Canad. Math. Bull. Vol. 66 (2), 2023, pp. 443–449
http://dx.doi.org/10.4153/S0008439522000418

Groups whose Chermak–Delgado lattice is a subgroup lattice of an abelian group

Lijian An

Abstract. The Chermak–Delgado lattice of a finite group G is a self-dual sublattice of the subgroup lattice of G. In this paper, we prove that, for any finite abelian group A, there exists a finite group G such that the Chermak–Delgado lattice of G is a subgroup lattice of A.

1 Introduction

Suppose that G is a finite group, and H is a subgroup of G. The Chermak–Delgado measure of H (in G) is denoted by $m_G(H)$, and defined as $m_G(H) = |H| \cdot |C_G(H)|$. The maximal Chermak–Delgado measure of G is denoted by $m^*(G)$, and defined as

$$m^*(G) = \max\{m_G(H) \mid H \leq G\}.$$

Let

$$\mathcal{CD}(G) = \{H \mid m_G(H) = m^*(G)\}.$$

Then the set $\mathcal{CD}(G)$ forms a sublattice of $\mathcal{L}(G)$ (the subgroup lattice of G), which is called the Chermak–Delgado lattice of G. It was first introduced by Chermak and Delgado [9], and revisited by Isaacs [12]. In the last years, there has been a growing interest in understanding this lattice (see, e.g., [1–11, 13–17, 19–22]).

A Chermak–Delgado lattice is always self-dual. So the question arises: Which types of self-dual lattices can be Chermak–Delgado lattices of finite groups? In [5], it is proved that, for any integer n, a chain of length n can be a Chermak–Delgado lattice of a finite p-group.

A quasi-antichain is a lattice consisting of a maximum, a minimum, and the atoms of the lattice. The width of a quasi-antichain is the number of atoms. For a positive integer $w \geq 3$, a quasi-antichain of width w is denoted by $\mathcal{M}_w$. In [6], it was proved that $\mathcal{M}_w$ can be a Chermak–Delgado lattice of a finite group if and only if $w = 1 + p^a$ for some positive integer a and some prime p.

An m-diamond is a lattice with subgroups in the configuration of an m-dimensional cube. A mixed n-string is a lattice with n components, adjoined end to end, so that the maximum of one component is identified with the minimum

Received by the editors January 22, 2022; revised June 13, 2022, accepted June 13, 2022.
Published online on Cambridge Core June 17, 2022.
This work was supported by NSFC (Grant No. 11971280).
AMS subject classification: **20D30**, 20D15.
Keywords: Chermak–Delgado lattice, subgroup lattice, finite p-groups.

of the other component. The following theorem gives more self-dual lattices which can be Chermak–Delgado lattices of finite groups.

Theorem 1.1 [4] *If $\mathcal{L}$ is a Chermak–Delgado lattice of a finite p-group G such that both $G/Z(G)$ and G' are elementary abelian, then so are $\mathcal{L}^+$ and $\mathcal{L}^{++}$, where $\mathcal{L}^+$ is a mixed 3-string with center component isomorphic to $\mathcal{L}$ and the remaining components being m-diamonds, and $\mathcal{L}^{++}$ is a mixed 3-string with center component isomorphic to $\mathcal{L}$ and the remaining components being lattice isomorphic to $\mathcal{M}_{p+1}$.*

By [18, Theorem 8.1.4], $\mathcal{L}(A)$ is always self-dual for any finite abelian group A. If A is a cyclic p-group, then $\mathcal{L}(A)$ is chain, and hence can be a Chermak–Delgado lattice of a finite p-group. In [2], it is proved that, if A is an elementary abelian p-group, then $\mathcal{L}(A)$ can be a Chermak–Delgado lattice of a finite p-group. In this paper, we prove that, for any finite abelian group A, $\mathcal{L}(A)$ can be a Chermak–Delgado lattice of a finite group. The main results are the following theorems.

