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$\mathcal{S}$-ASYMPTOTICALLY PERIODIC SOLUTIONS FOR ABSTRACT EQUATIONS WITH STATE-DEPENDENT DELAY

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Abstract

We study the existence and uniqueness of $\mathcal{S}$-asymptotically periodic solutions for a general class of abstract differential equations with state-dependent delay. Some examples related to problems arising in population dynamics are presented.

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1. Introduction

This paper continues our study [5] of abstract differential equations with state-dependent delay. Specifically, we study the existence and uniqueness of $\mathcal{S}$-asymptotically ω-periodic solutions for abstract problems of the form

$$u'(t) = Au(t) + F(t, u_{\sigma(t,u_t)}), \quad t \geq 0, \tag{1.1}$$

$$u_0 = \varphi \quad \text{with } \varphi \in \mathcal{B}_X = C([-p, 0]; X)), \tag{1.2}$$

where $A : D(A) \subset X \to X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$ and F, σ are functions to be specified later.

The theory of differential equations with state-dependent delay is a field of intense research because of its many applications and the fact that the qualitative theory is different from those for equations with discrete and time-dependent delays. For differential equations on finite-dimensional spaces, we cite the survey by Hartung *et al.* [3], and for related differential equations on abstract Banach spaces, the recent papers [4, 5, 7–9, 12]. For global solutions of problems on unbounded intervals, see [6, 11, 13] for equations on finite-dimensional spaces, [1, 10, 20] for abstract and partial differential equations and [1, 10] for problems with almost periodic solutions.

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The concept of $\mathcal{S}$-asymptotically periodic functions was introduced recently in [17, 18]. For the existence of $\mathcal{S}$-asymptotically ω-periodic solutions of differential equations, see [2, 21, 22] for equations on finite-dimensional spaces and [16–19] for equations on abstract spaces. We present here a unified approach which is of interest in itself and can be applied to study other types of abstract and applicable differential problems and the existence of other types of special solutions (for example, almost periodic, asymptotically almost periodic, or almost automorphic solutions) for problems similar to (1.1)–(1.2).

The problem of the existence and uniqueness of solutions for (1.1)–(1.2) is non-trivial because functions of the form $u \mapsto u_{\eta(\cdot,u_{(\cdot)})}$ are (in general) nonlinear and non-Lipschitz on spaces of continuous functions. When the functions involved are Lipschitz,

$$\|u_{\sigma(\cdot,u_{(\cdot)})} - v_{\sigma(\cdot,v_{(\cdot)})}\|_{C([0,a];\mathcal{B}_X)} \leq (1 + [v]_{C_{\text{Lip}}([-p,a];X)}[\sigma]_{C_{\text{Lip}}})\|u - v\|_{C([-p,a];X)},$$

so we study the existence of solutions for (1.1)–(1.2) on spaces of Lipschitz functions, a hard problem in the framework of semigroup theory.

We next describe some notation and results used in this work. Let $(Z, \|\cdot\|_Z)$, $(W, \|\cdot\|_W)$ be Banach spaces and $l > 0$. Let $B_l(z, Z) = \{x \in Z : \|x - z\|_Z \leq l\}$. Denote by $\mathcal{B}_Z$ the space $C([-p, 0]; Z)$ endowed with the uniform norm $\|\cdot\|_{\mathcal{B}_Z}$, and by $\mathcal{L}(Z, W)$ the space of bounded linear operators from Z into W endowed with the operator norm $\|\cdot\|_{\mathcal{L}(Z,W)}$. If $Z = W$, we write $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$. Let $C([0,\infty), Z)$ denote the space of all bounded continuous functions from $[0,\infty)$ into Z endowed with the uniform norm $\|\cdot\|_{C([0,\infty),Z)}$, and let $C_0([0,\infty), Z)$ be the subspace of all functions $f \in C([0,\infty), Z)$ such that $\lim_{t\to\infty} f(t) = 0$. Let $C_{\text{Lip}}([0,\infty); Z)$ be the subspace of all functions $\xi \in C([0,\infty), Z)$ such that

$$[\xi]_{C_{\text{Lip}}([0,\infty);Z)} = \sup_{t,s\in[0,\infty),t\neq s} \|\xi(s) - \xi(t)\|_Z/|t - s| < \infty,$$

