{e(\zeta,\lambda)}{2\pi in}\,d\lambda << q^{1/2}(M')^{-1}|\zeta_2|^{-1/2} << q^{1/2}|\zeta_2|^{1/2}.$$

Next we turn to the estimates of the remainders

$$R=\sum_{|n|\geq M}\frac{e(x,n)}{2\pi in},\quad R'=\sum_{|n|\geq M'}\frac{e(x',n)}{2\pi in}.$$

Since we deal with the case (4.13i), it follows from (4.8iv) and (4.8iii) that there is such a j that $q=Q_j(x_2)$, i.e., q belongs to the sequence of denominators of the continued fractions for x_2. Then by (4.8ii) we also have

$$\frac{1}{2qQ_{j+1}(x_2)}\leq|z_{2j}(x_2)|=|z_2|\leq\frac{1}{qQ_{j+1}(x_2)}.$$

Moreover

$$M=\frac{1}{8q|z_2|}\geq\frac{Q_{j+1}(x_2)}{8}.$$

Thus using (4.12iii) with $j+1$ instead of j and M instead of N we see that

$$(4.19i)\qquad R << Q_{j+1}^{-1/2}(x_2) << q^{1/2}|z_2|^{1/2}.$$

Exactly the same observations show that

$$(4.19ii)\qquad R' << q^{1/2}|\zeta_2|^{1/2}.$$

Now (4.16)–(4.19) imply (4.5), which completes the proof in the case (4.13i). As for the case (4.13ii), we notice that (4.5) is equivalent to

$$(4.20i)\qquad H(y+z)-H(y+\zeta) << q^{-1/2}.$$

The proof follows the same lines as in the case (4.13i) and uses the property (4.8v) instead of (4.8iv) when estimating the remainders R and R'. We omit the details.

REMARK 2. Let

$$y=\left(\frac{a}{q},\frac{b}{q}\right)\in R(q),\ y'=\left(\frac{a}{q},\frac{b+\theta}{q}\right)\text{ where }\theta=0\text{ or }\pm 1,$$

and let $x\in\sqsubset\!\sqsupset(y)$, $x'\in\sqsubset\!\sqsupset(y')$. Then

$$(4.20ii)\qquad H(x)-H(x') << q^{-1/2}.$$

This is an easy consequence of (4.20i).

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5. **Proofs of Theorem 1 and Proposition 1 (completed).** Given x_2, a collection $\Omega = \{\omega_k\}$ of nonoverlapping intervals of $[0, 1)$ and a positive number ε, let

$$h(\xi) = H(x_2, \xi), \quad \Omega(\varepsilon) = \{\omega_k \in \Omega, \operatorname{osc}(h, \omega_k) > \varepsilon\},$$
$$\Omega_j(\varepsilon) = \{\omega_k \in \Omega(\varepsilon), \quad Q_{j+1}^{-1} < |\omega_k| \leq Q_j^{-1}\}. \tag{5.1}$$

Clearly, we can assume that $\operatorname{osc}(h, \omega_k) = |h(\beta_k) - h(\alpha_k)|$, where $\omega_k = [\alpha_k, \beta_k]$. In (5.1) Q_j denotes the denominator of the continued fraction for x_2 under consideration (cf. Section 4). Note, that we may due to (2.4), assume that $0 \leq x_2 \leq 1/2$, and moreover that x_2 is an irrational number (the case of rational x_2 can be obviously treated by a continuity argument). Let $y_{2j} = a_j Q_j^{-1}$ denote the continued fractions themselves, and for each fixed j and $b = 0, \pm 1, \ldots$. Let

$$\sqsubset\!\sqsupset_{j,b} = \{(\xi_2, x_1) : |\xi_2 - y_{2j}| \leq (Q_j Q_{j+1})^{-1}, 2|x_1 - bQ_j^{-1}| \leq Q_j^{-1}\}$$

Theorem 2 can be applied to estimate $\operatorname{osc}(h, \omega_k)$ for the intervals ω_k in $\Omega_j(\varepsilon)$, and thus due to (5.1), (3.6iv) and (4.20ii), the set $\omega_j(\varepsilon)$ is empty if $Q_j \geq c\varepsilon^{-2}$, where c is a suitable, sufficiently large, but absolute, constant. On the other hand, since $|\omega_k| > Q_{j+1}^{-1}$ for $\omega_k \in \Omega_j(\varepsilon)$, we trivially have

$$\operatorname{card} \Omega_j(\varepsilon) << Q_{j+1}$$

which by (4.8i) implies that

$$\sum_{j : Q_{j+1} << \varepsilon^{-2}} \operatorname{card} \Omega_j(\varepsilon) << \varepsilon^{-2}. \tag{5.2}$$

Therefore, we are left to consider only those j's for which the two-sided restrictions

$$Q_j << \varepsilon^{-2} << Q_{j+1} \tag{5.3i}$$

hold true. By (4.8i) the number of such j's is bounded uniformly in ε:

$$\operatorname{card}\{j : Q_j << \varepsilon^{-2} << Q_{j+1}\} << 1. \tag{5.3ii}$$