Theorem 1.2 *For any finite abelian p-group A, there exists a finite p-group G such that $\mathcal{CD}(G)$ is isomorphic to $\mathcal{L}(A)$.*

Theorem 1.3 *For any finite abelian group A, there exists a finite group G such that $\mathcal{CD}(G)$ is isomorphic to $\mathcal{L}(A)$.*

2 Preliminary

We gather next some basic properties of the Chermak–Delgado lattice, which will be used often throughout the paper without further reference.

Theorem 2.1 [9] *Suppose that G is a finite group and $H, K \in \mathcal{CD}(G)$.*

(1) *$\langle H, K\rangle = HK$. Hence, a Chermak–Delgado lattice is modular.*
(2) *$C_G(H \cap K) = C_G(H)C_G(K)$.*
(3) *$C_G(H) \in \mathcal{CD}(G)$ and $C_G(C_G(H)) = H$. Hence, a Chermak–Delgado lattice is self-dual.*
(4) *Let M be the maximal member of $\mathcal{CD}(G)$. Then M is characteristic in G and $\mathcal{CD}(M) = \mathcal{CD}(G)$.*
(5) *The minimal member of $\mathcal{CD}(G)$ is characteristic, abelian, and contains $Z(G)$.*

We also need the following lemmas.

Theorem 2.2 [7, Theorem 2.9] *For any finite groups G and H, $\mathcal{CD}(G \times H) = \mathcal{CD}(G) \times \mathcal{CD}(H)$.*

Lemma 2.3 [2, Lemma 3.3] *Suppose that G is a finite group and $H \le G$ such that $G = HC_G(H)$. If $H \in \mathcal{CD}(H)$, then H is contained in the unique maximal member of $\mathcal{CD}(G)$.*

Lemma 2.4 [20, Lemma 5] *Let G be a finite p-group. Then $\mathcal{CD}(G) = [G/Z(G)]$ if and only if the interval $[G/Z(G)]$ of $\mathcal{L}(G)$ is modular and G' is cyclic.*

https://doi.org/10.4153/S0008439522000418 Published online by Cambridge University Press

In this section, we prove that, for any finite abelian group A, $\mathcal{L}(A \times A)$ can be a Chermak–Delgado lattice of a finite group. Although this result can be deduced from our main theorem, the proof is independent and short.

Lemma 2.5 *Let A be a finite abelian p-group. Then there exists a finite p-group G such that $\mathcal{CD}(G)$ is isomorphic to $\mathcal{L}(A \times A)$.*

Proof Assume that the type of A is $(p^{e_1}, p^{e_2}, \dots, p^{e_m})$, where $e_1 \geq e_2 \geq \cdots \geq e_m$. Let G be the group generated by $2m$ elements $x_1, \dots, x_m, y_1, \dots, y_m$ subject to the defining relations:

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ if } i \neq j,$$

$$x_i^{p^{e_i}} = y_i^{p^{e_i}} = z^{p^{e_1}} = 1, [x_i, y_i] = z^{p^{e_1 - e_i}}, [z, x_i] = [z, y_i] = 1 \text{ for } 1 \leq i \leq m.$$

Let $P_i = \langle x_i, y_i, z \rangle$. Then $Z(P_i) = \langle z \rangle$. Thus, G is also the central product of P_i. It is easy to see that $G' = Z(G) = \langle z \rangle$ and $G/Z(G) \cong A \times A$. By Lemma 2.4, $\mathcal{CD}(G)$ is just the interval $[G/Z(G)]$. Hence, $\mathcal{CD}(G) \cong \mathcal{L}(G/Z(G)) \cong \mathcal{L}(A \times A)$. ∎

Theorem 2.6 *For any finite abelian group A, there exists a finite group G such that $\mathcal{CD}(G)$ is isomorphic to $\mathcal{L}(A \times A)$.*

Proof Let $A = A_1 \times \cdots \times A_n$, where A_i is the Sylow p_i-subgroup of A. By Lemma 2.5, there exist finite group P_i such that $\mathcal{CD}(P_i)$ is isomorphic to $\mathcal{L}(A_i \times A_i)$. Let $G = P_1 \times \cdots \times P_n$. By Theorem 2.2,

$$\begin{aligned}\mathcal{CD}(G) &= \mathcal{CD}(P_1) \times \cdots \times \mathcal{CD}(P_n) \\ &\cong \mathcal{L}(A_1 \times A_1) \times \cdots \times \mathcal{L}(A_n \times A_n) \\ &= \mathcal{L}(A \times A).\end{aligned}$$