endowed with the norm $\|\cdot\|_{C_{\text{Lip}}([0,\infty),Z)} = \|\cdot\|_{C([0,\infty),Z)} + [\cdot]_{C_{\text{Lip}}([0,\infty);Z)}$. Similarly, define $(C([0,\infty) \times Z; W), \|\cdot\|_{C([0,\infty)\times Z;W)})$ and $(C_{\text{Lip}}([0,\infty) \times Z; W), \|\cdot\|_{C_{\text{Lip}}([0,\infty)\times Z;W)})$. We write simply $[g]_{C_{\text{Lip}}}$ for the Lipschitz seminorm of a function g.

For simplicity, we assume that $0 \in \rho(A)$ and we use the notation $(-A)^\beta$ ($\beta > 0$) for the β-fractional power $(-A)^\beta : D(-A)^\beta \subset X \mapsto X$ of A. Let X_β denote the domain of $(-A)^\beta$ endowed with the norm $\|x\|_\beta = \|(-A)^\beta x\|$. We assume that there are $\gamma > 0$ and constants $C_{i,\beta} > 0$ such that $\|(-A)^{i+\beta}T(t)\| \leq C_{i,\beta}e^{-\gamma t}/t^{i+\beta}$ for all $\beta > 0$, $i \in \{0\} \cup \mathbb{N}$ and $t > 0$.

The following useful lemma follows from [5, Lemma 1].

Lemma 1.1. *If $\zeta_1 \in C_{\text{Lip}}([0,\infty) \times \mathcal{B}_Z; [0,\infty))$ and $u, v \in C_{\text{Lip}}([-p,\infty); Z)$, then*

$$[u_{\zeta_1(\cdot,u_{(\cdot)})}]_{C_{\text{Lip}}([0,\infty);\mathcal{B}_Z)} \leq [u]_{C_{\text{Lip}}}[\zeta_1]_{C_{\text{Lip}}}(1 + [u]_{C_{\text{Lip}}}),$$
$$\|u_{\zeta_1(\cdot,u_{(\cdot)})} - v_{\zeta_1(\cdot,v_{(\cdot)})}\|_{C([0,\infty);\mathcal{B}_Z)} \leq (1 + [v]_{C_{\text{Lip}}}[\zeta_1]_{C_{\text{Lip}}})\|u - v\|_{C([-p,\infty);Z)}.$$

We now give the definitions of some well-known concepts.

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Definition 1.2. A function $f \in C([0,\infty), Z)$ is called $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that ω is an asymptotic period of $f(\cdot)$ and that $f(\cdot)$ is $\mathcal{S}$-asymptotically ω-periodic.

We denote by $SAP_\omega(Z)$ the subspace of $C([-p,\infty), Z)$ formed by all the $\mathcal{S}$-asymptotically ω-periodic functions. It is well known that $SAP_\omega(Z)$ is a Banach space.

Definition 1.3 [17]. A function $G \in C([0,\infty)\times Z; W)$ is said to be uniformly $\mathcal{S}$-asymptotically ω-periodic on bounded sets if for every bounded subset K of Z, the set $\{G(t,x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(G(t,x) - G(t+\omega, x)) = 0$ uniformly for $x \in K$.

Definition 1.4 [17]. A function $G \in C([0,\infty)\times Z; W)$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and all bounded sets $K \subseteq Z$, there exist $L_{\varepsilon,K} \geq 0$ and $\delta_{\varepsilon,K} > 0$ such that $\|G(t,x) - G(t,y)\| \leq \varepsilon$, for all $t \geq L_{\varepsilon,K}$ and all $x, y \in K$ with $\|x - y\| \leq \delta_{\varepsilon,K}$.

For additional details on such almost periodic functions, we refer the reader to [23].

2. Existence of $\mathcal{S}$-asymptotically ω-periodic solutions

To begin we give two definitions for types of solutions.

Definition 2.1. A function $u : [-p,\infty) \to X$ is called a strict solution of (1.1)–(1.2) if $u|_{[0,\infty)} \in C^1([0,\infty); X) \cap C([0,\infty); X_1)$, $u_0 = \varphi$ and $u(\cdot)$ satisfies (1.1) on $[0,\infty)$.