Fix a j with (5.3i) and first consider all those $\omega_k \in \Omega_j(\varepsilon)$ which contain points of the type $(b - 0.5)Q_j^{-1}$, where $b = 0, \pm 1 \ldots$. Clearly, the number of such ω_k's does not exceed Q_j, and thus their total number over all j with (5.3i) is $<< \varepsilon^{-2}$ on account of (5.3ii). For each of the remaining $\omega_k \in \Omega_j(\varepsilon)$ there is such an integer b that

$$\omega_k \subset I_{j,b} = [(b - 0.5)Q_j^{-1}, (b + 0.5)Q_j^{-1}]. \tag{5.4}$$

In order to estimate $\operatorname{osc}(h, \omega_k)$ from above, we make use of Theorem 2 with

$$y = y_{j,b} = (a_j Q_j^{-1}, bQ_j^{-1}), z_2 = \zeta_2 = x_2 - a_j Q_j^{-1},$$
$$z_1 = \alpha_k - bQ_j^{-1}, \zeta_1 = \beta_k - bQ_j^{-1}, \text{ where } \omega_k = [\alpha_k, \beta_k].$$

Then by (4.5) we see that for each $\omega_k \in \Omega_j(\varepsilon), \omega_k \subset I_{j,b}$, one of the following three estimates is valid:

$$\varepsilon << Q_j^{1/2}(|z_2|^{1/2} + |\zeta|^{1/2}) << Q_{j+1}^{-1/2} \tag{5.5i}$$

(cf. (4. 8*ii*));

$$(5.5ii) \qquad \varepsilon << Q_j^{1/2}|\beta_k - \alpha_k| = Q_j^{1/2}|\omega_k|;$$

$$(5.5iii) \qquad \varepsilon << Q_j^{-1/2}|G(z_2, \alpha_k - bQ_j^{-1}) - G(z_2, \beta_k - bQ_j^{-1}|.$$

Due to (5.3i) relation (5. 5*i*) is either impossible or it means that $Q_{j+1} << Q_j << \varepsilon^{-2}$ and thus

$$(5.6i) \qquad \sum_{1\le b\le Q_j} \text{card}\,\{\,\omega_k : (5.5i)\text{ holds },\omega_k \in \Omega_j(\varepsilon), \omega_k \subset I_{j,b}\} << \varepsilon^{-2}$$

Furthermore, since $\sum_k |\omega_k| \le 1$, by Chebyshev's inequality and (5.3i) we have

$$(5.6ii) \qquad \text{card}\{\,\omega_k : (5.5ii)\text{ holds }\} << Q_j^{1/2}\varepsilon^{-1} << \varepsilon^{-2}.$$

Finally, to estimate the number of those k for which (5.5iii) is valid with a fixed b, we apply (3.7). It follows that this number is $<< (\varepsilon Q_j^{1/2})^{-2} = \varepsilon^{-2}Q_j^{-1}$ and consequently

$$(5.6iii) \qquad \sum_{1\le b\le Q_j} \text{card}\{\,k : (5.5iii)\text{ holds }\} << \varepsilon^{-2}.$$

Estimates (5.6) and (5.3) imply that card $\Omega(\varepsilon) << \varepsilon^{-2}$, and thus

$$\text{war}_2(h, [0,1)) = \text{war}_2(H(x_2,\cdot), [0,1)) << 1$$

which completes the proof of Theorem 1 ((1.6ii) is a corollary of (1.6i) and Lemma 1).

Furthermore, to prove (1.7), we make use of the following well known asymptotic formula (cf. Lemma 4 with $q = 1, \kappa = 0$)

$$\sum_{m=1}^{n} e^{\frac{2\pi i m^2}{Q}} - \int_0^n e^{\frac{2\pi i m^2}{Q}}\,d\lambda << 1 \quad (8n \le Q,\ Q = 1, 2, \ldots).$$

If we take $\alpha_k = (Qk/2)^{1/2}, k = 0, 1, 2, \ldots, 32k \le Q$, and put $\omega_k = [\alpha_{k-1}, \alpha_k)$ we readily see that (cf. (3.15))

$$\sum_k \left| \sum_{n\in Q\omega_k} e^{\frac{2\pi i m^2}{Q}} \right|^2 >> Q \log Q,$$

and (1.7) follows.

We proceed to the proof of Proposition 1. (We have already proved (1.9i), cf. relations (3.3)). First of all note, that

$$g(+\infty) = e^{\frac{\pi i}{4}} \int_0^{+\infty} e^{-\pi i \lambda^2}\,d\lambda = \frac{1}{2},$$

and thus, using the asymptotical formula (3.15) we see that the following strict inequalities are valid for the quantities F_1, F_2, F_3:

$$(5.7) \qquad \begin{aligned} &F_1 > g(+\infty) = \frac{1}{2}; \quad F_2 > \frac{1}{2} + g(\infty) = 1; \\ &F_3 > \left|1 - \frac{1}{\sqrt{3}} g(\infty)\right| = \left|1 - \frac{i}{2\sqrt{3}}\right| = \sqrt{\frac{13}{12}}. \end{aligned}$$

Now denote by ξ_1, ξ_2, ξ_3 respectively, those values of ξ for which the max in the definitions of the quantities F_1, F_2, F_3 is attained. We present below explicit (up to the numerical values of ξ_1, ξ_2, ξ_3) examples of "bad" denominators q, arguments $\frac{a}{q}$ and intervals ω, for which the corresponding incomplete Gaussian sums are "large" in the asymptotical sense as $q \longrightarrow \infty$. We consider odd denominators q only, and use the notation $Q = (q+1)/2$ (cf. (2.7)).