∎

3 The groups $G(p, e)$

For any prime p and an integer $e \geq 1$, we use $G(p, e)$ to denote the finite p-group generated by three elements x, y, w subject to the following defining relations:

- $[x, y] = z_1$, $[y, w] = z_2$, $[w, x] = z_3$,
- $x^{p^e} = y^{p^e} = w^{p^e} = z_1^{p^e} = z_2^{p^e} = z_3^{p^e} = 1$, and
- $[z_i, x] = [z_i, y] = [z_i, w] = 1$ for all $i = 1, 2, 3$.

In this section, we prove that the Chermak–Delgado lattice of $G(p, e)$ is isomorphic to a subgroup lattice of a cyclic group of order p^e. This group will be used to construct an example in the proof of Theorem 1.2. Let $G = G(p, e)$. Then it is easy to check the following results:

- $d(G) = 3$, $\exp(G) = p^e$, $Z(G) = G' = \langle z_1, z_2, z_3 \rangle$, and
- $|Z(G)| = p^{3e}$, $|G/Z(G)| = p^{3e}$, $m_G(G) = m_G(Z(G)) = p^{9e}$.

https://doi.org/10.4153/S0008439522000418 Published online by Cambridge University Press

Lemma 3.1 *Assume that $G = G(p, e)$ and $Z(G) < H < G$.*

(1) *If $H/Z(G)$ is cyclic, then $m_G(H) < m_G(G)$.*
(2) *If $H/Z(G)$ is not cyclic, then $m_G(H) \leq m_G(G)$, where " = " holds if and only if the type of $H/Z(G)$ is $(p^{e_1}, p^{e_1}, p^{e_1})$ for some $1 \leq e_1 < e$.*

Proof (1) Let $H = \langle h, Z(G)\rangle$ and $H/Z(G)$ be of order p^{e_1}. Then we may let

$$h = x^{k_1 p^{e-e_1}} y^{k_2 p^{e-e_1}} w^{k_3 p^{e-e_1}},$$

where $p \nmid k_i$ for some i. Without loss of generality, we may assume that $p \nmid k_1$. Replacing x with $x^{k_1} y^{k_2} w^{k_3}$, we have $h = x^{p^{e-e_1}}$. It is easy to check that $C_G(H) = \langle x, y^{p^{e_1}}, w^{p^{e_1}}\rangle Z(G)$. Since $|C_G(H)/Z(G)| = p^{3e-2e_1}$,

$$|H/Z(G)| \cdot |C_G(H)/Z(G)| = p^{3e-e_1} < p^{3e} = |G/Z(G)|.$$

Hence, $m_G(H) = |H| \cdot |C_G(H)| < |G| \cdot |Z(G)| = m_G(G)$.

(2) Let $H = \langle h_1, h_2, h_3\rangle Z(G)$ and $H/Z(G)$ be of type $(p^{e_1}, p^{e_2}, p^{e_3})$, where $e_1 \geq e_2 \geq e_3 \geq 0$. Since $H/Z(G)$ is not cyclic, $e_2 \geq 1$. By a similar argument as (1), we may assume that $h_1 = x^{p^{e-e_1}}$. We may let

$$h_2 = x^{k_1 p^{e-e_2}} y^{k_2 p^{e-e_2}} w^{k_3 p^{e-e_2}},$$

where $p \nmid k_i$ for some $2 \leq i \leq 3$. Without loss of generality, we may assume that $p \nmid k_2$. Replacing y with $x^{k_1} y^{k_2} w^{k_3}$, we have $h_2 = y^{p^{e-e_2}}$. It is easy to check that

$$C_G(H) = C_G(h_1) \cap C_G(h_2) = \langle x^{p^{e_2}}, y^{p^{e_1}}, w^{p^{e_1}}\rangle Z(G).$$