Definition 2.2. A function $u : [-p,\infty) \to X$ is said to be a mild solution of (1.1)–(1.2) if $u \in C([-p,\infty); X)$, $u_0 = \varphi$ and

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_{\sigma(s,u_s)})\,ds \quad \text{for all } t \geq 0.$$

The next lemma follows from the proof of [17, Lemma 4.1 and Theorem 4.3].

Lemma 2.3. *Assume that $F \in C([0,\infty)\times \mathcal{B}_{X_\alpha}; X)$, $\sigma \in C([0,\infty)\times \mathcal{B}_{X_\alpha}; \mathbb{R}^+)$, $0 < \alpha < 1$, and that $F(\cdot)$ and $\sigma(\cdot)$ are uniformly $\mathcal{S}$-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets. If $u \in SAP_\omega(X_\alpha)$ and $v, w : [0,\infty) \to X$ are defined by $v(t) = F(t, u_{\sigma(t,u_t)})$ and $w(t) = \int_0^t T(t-s)v(s)\,ds$, then $v \in SAP_\omega(X)$ and $w \in SAP_\omega(X_\alpha)$.*

Proof. From the proof of [17, Lemma 4.1], $u_{(\cdot)} \in SAP_\omega(\mathcal{B}_{X_\alpha})$, $\sigma(\cdot, u_{(\cdot)}) \in SAP_\omega(\mathbb{R}^+)$ and $v \in SAP_\omega(X)$. Since $v \in SAP_\omega(X)$, given $\varepsilon > 0$, we select $L_\varepsilon^2 > L_\varepsilon^1 > 0$ such that

$$\begin{aligned} \Theta(\alpha,\gamma)\|v(s+\omega) - v(s)\| &\leq \varepsilon \quad \text{for all } s \geq L_\varepsilon^1, \\ \|v\|_{C([0,\infty);X)}\Theta(\alpha,\gamma)(e^{-\gamma L_\varepsilon^2} + 2e^{-\gamma(L_\varepsilon^2 - L_\varepsilon^1)}) &\leq \varepsilon, \end{aligned} \tag{2.1}$$

where $\Theta(\alpha,\gamma)=C_{0,\alpha}[1/(1-\alpha)+1/\gamma]$. Now $\int_0^l (e^{-\gamma(l-s)}/(l-s)^{\alpha})\,ds \le [1/(1-\alpha)+1/\gamma]$ for all $l \ge 0$, so for $t \ge L_{\varepsilon}^2$,

$$\begin{aligned}
\|w(t+\omega)-w(t)\|_{\alpha} &\le \int_0^{\omega} \|(-A)^{\alpha}T(t+\omega-s)v(s)\|\,ds \\
&\quad + \int_0^{L_{\varepsilon}^1} \|(-A)^{\alpha}T(t-s)(v(s+\omega)-v(s))\|\,ds \\
&\quad + \int_{L_{\varepsilon}^1}^{t} \|(-A)^{\alpha}T(t-s)(v(s+\omega)-v(s))\|\,ds \\
&\le C_{0,\alpha}\|v\|_{C([0,\infty);X)}e^{-\gamma t}\int_0^{\omega}\frac{e^{-\gamma(\omega-s)}}{(\omega-s)^{\alpha}}\,ds \\
&\quad + 2C_{0,\alpha}\|v\|_{C([0,\infty);X)}e^{-\gamma(t-L_{\varepsilon}^1)}\int_0^{L_{\varepsilon}^1}\frac{e^{-\gamma(L_{\varepsilon}^1-s)}}{(L_{\varepsilon}^1-s)^{\alpha}}\,ds \\
&\quad + C_{0,\alpha}\int_{L_{\varepsilon}^1}^{t}\frac{e^{-\gamma(t-s)}}{(t-s)^{\alpha}}\,ds\cdot\sup_{s\ge L_{\varepsilon}^1}\|v(s+\omega)-v(s)\| \\
&\le \|v\|_{C([0,\infty);X)}\Theta(\alpha,\gamma)(e^{-\gamma L_{\varepsilon}^2}+2e^{-\gamma(L_{\varepsilon}^2-L_{\varepsilon}^1)})+\varepsilon,
\end{aligned}$$