1) In the case of the short sums, i.e., $W_\alpha(aq^{-1})$ with $\alpha q = o(q), (\alpha > 0)$ as $q \longrightarrow \infty$, fix an integer $k = 0, 1, \ldots$, and let

$$(5.8i) \qquad a = Q^{k+2} \text{ (i.e., } (4a)' = 2^k), \alpha q = \xi_1\sqrt{2^{k+1}q}.$$

1i) In the case of short sums, symmetrical with respect to 0, i.e., $W(\omega, aq^{-1})$ with $\omega = [-\alpha q, \alpha q], \alpha q = o(q)(\alpha > 0)$, define a and α as above, cf. (5.8i).

3) In the case of middle-length sums, i.e., $W_\alpha(aq^{-1})$ with $\alpha = 1/2 + \beta$, where $\beta > 0, \beta q = o(q)\,(q \longrightarrow \infty)$, let

$$(5.8ii) \qquad a = Q, \text{ (i.e., } (4a)' = Q)\ \alpha q = \frac{q}{2} + \xi_2\sqrt{q}.$$

4) In the case of almost complete sums, i.e., $W_\alpha(aq^{-1})$ with $\alpha = 1 - \beta$, where $\beta > 0, \beta q = o(q)\,(q \longrightarrow \infty)$, let

$$(5.8iii) \qquad \begin{gathered} q \equiv -1 \pmod 6; a = -3Q^2 \quad \text{(i.e., } (4a)' = -\frac{2Q}{3}), \\ \alpha q = q - \xi_3\sqrt{\frac{2q}{3}}. \end{gathered}$$

In all these three cases we use the identity (1.5i), and the following asymptotic relations, which are special cases of Theorem 2 (see also (3.6ii)) ($z_2 > 0, z_1 \geq 0$).

$$(5.9i) \qquad H(z_2, z_1) - g\left(\frac{z_1}{(2z_2)^{1/2}}\right) << z_2^{1/2} + z_1$$

$$(5.9ii) \quad H(\frac{1}{3} + z_2, z_1) - \frac{i}{\sqrt{3}}g\left(\frac{z_1}{(2z_2)^{1/2}}\right) << z_2^{1/2} + z_1\left(\frac{i}{\sqrt{3}} = s\left(\frac{1}{3}, 0\right)\right).$$

Case 1). Let (cf. (5.8i)), for some fixed $k = 0, 1, \ldots$

$$z_2 = \frac{(4a)'}{q} = \frac{2^k}{q}, \quad z_1 = \alpha = \xi_1\sqrt{\frac{2^{k+1}}{q}}, \quad \text{i.e., } z_1(2z_2)^{-1/2} = \xi_1.$$

Then we see from (1.5i) and (5.9i) that

$$W_\alpha\left(\frac{a}{q}\right) - S\left(\frac{a}{q}\right)\bar{g}(\xi_1) <<_k 1,$$

and the result follows by the definition of ξ_1.

Case 2). Let (cf. (5.8ii))

$$z_2 = \frac{(4a)'}{q} - \frac{1}{2} = \frac{Q}{q} - \frac{1}{2} = \frac{1}{2q}; \quad z_1 = \alpha - \frac{1}{2} = \frac{\xi_2}{\sqrt{q}},$$

i.e.,

$$z_1(2z_2)^{-1/2} = \xi_2,$$

and use the identity (cf.(2.4))

$$H(x_2 + \frac{1}{2}, x_1 + \frac{1}{2}) \equiv H(x_2, x_1).$$

Again by (1.5i) and (5.9i) we have

$$W_\alpha\left(\frac{a}{q}\right) - S\left(\frac{a}{q}\right)\left(\frac{1}{2} + \bar{g}(\xi_2)\right) << 1,$$

and the result follows from the definition of ξ_2.

Case 3). Let (cf. (5.8iii))

$$z_2 = \frac{(4a)'}{q} + \frac{1}{3} = -\frac{2Q}{3q} + \frac{1}{3} = -\frac{1}{3q},$$
$$z_1 = 1 - \alpha = \xi_3\sqrt{\frac{2}{3q}}, \qquad \text{i.e., } z_1(2|z_2|)^{-1/2} = \xi_3.$$

Then by (1.5i) and (2.4), (2.5) we have

$$W_{1-z_1}\left(\frac{a}{q}\right) = S\left(\frac{a}{q}\right)(1 - z_1 + \bar{H}(z_2 - \frac{1}{3q}, 1 - z_1))$$
$$= S\left(\frac{a}{q}\right)(1 - z_1 - \bar{H}(\frac{1}{3} + \frac{1}{3q}, z_1)),$$

and therefore, by (5.9ii),

$$W_\alpha\left(\frac{a}{q}\right) - S\left(\frac{a}{q}\right)\left(1 - \frac{i}{\sqrt{3}}g(\xi_3)\right) << 1,$$

which concludes the proof of the estimates $\kappa_k \geq F_k, k = 1, 2, 3$.