Since $|H/Z(G)| = p^{e_1+e_2+e_3}$ and $|C_G(H)/Z(G)| = p^{3e-e_2-2e_1}$,

$$|H/Z(G)| \cdot |C_G(H)/Z(G)| = p^{3e+e_3-e_1} \leq p^{3e} = |G/Z(G)|,$$

where " = " holds if and only if $e_3 = e_1$. Hence, $m_G(H) = |H| \cdot |C_G(H)| \leq |G| \cdot |Z(G)| = m_G(G)$, where " = " holds if and only if $e_1 = e_2 = e_3$. ■

Theorem 3.2 *Let $G = G(p, e)$. Then $G \in \mathcal{CD}(G)$ and $\mathcal{CD}(G)$ is a chain of length e. Moreover, $H \in \mathcal{CD}(G)$ if and only if $H = \langle x^{p^{e-e_1}}, y^{p^{e-e_1}}, w^{p^{e-e_1}}\rangle Z(G)$ for some $0 \leq e_1 \leq e$.*

Proof By Lemma 3.1, $m^*(G) = m_G(G) = p^{9e}$, and $H \in \mathcal{CD}(G)$ if and only if the type of $H/Z(G)$ is $(p^{e_1}, p^{e_1}, p^{e_1})$ for some $0 \leq e_1 \leq e$. Hence, all elements of $\mathcal{CD}(G)$ are $\langle x^{p^{e-e_1}}, y^{p^{e-e_1}}, w^{p^{e-e_1}}\rangle Z(G)$ where $0 \leq e_1 \leq e$. ■

4 The proof of main results

For any prime p and an abelian p-group A with type $(p^{e_1}, p^{e_2}, \ldots, p^{e_m})$, where $e_1 \geq e_2 \geq \cdots \geq e_m$, we use G_A to denote the finite p-group generated by $3m$ elements $x_1, \ldots, x_m, y_1, \ldots, y_m, w_1, \ldots, w_m$ subject to the following defining relations:

- $x_i^{p^{e_i}} = y_i^{p^{e_i}} = w_i^{p^{e_i}} = z_1^{p^{e_1}} = z_2^{p^{e_1}} = z_3^{p^{e_1}} = 1$ for $1 \leq i \leq m$,
- $[x_i, x_j] = [y_i, y_j] = [w_i, w_j] = [x_i, y_j] = [y_i, w_j] = [w_i, x_j] = 1$ if $i \neq j$,

https://doi.org/10.4153/S0008439522000418 Published online by Cambridge University Press

- $[x_i, y_i] = z_1^{p^{e_1-e_i}}$, $[y_i, w_i] = z_2^{p^{e_1-e_i}}$, $[w_i, x_i] = z_3^{p^{e_1-e_i}}$ for $1 \le i \le m$, and
- $[z_j, x_i] = [z_j, y_i] = [z_j, w_i] = 1$ for $1 \le i \le m$ and $j = 1, 2, 3$.

In this section, we require the following notation and straightforward results for a finite p-group $G = G_A$.

- $Z(G) = G' = \langle z_1, z_2, z_3\rangle$ is of order p^{3e_1}.
- Let $P_i = \langle x_i, y_i, w_i\rangle$ for $1 \le i \le m$. Then $P_i \cong G(p, e_i)$, $|P_i Z(G)/Z(G)| = p^{3e_i}$, and G is the central product $P_1 * P_2 * \cdots * P_m$.
- Let $X = \langle x_1, x_2, \ldots, x_m\rangle$, $Y = \langle y_1, y_2, \ldots, y_m\rangle$, and $W = \langle w_1, w_2, \ldots, w_m\rangle$. Then $X \cong Y \cong W \cong A$.
- Let $n = e_1 + e_2 + \cdots + e_m$. Then $|A| = p^n$, $|G/Z(G)| = p^{3n}$, $|G| = p^{3n+3e_1}$, and $m_G(G) = p^{3n+6e_1}$.
- Let α, β, γ be isomorphisms from A to X, Y, W, respectively, such that $x_i^{\alpha^{-1}} = y_i^{\beta^{-1}} = w_i^{\gamma^{-1}}$ for all $1 \le i \le m$.
- For $a \in A$, let $a^\varphi = \langle a^\alpha, a^\beta, a^\gamma\rangle Z(G)$.
- For $B \le A$, let $B^\varphi = \langle B^\alpha, B^\beta, B^\gamma\rangle Z(G) = \prod_{b\in B} b^\varphi$.