which proves that $w \in SAP_{\omega}(X_{\alpha})$ and completes the proof. □

Theorem 2.4. *Assume $F \in C_{\mathrm{Lip}}([0,\infty)\times\mathcal{B}_{X_{\alpha}};X)$, $\sigma \in C_{\mathrm{Lip}}([0,\infty)\times\mathcal{B}_{X_{\alpha}};\mathbb{R}^{+})$, the functions $F(\cdot)$ and $\sigma(\cdot)$ are uniformly $\mathcal{S}$-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets, and $\varphi \in C_{\mathrm{Lip}}([-p,0];X_{\alpha})$, $\sigma(0,\varphi)=0$, $T(\cdot)\varphi(0) \in C_{\mathrm{Lip}}([0,\infty);X_{\alpha})$, $T(\cdot)F(0,\varphi) \in L^{\infty}([0,\infty);X_{\alpha})$, and*

$$2[F]_{C_{\mathrm{Lip}}}\Theta(\alpha,\gamma)(2[\sigma]_{C_{\mathrm{Lip}}}(2\Lambda+1)+1) < 1, \tag{2.2}$$

where $\Theta(\alpha,\gamma)=C_{0,\alpha}[1/(1-\alpha)+1/\gamma]$ and

$$\Lambda = [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty);X_{\alpha})} + \|T(\cdot)F(0,\varphi)\|_{L^{\infty}([0,\infty);X_{\alpha})} + 2[F]_{C_{\mathrm{Lip}}}\Theta(\alpha,\gamma).$$

Then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,\infty);X_{\alpha})\cap SAP_{\omega}(X_{\alpha})$ of (1.1)–(1.2). Moreover, $u(\cdot)$ is a strict solution if $\varphi(0)\in X_1$.

Proof. Let $P:\mathbb{R}\to\mathbb{R}$ be the polynomial given by

$$P(x) = \Lambda + \Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}}}(2([\sigma]_{C_{\mathrm{Lip}}}+1)-1)x + 2\Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}}}[\sigma]_{C_{\mathrm{Lip}}}x^2.$$

From (2.2) and noting that $\Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}}}(2([\sigma]_{C_{\mathrm{Lip}}}+1)-1) < 0$, we infer that $P(\cdot)$ has a root $R_1 > 0$. Thus, there exists $R > 0$ such that $P(R) < 0$, which implies that

$$\begin{aligned}
\Lambda + 2[F]_{C_{\mathrm{Lip}}}\Theta(\alpha,\gamma)[\sigma]_{C_{\mathrm{Lip}}}R(1+R) &\le R, \\
\Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}}}(1+R[\sigma]_{C_{\mathrm{Lip}}}) &< 1.
\end{aligned}$$

Let $\mathcal{S}(R)=\{u \in SAP_{\omega}(X_{\alpha}) : u_0=\varphi, [u]_{C_{\mathrm{Lip}}([-p,\infty);X_{\alpha})} \le R\}$ endowed with the metric $d(u,v)=\|u-v\|_{C([0,\infty);X_{\alpha})}$ and let $\Gamma:\mathcal{S}(R)\to C([-p,\infty);X)$ be the map defined by

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$(\Gamma u)_0 = \varphi$ and

$$\Gamma u(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(s, u_{\sigma(s,u_s)})\,ds \quad \text{for } t \in [0,\infty).$$

Let $u \in \mathcal{S}(R)$. From Lemma 2.3 it is easy to see that $\Gamma(u) \in SAP_\omega(X_\alpha)$. To estimate $[\Gamma u]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)}$, note that from Lemma 1.1,

$$[F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{\mathrm{Lip}}([0,\infty);X)} \le [F]_{C_{\mathrm{Lip}}}(1 + R[\sigma]_{C_{\mathrm{Lip}}}(1+R)).$$