Finally we show that in the case of $k = 1$ we actually have equality $\kappa_1 = F_1$ (cf. (1.9i-ii)). This assertion means that the *maximum F_1 of incomplete Fresnel integrals equals the best possible value of the constant $\times\sqrt{q}$ in the estimate*

$$W_\alpha\left(\frac{a}{a}\right) << \sqrt{q}$$

for incomplete Gaussian sums of short length (i.e., $\alpha > 0, \alpha q \longrightarrow 0\,(q \longrightarrow \infty)$; we consider only odd denominators q, but an appropriate modification seems to be also true for even q). We already have the inequality $\kappa_1 \geq F_1$ and thus (cf. (1.5i)) to prove the assertion, it suffices to establish the following estimate for $H(x_2, x_1)$:

$$|H(x_2, x_1)| \leq F_1 + c|x_1|^{1/2}, \tag{5.10}$$

where c is an absolute positive constant. Clearly, we can assume that $x_2 \in (0, 1/2), x_1 \in (0, 1/2)$. Consider the continued fractions $a_j Q_j^{-1}, j = 0, 1, 2, \ldots$, corresponding to x_2 (see section 4; we let $a_0 = 0, Q_0 = 1$), and find a number j such that

$$x_1 < \frac{1}{2Q_j}, x_1 \geq \frac{1}{2Q_{j+1}}.$$

Let $\mathbf{y}_j = (a_j Q_j^{-1}, 0)$; then $\mathbf{x} = (x_2, x_1) \in \square(y_j)$ (cf. (4.8ii) and (4.4)) and $\mathbf{H}(y_j) = 0$. Thus, we can apply Theorem 2, and it follows from (4.5) and (4.8ii) that

$$\begin{aligned}\mathbf{H}(x_2, x_1) - s(\mathbf{y}_j)G(x_2 - a_j Q_j^{-1}, x_1) &<< Q_j^{1/2}(|x_2 - a_j Q_j^{-1}|^{1/2} + |x_1|) \\ &<< Q_{j+1}^{-1/2} + Q_j^{1/2}|x_1| << |x_1|^{1/2}.\end{aligned}$$

Since $|s(\mathbf{y})| \leq 1$ for all $\mathbf{y} \in \mathbb{R}$ and $\max |G| = F_1$ (cf. (3.6iv)), (5.10) follows, thereby completing the proof.

6. **Some formulae of summation.** The main goal of this Section is to prove Lemma 4, which was used above in the proof of Theorem 2. This is achieved by auxiliary statements—Lemmas 5 and 6, which are of a well known type in Analytic Number Theory: oscillatory sums are substuted by appropriate integrals, and the corresponding error terms are estimated. This general scheme originates from early papers of Vinogradov and the work of Van der Corput.

Let $B(\lambda)$ denote the Bernoulli kernel of the first order, (cf. (2.2)), i.e., $B(\lambda) = 1/2 - \{\lambda\}$ for real nonintegral λ, and it is again convenient to assume, that $B(m) = 1/2$, if $m = 0, \pm 1, \ldots$. Fix a complex parameter $\varphi, \varphi \neq \pm 1, \pm 2, \ldots$, and let for the real variable λ (cf. also [5i, p. 234])

$$\Delta(\varphi, \lambda) = \text{p. v.} \sum_{n \neq 0} \frac{e^{2\pi i n \lambda}}{2\pi i(n + \varphi)} = \frac{1}{2\pi i}\left(\frac{\pi e^{2\pi i \varphi B(\lambda)}}{\sin \pi \varphi} - \frac{1}{\varphi}\right), \tag{6.1i}$$

$$\Delta^{(1)}(\varphi, \lambda) = \frac{\partial \Delta(\varphi, \lambda)}{\partial \varphi} = -\sum_{n \neq 0} \frac{e^{2\pi i n \lambda}}{2\pi i(n + \varphi)^2}, \tag{6.1ii}$$

$$\Delta^{(2)}(\varphi, \lambda) = \frac{\partial^2 \Delta(\varphi, \lambda)}{\partial \varphi^2} = \sum_{n \neq 0} \frac{e^{2\pi i n \lambda}}{\pi i(n + \varphi)^3}, \tag{6.1iii}$$

LEMMA 5. *Let $\omega = [\alpha, \beta]$ be a closed interval on the real axis, $|\omega| = \beta - \alpha > 0$, and $\Phi(\lambda)$ a complex valued function, twice continuously differentiable on ω, $|\Phi(\lambda)| > 0 (\lambda \in \omega)$, and possessing the property: the logarithmic derivative*

$$\varphi(\lambda) = \frac{1}{2\pi i} \frac{\Phi'(\lambda)}{\Phi(\lambda)} \tag{6.3}$$

does not take on any integral values on ω, with the possible exception of zero, i.e.,

$$\min_{\lambda \in \omega} \min_{|n| = 1, 2, \ldots} |\varphi(\lambda) - n| = \rho > 0 \tag{6.4}$$

Furthermore, let (cf. (6.2))

$$D(\phi) = \sum_{\nu\in\omega}{}'\Phi(\nu) - \int_\omega \Phi(\lambda)\, d\lambda,$$

($\sum'$ denotes the sum in which the summand, corresponding to the end point of ω is taken with the factor $1/2$ whenever this end is an integer),

$$a(\lambda) = \Delta(\varphi(\lambda), \lambda), \quad b(\lambda) = \Delta^{(1)}(\varphi(\lambda), \lambda) \quad (\lambda \in \omega).$$