The proof of Theorem 1.2 Assume that the type of A is $(p^{e_1}, p^{e_2}, \ldots, p^{e_m})$, where $e_1 \ge e_2 \ge \cdots \ge e_m$. Let $G = G_A$. We will prove $\mathcal{CD}(G) \cong \mathcal{L}(A)$ in six steps.

(1) $G \in \mathcal{CD}(G)$ and $m^*(G) = p^{3n+6e_1}$.

By Theorem 3.2, $P_i \in \mathcal{CD}(P_i)$. Since $G = P_i C_G(P_i)$, by Lemma 2.3, P_i is contained in the unique maximal member of $\mathcal{CD}(G)$. Hence, G is the unique maximal member of $\mathcal{CD}(G)$ and $m^*(G) = m_G(G) = p^{3n+6e_1}$.

(2) For any $a \in A$, there exists a subgroup C_a of A such that $C_X(a^\beta) = C_X(a^\gamma) = (C_a)^\alpha$, $C_Y(a^\alpha) = C_Y(a^\gamma) = (C_a)^\beta$, and $C_W(a^\alpha) = C_W(a^\beta) = (C_a)^\gamma$.

Notice that for $x \in X$, $[x, a^\beta] = 1$ if and only if $[x, a^\gamma] = 1$. We have $C_X(a^\beta) = C_X(a^\gamma)$. Let $C_a = (C_X(a^\beta))^{\alpha^{-1}}$. Then $C_X(a^\beta) = C_X(a^\gamma) = (C_a)^\alpha$. Notice that for $c \in A$, $[c^\alpha, a^\gamma] = 1$ if and only if $[c^\beta, a^\gamma] = 1$. We have

$$c \in C_a \Longleftrightarrow c^\alpha \in C_X(a^\gamma) \Longleftrightarrow c^\beta \in C_Y(a^\gamma).$$

It follows that $C_Y(a^\gamma) = (C_a)^\beta$. By the symmetry, the conclusions hold.

(3) $C_G(a^\varphi) = (C_a)^\varphi$ and $a^\varphi \in \mathcal{CD}(G)$.

Suppose that a is of order p^t. Then $|a^\varphi/Z(G)| = p^{3t}$. Since $[a^\alpha, G] \le \langle z_1^{p^{e_1-t}}, z_3^{p^{e_1-t}}\rangle$, the length of the conjugacy class of a^α does not exceed p^{2t}. Hence, $|C_G(a^\alpha)| \ge p^{3n+3e_1-2t}$ and $|C_G(a^\alpha)/Z(G)| \ge p^{3n-2t}$. Notice that

$$C_G(a^\alpha)/Z(G) = XZ(G)/Z(G) \times C_Y(a^\alpha)Z(G)/Z(G) \times C_W(a^\alpha)Z(G)/Z(G),$$

$|XZ(G)/Z(G)| = |X| = p^n$, and by (2),

$$|C_a| = |C_Y(a^\alpha)| = |C_W(a^\alpha)| = |C_Y(a^\alpha)Z(G)/Z(G)| = |C_W(a^\alpha)Z(G)/Z(G)|.$$

We have $|C_a| \ge p^{n-t}$. Hence, $|(C_a)^\varphi/Z(G)| \ge p^{3n-3t}$. By (2), $(C_a)^\varphi \le C_G(a^\varphi)$. Hence,

$$|a^\varphi/Z(G)| \cdot |C_G(a^\varphi)/Z(G)| \ge |a^\varphi/Z(G)| \cdot |(C_a)^\varphi/Z(G)| \ge p^{3n} = |G/Z(G)|.$$

https://doi.org/10.4153/S0008439522000418 Published online by Cambridge University Press

It follows that

$$m_G(a^\varphi) = |a^\varphi| \cdot |C_G(a^\varphi)| \geq |G| \cdot |Z(G)| = m^*(G).$$

Thus, " = " holds, $C_G(a^\varphi) = (C_a)^\varphi$, and $a^\varphi \in \mathcal{CD}(G)$.