Using this estimate, for $t, h \in [0,\infty)$,

$$\begin{aligned}
&\|\Gamma u(t+h) - \Gamma u(t)\|_{X_\alpha} \\
&\quad\le [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)}h + \int_0^h \|(-A)^\alpha T(t+h-s)F(0,\varphi)\|\,ds \\
&\qquad + \int_0^h \|(-A)^\alpha T(t+h-s)\|_{\mathcal{L}(X)}\|F(s, u_{\sigma(s,u_s)}) - F(0,\varphi)\|\,ds \\
&\qquad + \int_0^t \|(-A)^\alpha T(t-s)\|_{\mathcal{L}(X)}\|F(s+h, u_{\sigma(s+h,u_{s+h})}) - F(s, u_{\sigma(s,u_s)})\|\,ds \\
&\quad\le [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)}h + \|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);X_\alpha)}h \\
&\qquad + [F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{\mathrm{Lip}}([0,\infty);X)}h \int_0^h \frac{C_{0,\alpha}e^{-\gamma(h-s)}}{(h-s)^\alpha}\,ds \\
&\qquad + [F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{\mathrm{Lip}}([0,\infty);X)}h \int_0^t \frac{C_{0,\alpha}e^{-\gamma(t-s)}}{(t-s)^\alpha}\,ds \\
&\quad\le [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)}h + \|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);X_\alpha)}h \\
&\qquad + 2[F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})})]_{C_{\mathrm{Lip}}([0,\infty);X)}\Theta(\alpha,\gamma)h,
\end{aligned}$$

and hence, $[\Gamma u]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)} \le \Lambda + 2[F]_{C_{\mathrm{Lip}}}\Theta(\alpha,\gamma)[\sigma]_{C_{\mathrm{Lip}}}R(1+R) \le R$. From this estimate, $[\Gamma u]_{C_{\mathrm{Lip}}([-p,\infty);X_\alpha)} \le R$ since $[\varphi]_{C_{\mathrm{Lip}}([-p,0];X_\alpha)} \le R$, which shows that $\Gamma u \in \mathcal{S}(R)$.

On the other hand, for $u, v \in \mathcal{S}(R)$ and $t \ge 0$,

$$\begin{aligned}
\|\Gamma u(t) - \Gamma v(t)\|_{X_\alpha} &\le \int_0^t \|(-A)^\alpha T(t-s)\|_{\mathcal{L}(X)}[F]_{C_{\mathrm{Lip}}}\|u_{\sigma(s,u_s)} - v_{\sigma(s,v_s)}\|_\alpha\,ds \\
&\le C_{0,\alpha}\int_0^t \frac{e^{-\gamma(t-s)}}{(t-s)^\alpha}[F]_{C_{\mathrm{Lip}}}(1 + R[\sigma]_{C_{\mathrm{Lip}}})\,d(u,v)\,ds \\
&\le \Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}}}(1 + R[\sigma]_{C_{\mathrm{Lip}}})\,d(u,v),
\end{aligned}$$

which proves that Γ is a contraction on $\mathcal{S}(R)$ and so there exists a unique mild solution $u \in C_{\mathrm{Lip}}([0,\infty); X_\alpha) \cap SAP_\omega(X_\alpha)$ of (1.1)–(1.2). In addition, from [15, Theorem 4.3.2], we infer that $u(\cdot)$ is a strict solution if $\varphi(0) \in X_1$. □

Remark 2.5. The proofs of Lemma 2.3 and Theorem 2.4 depend on the asymptotic stability of $(T(t))_{t\ge0}$. The relation between stability of C_0-semigroups, the spectrum of the associated generator and the existence of almost periodic solutions is an important

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field of research. In this connection, an interesting question arises: is there a result similar to Theorem 2.4 if $(T(t))_{t\geq 0}$ is strongly S-asymptotically periodic (that is, for each $x \in X$ there is $\omega_x > 0$ such that $T(\cdot)x$ is S-asymptotically ω_x-periodic)? We will address this question in a forthcoming paper.

Next, we study asymptotically ω-periodic solutions. Let $u \in C([0,\infty); Z)$. We say that u is $\mathbb{N}\mathcal{S}$-asymptotically ω-periodic if $\lim_{t\to\infty}(u(t+n\omega) - u(t)) = 0$ uniformly for $n \in \mathbb{N}$ and we denote by $\mathbb{N}SAP_{\omega}(Z)$ the space of functions of this type endowed with the norm $\|\cdot\|_{C([0,\infty);Z)}$. It is easy to see that $\mathbb{N}SAP_{\omega}(Z)$ is a Banach space. In addition, we say that $G \in C([0,\infty) \times Z; W)$ is uniformly $\mathbb{N}\mathcal{S}$-asymptotically ω-periodic on bounded sets if, for every bounded set $K \subset W$, $\lim_{t\to\infty}(G(t+n\omega, x) - G(t, x)) = 0$ uniformly for $x \in K$ and $n \in \mathbb{N}$.