Then

$$D(\Phi) = \Phi(\lambda)a(\lambda)|_{\lambda\in\omega} - \int_\omega \Phi(\lambda)\varphi'(\lambda)b(\lambda)\, d\lambda. \tag{6.5i}$$

In particular, the following estimate is valid:

$$\begin{aligned} D(\Phi) &<<_\rho \max_\omega |\Phi|(1 + \operatorname{var}_1(\varphi, \omega)) \\ &= \max_\omega |\Phi|(1 + \int_\omega |\varphi'(\lambda)| d\lambda). \end{aligned} \tag{6.5ii}$$

PROOF. For each fixed $\varphi \neq \pm 1, \pm 2, \ldots, \Delta(\varphi, \lambda)$ is differentiable for $\lambda \neq 0, \pm 1, \pm 2, \ldots$, and

$$\frac{\partial\Delta(\varphi, \lambda)}{\partial\lambda} + 2\pi i\varphi\Delta(\varphi, \lambda) = -1 \quad (\lambda \neq 0, \pm 1, \ldots). \tag{6.6}$$

This may be easily checked using the explicit representation of Δ in (6.1i). Moreover, keeping (6.4) in mind and using the series representations of $B(\lambda)$ and $a(\lambda)$ (cf. (6.1i)), we see that the function

$$c(\lambda) = B(\lambda) - a(\lambda) = \varphi(\lambda) \sum_{n\neq 0} \frac{e^{2\pi i n\lambda}}{2\pi i n(n + \varphi(\lambda))}$$

is continuous on ω, and is continuously differentiable everywhere on that interval except for the integral points $\lambda \in \omega$. On account of (6.6) and definitions of $a(\lambda), b(\lambda)$ the derivative $c'(\lambda)$ is computed as follows:

$$\begin{aligned} c'(\lambda) = B'(\lambda) - a'(\lambda) &= -1 - \frac{\partial\Delta(\varphi, \lambda)}{\partial\lambda}\Big|_{\varphi=\varphi(\lambda)} - \varphi'(\lambda)\frac{\partial\Delta(\varphi, \lambda)}{\partial\varphi}\Big|_{\varphi=\varphi(\lambda)} \\ &= 2\pi i\varphi(\lambda)a(\lambda) - \varphi'(\lambda)b(\lambda) \quad (\lambda \neq 0, \pm 1, \ldots, \lambda \in \omega). \end{aligned}$$

In particular, $c(\lambda)$ is absolutely continuous on ω, and integration by parts shows that

$$\begin{aligned} D(\Phi) &= \int_\omega \Phi\, dB = \int_\omega \Phi\, d(a + c) \\ &= \int_\omega \Phi\, da + \int_\omega \Phi c'\, d\lambda = \Phi a|_\omega + \int_\omega (\Phi c' - \Phi' a)\, d\lambda \\ &= \Phi a|_\omega + \int_\omega \Phi \cdot (2\pi i\varphi a - \varphi' b - 2\pi i\varphi a)\, d\lambda \\ &= \Phi a|_\omega - \int_\omega \Phi\varphi' b\, d\lambda, \end{aligned}$$

and (6.5i) follows. The estimate (6.5ii) is a consequence of (6.5i) and (6.4), since

$$|a(\lambda)| + |b(\lambda)| <<_\rho 1 \quad (\lambda \in \omega). \tag{6.7}$$

LEMMA 6. *Let q be a positive integer, $e = e_m$ $(m = 0, \pm 1, \ldots)$ a sequence of complex numbers, periodic in m with period q i.e.,*

$$e_{m+q} \equiv e_m \quad (m = 0, \pm 1, \ldots) \tag{6.8}$$

and let

$$s = \frac{1}{q} \sum_{m=1}^{q} e_m, \quad V = \max_{1 \le n \le q} \left| \sum_{m=1}^{n} e_m \right|.$$

Furthermore, let ω be an interval in $(-\infty, \infty)$, and $F(\lambda)$ a complex valued function on ω, three times continuously differentiable and vanishing nowhere on ω, and assume that the function

$$\varphi(\lambda) = \frac{1}{2\pi i} \frac{F'(\lambda)}{F(\lambda)}$$

does not take on any values of the form $\frac{a}{q}$, $a = \pm 1, \pm 2, \ldots$ on ω except perhaps the value 0, *i.e.,*

$$\min_{\lambda \in \omega} \min_{|a| = 1,2,\ldots} |q\varphi(\lambda) - a| = \rho > 0. \tag{6.9}$$

Then the following estimate holds true:

$$\sum_{m \in \omega} e_m F(m) - s \int_\omega F(\lambda)\, d\lambda <<_\rho V(\max_\omega |F|) \left(1 + \sum_{j=1}^{5} \delta_j\right) \tag{6.10}$$

where

$$\delta_1 = \delta_1(\varphi, q) = \max \left\{ \int_\alpha^\beta |\varphi|\, d\lambda; \quad (\alpha, \beta) \subset \omega, |\beta - \alpha| \le q \right\},$$

$$\delta_2 = \delta_2(\varphi', q) = q \int_\omega |\varphi'|\, d\lambda; \quad \delta_3 = \delta_3(\varphi\varphi', q) = q^2 \int_\omega |\varphi\varphi'|\, d\lambda;$$

$$\delta_4 = \delta_4(\varphi'', q) = q^2 \int_\omega |\varphi''|\, d\lambda; \quad \delta_5 = \delta_5(\varphi', q) = q^3 \int_\omega |\varphi'|^2\, d\lambda.$$