(4) For any $B \leq A$, $B^\varphi \in \mathcal{CD}(G)$ and there exists a subgroup C_B of A such that $C_G(B^\varphi) = (C_B)^\varphi$. Moreover, $|B| \cdot |C_B| = p^n$.

Let $C_B = \bigcap_{b\in B} C_b$. Since $B^\varphi = \prod_{b\in B} b^\varphi$, $B^\varphi \in \mathcal{CD}(G)$ and

$$C_G(B^\varphi) = \bigcap_{b\in B} C_G(b^\varphi) = \bigcap_{b\in B} (C_b)^\varphi = (C_B)^\varphi.$$

Since $|B^\varphi/Z(G)| = |B|^3$ and $|(C_B)^\varphi/Z(G)| = |C_B|^3$, we have

$$|B|^3 \cdot |C_B|^3 = |B^\varphi/Z(G)| \cdot |(C_B)^\varphi/Z(G)| = |G/Z(G)| = p^{3n}.$$

Hence, $|B| \cdot |C_B| = p^n$.

(5) If $K \in \mathcal{CD}(G)$, then there exists a subgroup B of A such that $K = B^\varphi$.

Let $H = C_G(K)$. Then $H \in \mathcal{CD}(G)$ and $K = C_G(H)$. Let

$$B_1 = \{a \in A \mid \text{there exist } y \in Y, w \in W, \text{ and } z \in Z(G) \text{ such that } a^\alpha ywz \in H\},$$

$$B_2 = \{a \in A \mid \text{there exist } x \in X, w \in W, \text{ and } z \in Z(G) \text{ such that } xa^\beta wz \in H\},$$

$$B_3 = \{a \in A \mid \text{there exist } x \in X, y \in Y, \text{ and } z \in Z(G) \text{ such that } xya^\gamma z \in H\}.$$

Then B_1, B_2, and B_3 are subgroups of A and $|H/Z(G)| \leq |B_1| \cdot |B_2| \cdot |B_3|$. By (2),

$$C_X(H) \leq C_X(B_2^\beta) = (C_{B_2})^\alpha.$$

Hence, $|C_X(H)| \leq |C_{B_2}|$. Similarly, $|C_Y(H)| \leq |C_{B_3}|$ and $|C_W(H)| \leq |C_{B_1}|$. It follows that

$$|H/Z(G)| \cdot |K/Z(G)| \leq |B_1| \cdot |B_2| \cdot |B_3| \cdot |C_{B_2}| \cdot |C_{B_3}| \cdot |C_{B_1}| = p^{3n} = |G/Z(G)|.$$

Since $H \in \mathcal{CD}(G)$, " = " holds. Hence,

$$K = C_G(H) = \langle (C_{B_2})^\alpha, (C_{B_3})^\beta, (C_{B_1})^\gamma \rangle Z(G)$$

and

$$C_X(H) = (C_{B_2})^\alpha, C_Y(H) = (C_{B_3})^\beta, \text{ and } C_W(H) = (C_{B_1})^\gamma.$$

By the symmetry, we also have

$$C_X(H) = (C_{B_3})^\alpha, C_Y(H) = (C_{B_1})^\beta, \text{ and } C_W(H) = (C_{B_2})^\gamma.$$

It follows that $C_{B_1} = C_{B_2} = C_{B_3}$. Let $B = C_{B_1}$. Then $K = C_G(H) = B^\varphi$.

(6) $\mathcal{CD}(G)$ is isomorphic to $\mathcal{L}(A)$.

It is a direct result of (4) and (5). ∎

The proof of Theorem 1.3 Let $A = A_1 \times \cdots \times A_n$, where A_i is the Sylow p_i-subgroup of A. By Theorem 1.2, there exist finite groups P_i such that $\mathcal{CD}(P_i)$ is isomorphic to $\mathcal{L}(A_i)$. Let $G = P_1 \times \cdots \times P_n$. By Theorem 2.2,

$$\mathcal{CD}(G) = \mathcal{CD}(P_1) \times \cdots \times \mathcal{CD}(P_n) \cong \mathcal{L}(A_1) \times \cdots \times \mathcal{L}(A_n) = \mathcal{L}(A).$$ ∎

https://doi.org/10.4153/S0008439522000418 Published online by Cambridge University Press

Acknowledgment I cordially thank the referee for detailed reading and helpful comments, which helped me to improve the whole paper considerably.