Lemma 2.6. *Assume that $F \in C([0,\infty) \times \mathcal{B}_{X_\alpha}; X)$ and $\sigma \in C([0,\infty) \times \mathcal{B}_{X_\alpha}; \mathbb{R}^+)$ for some $\alpha \in (0,1)$, and that $F(\cdot)$ and $\sigma(\cdot)$ are uniformly $\mathbb{N}\mathcal{S}$-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets. If $u \in \mathbb{N}SAP_{\omega}(X_\alpha)$ and $v(\cdot)$, $w(\cdot)$ are the functions in Lemma 2.3, then $v \in \mathbb{N}SAP_{\omega}(X)$ and $w \in \mathbb{N}SAP_{\omega}(X_\alpha)$.*

Proof. Assuming that $v \in \mathbb{N}SAP_{\omega}(X)$ and proceeding as in the proof of Lemma 2.3, we select $L^2_\varepsilon > L^1_\varepsilon > 0$ such that (2.1) is satisfied and

$$\|v\|_{C([0,\infty);X)}\Theta(\alpha,\gamma)\|v(s+n\omega) - v(s)\| \leq \varepsilon \quad \text{for all } n \in \mathbb{N},\ s \geq L^1_\varepsilon.$$

For $t \geq L^2_\varepsilon$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|w(t+n\omega) - w(t)\|_{X_\alpha} &\leq \int_0^{n\omega} \|(-A)^\alpha T(t+n\omega-s)v(s)\|\,ds \\ &\quad + \int_0^{L^1_\varepsilon} \|(-A)^\alpha T(t-s)(v(s+n\omega)-v(s))\|\,ds \\ &\quad + \int_{L^1_\varepsilon}^{t} \|(-A)^\alpha T(t-s)(v(s+n\omega)-v(s))\|\,ds \\ &\leq \|v\|_{C([0,\infty);X)}\Theta(\alpha,\gamma)(e^{-\gamma L^2_\varepsilon} + 2e^{-\gamma(L^2_\varepsilon - L^1_\varepsilon)}) + \varepsilon,\end{aligned}$$

which allows us to conclude that $w \in \mathbb{N}SAP_{\omega}(X_\alpha)$. □

Proposition 2.7. *Assume that the conditions in Theorem 2.4 are satisfied and that $F(\cdot)$ and $\sigma(\cdot)$ are uniformly $\mathbb{N}\mathcal{S}$-asymptotically ω-periodic on bounded sets. Then there exists a unique asymptotically ω-periodic mild solution $u \in C_{\mathrm{Lip}}([-p,\infty); X_\alpha) \cap \mathbb{N}SAP_{\omega}(X_\alpha)$ of (1.1)–(1.2) and $u(\cdot)$ is a strict solution if $\varphi(0) \in X_1$.*

Proof. The existence of a mild solution $u \in C_{\mathrm{Lip}}([-p,\infty); X_\alpha) \cap \mathbb{N}SAP_{\omega}(X_\alpha)$ follows by combining Theorem 2.4, Lemmas 2.6 and 2.3 and using $\mathbb{N}SAP_{\omega}(X_\alpha)$ instead of $SAP_{\omega}(X_\alpha)$. The fact that $u(\cdot)$ is asymptotically ω-periodic follows from [17, Corollary 3.1]. The last assertion is a consequence of [15, Theorem 4.3.2]. □

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Finally, we study the existence of $\mathcal{S}$-asymptotically periodic solutions when $F(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz. To avoid additional notation and concepts, in the next result we say that $f \in C([0,\infty) \times Z; W)$ is locally Lipschitz if f is Lipschitz on $[0,\infty) \times B_r(0, Z)$ for all $r > 0$ and we use the notation $[f]_{C_{\mathrm{Lip}},r}$ for the Lipschitz constant of f on $[0,\infty) \times B_r(0, Z)$. We can prove the following theorem by proceeding as in the proof of Theorem 2.4.