In particular, if $\Re(\varphi')$ and $\Im(\varphi')$ are piecewise monotonous and the number of intervals of monotonicity does not exceed an absolute constant, then

$$\sum_{m \in \omega} e_m F(m) - s \int_\omega F(\lambda)\, d\lambda <<_\rho V(\max_\omega |F|)(1 + \delta_6^2 + \delta_7 + \delta_6\delta_7), \tag{6.11}$$

where $\delta_6 = q \max_\omega |\varphi|, \delta_7 = q^2 \max_\omega |\varphi'|$.

PROOF. With no loss of generality we can assume that: a) $\max_\omega |F| = 1$; b) $q \ge 2$ (cf. Lemma 5); c) ω is of the form $[1, \alpha]$, where $\alpha > 1$.

First consider the case of short intervals ω, namely $\alpha << q$. Then we obviously have

$$s \int_\omega F\, d\lambda << q^{-1} V|\omega| << V. \tag{6.12i}$$

Let $p = [\alpha]$ and

$$W_n = \sum_{m=1}^{n} e_m, \quad 1 \leq n \leq p.$$

Since $p << q$, we have $W_n << V$, and thus using summation by parts we see that

$$\text{(6.12}ii\text{)} \qquad \sum_{m \in \omega} e_m F(m) = \sum_{m=1}^{p-1} W_m(F(m) - F(m+1)) + W_p F(p)$$
$$= -\sum_{m=1}^{p-1} W_m \int_m^{m+1} F'(\lambda)\, d\lambda + W_p F(p)$$
$$<< V(\int_\omega |F'(\lambda)|\, d\lambda + 1) << V(\int_\omega |\varphi(\lambda)|\, d\lambda + 1)$$
$$<< V(1 + \delta_1)$$

Estimates (6.12) imply (6.10) in the case when $|\omega| << q$. Now, it suffices to consider intervals ω of the form $\omega = [1, Nq]$, where N is a positive integer. For such an interval let

$$\tilde{\omega} = [0, N-1], \Phi(\mu, \lambda) = F(\mu + \lambda q) \quad (\lambda \in \tilde{\omega},\ 1 \leq \mu \leq q),$$
$$I_m = \int_{\tilde{\omega}} \Phi(m, \lambda)\, d\lambda \quad (m = 1, 2, \ldots, q).$$

Using (6.8) we see that

$$\text{(6.13}i\text{)} \qquad \sum_{m \in \omega} e_m F(m) = \sum_{m=1}^{Nq} e_m F(m) = \sum_{m=1}^{q} e_m \left(\sum_{j \in \tilde{\omega}} \Phi(m, \nu) \right)$$

Summation by parts as in (6.12ii) easily shows, that for each fixed $\nu \in \tilde{\omega}$ we have

$$\sum_{m=1}^{q} e_m \Phi(m, \nu) << V(1 + \delta_1),$$

and thus, up to the same error, we can substitute Σ by Σ' in (6.13i) (cf. Lemma 5)

$$\text{(6.13}ii\text{)} \qquad \sum_{m \in \omega} e_m F(m) - \sum_{m=1}^{q} e_m \sum_{\nu \in \tilde{\omega}}{}' \Phi(m, \nu) << V(1 + \delta_1).$$

Furthermore,

$$I_m = \int_0^{N-1} F(m + \lambda q)\, d\lambda = \frac{1}{q} \int_m^{(N-1)q+m} F(\lambda)\, d\lambda,$$
$$|I_m - I_{m+1}| \leq 2q^{-1}; \quad \left| I_q - \frac{1}{q} \int_\omega F(\lambda)\, d\lambda \right| \leq 1,$$

and thus after summation by parts we obtain:

$$\sum_{m=1}^{q} e_m I_m = \sum_{m=1}^{q-1} W_m (I_m - I_{m+1}) + W_q I_q,$$

$$\text{(6.14)} \qquad \sum_{m=1}^{q} e_m I_m - s \int_\omega f(\lambda)\, d\lambda << V \left(\sum_{m=1}^{q-1} | I_m - I_{m+1} | \right.$$
$$\left. + \left| I_q - \frac{1}{q} \int_\omega F(\lambda) d\lambda \right| \right) << V.$$

Now we apply the identity (6.5i) to the functions $\Phi(m,\lambda)$ for $m = 1,\ldots,q$ and $\lambda \in \tilde{\omega}$. By the assumption (6.9) the requirement (6.4) is satisfied for all these functions, since

$$(6.15)\qquad (2\pi i\Phi(m,\lambda))^{-1}\frac{\partial\Phi(m,\lambda)}{\partial\lambda} = q\varphi(\lambda q+m) \quad (m=1,\ldots,q; \quad \lambda\in\tilde{\omega}).$$