References

[1] L. An, *Groups whose Chermak–Delgado lattice is a quasi-antichain*. J. Group Theory **22**(2019), no. 3, 529–544.
[2] L. An, *Groups whose Chermak–Delgado lattice is a subgroup lattice of an elementary abelian p-group*. Comm. Algebra **50**(2022), no.7, 2846–2853.
[3] L. An, *Twisted centrally large subgroups of finite groups*. J. Algebra **604**(2022), 87–106.
[4] L. An, J. Brennan, H. Qu, and E. Wilcox, *Chermak–Delgado lattice extension theorems*. Comm. Algebra **43**(2015), no. 5, 2201–2213.
[5] B. Brewster, P. Hauck, and E. Wilcox, *Groups whose Chermak–Delgado lattice is a chain*. J. Group Theory **17**(2014), no. 2, 253–265.
[6] B. Brewster, P. Hauck, and E. Wilcox, *Quasi-antichain Chermak–Delgado lattices of finite groups*. Arch. Math. **103**(2014), no. 4, 301–311.
[7] B. Brewster and E. Wilcox, *Some groups with computable Chermak–Delgado lattices*. Bull. Aust. Math. Soc. **86**(2012), no. 1, 29–40.
[8] E. Brush, J. Dietz, K. Johnson-Tesch, and B. Power, *On the Chermak–Delgado lattices of split metacyclic p-groups*. Involve **9**(2016), no. 5, 765–782.
[9] A. Chermak and A. Delgado, *A measuring argument for finite groups*. Proc. Amer. Math. Soc. **107**(1989), no. 4, 907–914.
[10] W. Cocke, *Subnormality and the Chermak–Delgado lattice*. J. Algebra Appl. **19**(2020), no. 8, 2050141, 7 pp.
[11] G. Glauberman, *Centrally large subgroups of finite p-groups*. J. Algebra **300**(2006), 480–508.
[12] I. M. Isaacs, *Finite group theory*, American Mathematical Society, Providence, RI, 2008.
[13] R. McCulloch, *Chermak–Delgado simple groups*. Comm. Algebra **45**(2017), no. 3, 983–991.
[14] R. McCulloch, *Finite groups with a trivial Chermak–Delgado subgroup*. J. Group Theory **21**(2018), no. 3, 449–461.
[15] R. McCulloch and M. Tărnăuceanu, *Two classes of finite groups whose Chermak–Delgado lattice is a chain of length zero*. Comm. Algebra **46**(2018), no. 7, 3092–3096.
[16] R. McCulloch and M. Tărnăuceanu, *On the Chermak–Delgado lattice of a finite group*. Comm. Algebra **48**(2020), no. 1, 37–44.
[17] A. Morresi Zuccari, V. Russo, and C. M. Scoppola, *The Chermak–Delgado measure in finite p-groups*. J. Algebra **502**(2018), 262–276.
[18] R. Schmidt, *Subgroup lattices of groups*, Walter de Gruyter, Berlin, New York, 1994.
[19] M. Tărnăuceanu, *The Chermak–Delgado lattice of ZM-groups*. Results Math. **72**(2017), no. 4, 1849–1855.
[20] M. Tărnăuceanu, *A note on the Chermak–Delgado lattice of a finite group*. Comm. Algebra. **46**(2018), no. 1, 201–204.
[21] M. Tărnăuceanu, *Finite groups with a certain number of values of the Chermak–Delgado measure*. J. Algebra Appl. **19** (2020), no.5, 2050088, 7 pp.
[22] E. Wilcox, *Exploring the Chermak–Delgado lattice*. Math. Magazine **89**(2016), no. 1, 38–44.

Department of Mathematics, Shanxi Normal University, Linfen, Shanxi 041004, P. R. China
e-mail: anlj@sxnu.edu.cn