Proposition 2.8. *Suppose the conditions in Theorem 2.4 hold but assume that $F(\cdot)$ and $\sigma(\cdot)$ are locally Lipschitz and that there is an $r > 0$ such that (2.2) is valid with $[F]_{C_{\mathrm{Lip}},r}$ and $[\sigma]_{C_{\mathrm{Lip}},r}$ in place of $[F]_{C_{\mathrm{Lip}}}$ and $[\sigma]_{C_{\mathrm{Lip}}}$, and*

$$\Lambda(r) = [T(\cdot)\varphi(0)]_{C_{\mathrm{Lip}}([0,\infty);X_\alpha)} + \|T(\cdot)F(0,\varphi)\|_{L^\infty([0,\infty);X_\alpha)} + 2\Theta(\alpha,\gamma)[F]_{C_{\mathrm{Lip}},r}$$

in place of Λ. If $\max\{\|\varphi\|_{B_{X_\alpha}}, C_0\|\varphi(0)\|_{X_\alpha} + \|F\|_{B_r(0,\mathcal{B}_{X_\alpha})}\Theta(\alpha,\gamma)\} \le r$, then there exists a unique mild solution $u \in C_{\mathrm{Lip}}([-p,\infty); X_\alpha) \cap SAP_\omega(X_\alpha)$ of (1.1)–(1.2). Moreover, $u \in B_r(0, C([-p,\infty); X_\alpha))$ and $u(\cdot)$ is a strict solution if $\varphi(0) \in X_1$.

3. Examples

We now present some examples related to problems in population dynamics. For simplicity, we assume that $A : D(A) \subset X \to X$ is the generator of an exponentially asymptotically stable analytic semigroup of bounded linear operators $(T(t))_{t\ge 0}$ on X with $X = L^2(\Omega;\mathbb{R})$ or $X = C(\Omega;\mathbb{R})$, where $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary $\partial\Omega$. We use the notation and properties from the previous sections.

We assume that $\sigma(\cdot)$ is a Lipschitz function satisfying the general conditions in Section 2. We can think of $\sigma(\cdot)$ as defined via a threshold condition of the form

$$S(\sigma(u), u) = \int_{-p}^{\sigma(u)} \Big(\frac{D_1}{D_2 + \|u(s)\|^2_{L^2(\Omega)}} + D_3 \Big)\, ds = D_4$$

(see [3, 9]), where the D_i are fixed positive numbers. In particular, in the example presented in [9], the function $\sigma(\cdot)$ is a C^1 function, and hence, locally Lipschitz.

Motivated by the problems studied in [20], we consider

$$u'(t,x) = Au(t,x) + \int_\Omega b(u(\sigma(u_t), y)) f(x-y)\, dy + g(t)H(u(t,x)), \qquad \text{for } t \ge 0, x \in \Omega, \tag{3.1}$$

$$u(\theta, y) = \varphi(\theta, y), \qquad \theta \in [-p, 0],\ y \in \Omega, \tag{3.2}$$

where $\varphi \in C_{\mathrm{Lip}}([-p,0]; X)$, $X = L^2(\Omega;\mathbb{R})$, $f \in C(\mathbb{R}^n;\mathbb{R})$, $\sigma \in C_{\mathrm{Lip}}(\mathcal{B}_X; [0,\infty))$, $H \in C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})$, $b \in C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})$ and $g \in C_{\mathrm{Lip}}([0,\infty);\mathbb{R}) \cap \mathcal{S}AP_\omega(\mathbb{R})$. For the sake of brevity, we assume that g, H and b are bounded and $\mu = \big(\int_\Omega \int_\Omega f(x-y)^2 dy\, dx\big)^{1/2}$ is finite.

Define $F : \mathcal{B}_X \to X$ by $F(\psi)(x) = \int_\Omega b(\psi(0,y)) f(x-y)\, dy + g(t)H(\psi(0,x))$. Note that

$$\|F\|_{C(\mathcal{B}_X;X)} \le \|b\|_{C(\mathbb{R};\mathbb{R})}\mu + \|g\|_{C([0,\infty);\mathbb{R})}\|H\|_{C(\mathbb{R}^n;\mathbb{R})} m(\Omega),$$
$$[F]_{C_{\mathrm{Lip}}} \le L_F := [b]_{C_{\mathrm{Lip}}}\mu + \|g\|_{C_{\mathrm{Lip}}}\|H\|_{C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})},$$

where $m(\cdot)$ denotes the Lebesgue measure.