Then, using (6.13ii), (6.14) and (6.5i) we see that

$$(6.14i)\qquad \sum_{m\in\omega} e_m F(m) - s\int_\omega F(\lambda)\,d\lambda << V(1+\delta_1)+|R_1-R_2|,$$

where for $1\le\mu\le q$ (cf. (6.1))

$$R_1 = \sum_{m=1}^{q} e_m A(m),\quad A(\mu)+\Phi(\mu,\lambda)a(\mu,\lambda)|_{\lambda\in\tilde{\omega}};$$
$$a(\mu,\lambda) = \Delta(q\varphi(\lambda q+\mu),\lambda);$$
$$R_2 = \sum_{m=1}^{q} e_m B(m),\quad B(\mu) = \int_\omega \Phi(\mu,\lambda)q^2\varphi'(\lambda q+\mu)b(\mu,\lambda)\,d\lambda,$$
$$b(\mu,\lambda) = \Delta^{(1)}(q\varphi(\lambda q+\mu),\lambda).$$

To estimate R_1 and R_2, we use summation by parts once again. For convenience we will use the symbol ∂ to denote differentiation with respect to the parameter μ, i.e., $\partial = \frac{\partial}{\partial\mu}$. Then we have

$$(6.16i)\qquad |R_1| \le V\left(\int_1^q |\partial A(\mu)\,d\mu + \max_{1\le\mu\le q}|A(\mu)|\right),$$

$$(6.16ii)\qquad |R_2| \le V\left(\int_1^q |\partial B(\mu)\,d\mu + \max_{1\le\mu\le q}|B(\mu)|\right),$$

According to (6.9) (cf. also (6.7)) we see that

$$(6.17)\qquad |a(\mu,\lambda)|+|b(\mu,\lambda)| <<_\rho 1 \quad (\mu\in[1,q],\ \lambda\in\tilde{\omega})$$

and therefore

$$(6.18i)\qquad \max_{\mu\in[1,q]}|A(\mu)| <<_\rho 1;$$

$$(6.18ii)\qquad \max_{\mu\in[1,q]}|B(\mu)| <<_\rho q^2\max_{\mu\in[1,q]}\int_{\tilde{\omega}}|\varphi'(\lambda q+\mu)|\,d\lambda$$
$$\le q\int_\omega|\varphi'(\lambda)|\,d\lambda = \delta_2.$$

Fix a $\lambda\in\tilde{\omega}$ and introduce the notations

$$\Phi(\mu,\lambda)=\Phi(\mu),\quad a(\mu,\lambda)=a_\mu,\quad \varphi(\lambda q+\mu)=\varphi_\mu,$$
$$\varphi'(\lambda q+\mu)=\varphi'_\mu \quad \varphi''(\lambda q+\mu)=\varphi''_\mu,\quad b(\mu,\lambda)=b_\mu,$$
$$\Delta^{(2)}(q\varphi(\lambda q+\mu),\lambda)=h_\mu.$$

We have

$$\partial\Phi_\mu = \frac{\partial\Phi_\mu}{\partial\mu} = \frac{\partial F(\lambda q+\mu)}{\partial\mu} = 2\pi i\varphi_\mu\Phi_\mu;$$
$$\partial a_\mu = q\varphi'(\lambda q+\mu)\Delta^{(1)}(q\varphi(\lambda q+\mu),\lambda)$$
$$= q\varphi'(\lambda q+\mu)\, b(\mu,\lambda) = q\varphi'_\mu b_\mu;$$
$$\partial b_\mu = q\varphi'_\mu h_\mu;$$
$$\partial(\Phi_\mu, a_\mu) = a_\mu\partial\Phi_\mu + \Phi_\mu\partial a_\mu = \Phi_\mu(2\pi i\varphi_\mu a_\mu + q\varphi'_\mu b_\mu); \tag{6.19i}$$
$$\partial(\Phi_\mu q^2\varphi'_\mu b_\mu) = q^2\Phi_\mu(2\pi i\varphi_\mu\varphi'_\mu + \varphi''_\mu b_\mu + q(\varphi'_\mu)^2 h_\mu). \tag{6.19ii}$$

Keeping in mind (6.17) and the analogous estimate for h_μ, namely

$$|h_\mu| \le \max_{\lambda\in\omega}\max_{1\le\mu\le q}|\Delta^{(2)}(q\varphi(\lambda q+\mu),\lambda)| <<_\rho 1,$$

we see (6.19) that

$$|\partial(\Phi_\mu a_\mu)| <<_\rho |\varphi_\mu| + q|\varphi'_\mu|,$$
$$|\partial(\Phi_\mu q^2\Phi'_\mu b_\mu)| <<_\rho (|\varphi_\mu\varphi'_\mu| + |\varphi''_\mu + q|\varphi'_\mu|^2).$$

It follows from these estimates and from (6.16) that

$$|R_1| <<_\rho V\left(1+\max_{\lambda\in\tilde\omega}\int_1^q(|\varphi(\lambda q+\mu)| + q|\varphi'(\lambda q+\mu)|)d\mu\right)$$
$$<<_\rho V(1+\delta_1+\delta_2),$$
$$|R_2| <<_\rho V(\delta_2 + q^2\int_\omega(|\varphi\varphi'| + |\varphi''|)d\lambda + q^3\int_\omega|\varphi'|^2 d\lambda)$$
$$= V(\delta_2+\delta_3+\delta_4+\delta_5).$$

These estimates and (6.14i), (6.16), (6.18) imply the validity of (6.10) with δ_j, $1\le j\le 5$, in the remainder terms. Now, if $\Re\varphi'(\lambda)$ and $\Im\varphi'(\lambda)$ are piecewise monotonous, the same is true for $\Re\varphi(\lambda)$ and $\Im\varphi(\lambda)$. In this case we have

$$\delta_1 \le \delta_6: \quad \delta_2 \le q\ \mathrm{var}_1(\varphi,\omega) << \delta_6 \le 1+\delta_6^2;$$
$$\delta_3 \le (q\max_\omega|\varphi|)\quad (q\,\mathrm{var}_1(\varphi,\omega)) << \delta_6^2;$$
$$\delta_4 \le q^2\,\mathrm{var}_1(\varphi',\omega) << q^2\max_\omega|\varphi'| = \delta_7;$$
$$\delta_5 \le (q^2\max_\omega|\varphi'|)\cdot(q\ \mathrm{var}_1(\varphi,\omega)) << \delta_6\delta_7,$$

and (6.11) follows, completing the proof of Lemma 6.

Now we are in a position to complete the proof of Lemma 4. Let

$$e(m) = e(y,m): \quad F(\lambda) = \lambda^x e(z,\lambda) \quad (\lambda\in\omega)$$

Then (cf. (1.4))

$$(6.20i) \qquad s = \frac{1}{q}\sum_{m=1}^{q} e(y,m) = s(y);$$

$$\varphi(\lambda) = \frac{1}{2\pi i}\frac{F'(\lambda)}{F(\lambda)} = \frac{\kappa}{2\pi i\lambda} + 2\lambda z_2 + z_1;$$

$$\varphi'(\lambda) = \frac{x}{2\pi i\lambda^2} + 2z_2;$$

$$(6.20ii) \qquad V = \max_{1\le n\le q}\left|\sum_{m=1}^{n} e(y,m)\right| << q^{1/2}.$$

We first consider (4.7) for the intersection $\tilde{\omega} = \omega \cap [1,q]$. If $\tilde{\omega} \neq \emptyset$, we estimate the integrals trivially:

$$s(y)\int_{\tilde{\omega}} \lambda^\kappa e(z,\lambda)\,d\lambda << q^{-1/2}\max_\omega \lambda^\kappa; \quad q = q^{1/2}\max_\omega \lambda^\kappa$$

and apply summation by parts to $\sum_{\tilde{\omega}}$:

$$\sum_{n\in\tilde{\omega}} n^\kappa e(x,n) = \sum_{n\in\tilde{\omega}} e(y,n)n^\kappa e(z,\omega)$$
$$<< V\left(\sum_{n\in\tilde{\omega}} |n^\kappa e(z,n) - (n+1)^\kappa e(z,n+1)| + \max_{\lambda\in\omega}\lambda^\kappa\right)$$
$$<< q^{1/2}(\max_\omega \lambda^\kappa)\left(\int_1^{q+1}(|z_2\lambda| + |z_1|)\,d\lambda + 1\right).$$

Therefore we see from (4.6) that both the sum and the integral of (4.7) which correspond to the interval $\tilde{\omega}$ can be included in the remainder term $q^{1/2}\max\lambda^\kappa$. Thus we are left with the case when $\min_\omega \lambda \geq q$. In that case we have the following estimates for the remainder terms δ_6 and δ_7 of (6.11)(cf. (4.6)):

$$(6.21i) \qquad \delta_6 << q(\max_\omega \lambda^{-1} + \max_\omega |z_2| + |z_1|)) << 1,$$

$$(6.21ii) \qquad \delta_7 << q^2(\max_\omega \lambda^{-2} + |z_2|) << 1.$$

On the other hand (4.6) ensures that the condition (6.9) is satisfied with $\rho = \frac{1}{4}$, and therefore (4.7i) is a consequence of (6.11), (6.21) and (6.20).

As for (4.7ii), we let, for $\lambda \in \omega$,

$$e^*(m) = e(y,-m) + e(-m), \quad F^*(\lambda) = F(-\lambda) = (-\lambda)^\kappa e(z,-\lambda),$$

and from exactly the same considerations as above we conclude that

$$(6.22) \qquad \sum_{n\in\omega}(-n)^\kappa e(\mathbf{x},-n) - s^*(\mathbf{y})\int_{\lambda\in\omega}(-\lambda)^\kappa e(z,-\lambda)\,d\lambda << q^{1/2}\max_\omega \lambda^\kappa,$$

where

$$s^*(\mathbf{y}) = \frac{1}{q}\sum_{m=1}^{q} e^*(m) = \frac{1}{q}\sum_{m=1}^{q} e(y,-m).$$

It is a very simple, but essential consideration, that $s^*(\mathbf{y}) = s(\mathbf{y})$, since $(-m)$ runs over the same complete system of residues mod$q$ as m does. Thus, (4.7ii) is a consequence of (6.22) and (4.7i).

REMARK 3. In connection with Theorem 1, it seems to be of interest to consider variational properties of $H(x_2, x_1)$ on other curves on the plane E^2, in particular, those of $H(\xi, x_1), \xi \in [0, 1)$ for fixed x_1.

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