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In the next result, which follows from Theorem 2.4 with $\alpha = 0$, we say that $u \in C([-p,\infty); X)$ is a mild (strict) solution of (3.1)–(3.2), if $u(\cdot)$ is a mild (strict) solution of the associated problem (1.1)–(1.2). We adopt a similar convention for the second example.

Proposition 3.1. *Assume that* $\varphi(0) \in D(A)$ *and* $2L_F\gamma^{-1}(2[\sigma]_{C_{\mathrm{Lip}}}(2\Lambda + 1) + 1) < 1$ *where* $\Lambda = C_0\|A\varphi(0)\| + \|b\|_{C(\mathbb{R};\mathbb{R})}\mu + \|g\|_{C(\mathbb{R};\mathbb{R})}\|H\|_{C(\mathbb{R}^n;\mathbb{R})}m(\Omega) + (2C_0/\gamma)L_F$. *Then there exists a unique strict solution* $u \in C_{\mathrm{Lip}}([-p,\infty); X_\alpha) \cap SAP_\omega(X_\alpha)$ *of (3.1)–(3.2).*

The next example is motivated by the Fisher–Kolmogorov and Hutchinson equations (see [11, 14, 20] for details). Consider the diffusive equation with state-dependent delay

$$w'(t,\xi) = Aw(t,\xi) + \mu(t)w(\sigma(t,w_t),\xi)[1 - w(\sigma(t,w_t),\xi)], \quad t \in \mathbb{R},\ \xi \in \Omega, \tag{3.3}$$

$$w(\theta,y) = \varphi(\theta,y), \quad \theta \in [-p,0], y \in \Omega, \tag{3.4}$$

where $\sigma \in C_{\mathrm{Lip}}(\mathbb{R} \times \mathcal{B}_X; \mathbb{R}^+)$, $\mu \in C_{\mathrm{Lip}}(\mathbb{R},\mathbb{R})$ and $X = C(\Omega)$.

Let $F : \mathbb{R} \times \mathcal{B}_X \to X$ be defined by $F(t,\psi)(x) = \mu(t)\psi(0,x)[1 - \psi(0,x)]$. For $r > 0$, $t, s \in [0,\infty)$ and $\psi, \phi \in B_r(0,\mathcal{B}_X)$, we have $\|F(t,\psi)\| \le \|\mu\|_{C(\mathbb{R};\mathbb{R})}r(1 + r)$ and

$$\|F(t,\psi) - F(s,\phi)\| \le [\mu]_{C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})} \mid t - s \mid r(1 + r) + \|\mu\|_{C(\mathbb{R};\mathbb{R})}(1 + 2r)\|\psi - \phi\|_{\mathcal{B}_X},$$
$$[F]_{C_{\mathrm{Lip}},r} \le \|\mu\|_{C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})}(r^2 + 3r + 1).$$

Moreover, if $\varphi(0) \in D(A)$, the number $\Lambda(r)$ in Proposition 2.7 is given by

$$\Lambda(r) = C_0\|A\varphi(0)\| + C_0\|\mu\|_{C(\mathbb{R};\mathbb{R})}r(1 + r) + 2\|\mu\|_{C_{\mathrm{Lip}}(\mathbb{R};\mathbb{R})}(r^2 + 3r + 1)\gamma^{-1}.$$

The following result follows from Proposition 2.7.

Proposition 3.2. *If there is an* $r > 0$ *such that* $2[F]_{C_{\mathrm{Lip},r}}(2[\sigma]_{C_{\mathrm{Lip}},r}(2\Lambda(r) + 1) + 1) < \gamma$ *and* $\max\{\|\varphi\|_{B_X}, C_0\|\varphi(0)\| + \|F\|_{B_r(0,\mathcal{B}_X)}/\gamma\} \le r$, *then there exists a unique strict solution* $u \in C_{\mathrm{Lip}}([-p,\infty); X_\alpha) \cap SAP_\omega(X_\alpha)$ *of* (3.3)–(3.4) *such that* $u \in B_r(0, C([-p,\infty); X))$.

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