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On the Limit Cycles of Linear Differential Systems with Homogeneous Nonlinearities

Jaume Llibre and Xiang Zhang

Abstract. We consider the class of polynomial differential systems of the form $\dot{x} = \lambda x - y + P_n(x, y)$, $\dot{y} = x + \lambda y + Q_n(x, y)$, where P_n and Q_n are homogeneous polynomials of degree n. For this class of differential systems we summarize the known results for the existence of limit cycles, and we provide new results for their nonexistence and existence.

1 Introduction and Statement of the Main Results

One of the main problems in the qualitative theory of real planar differential systems is how to control the existence, non-existence, or uniqueness of limit cycles for a given class of polynomial differential systems.

Limit cycles of planar differential systems were defined by Poincaré [13] and started to be studied intensively at the end of the 1920s by van der Pol [14], Liénard [7], and Andronov [1].

In this work we study the real planar polynomial differential systems of the form

$$\dot{x} = \lambda x - y + P_n(x, y), \quad \dot{y} = x + \lambda y + +Q_n(x, y), \tag{1.1}$$

where P_n and Q_n are homogeneous polynomials of degree n.

In order to be more precise, we need to introduce some notation and basic results. Then, in polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$, system (1.1) can be written as

$$\dot{r} = \lambda r + f(\theta) r^n, \quad \dot{\theta} = 1 + g(\theta) r^{n-1}, \tag{1.2}$$

where

$$f(\theta) = \cos\theta P_n(\cos\theta, \sin\theta) + \sin\theta Q_n(\cos\theta, \sin\theta),$$
$$g(\theta) = \cos\theta Q_n(\cos\theta, \sin\theta) - \sin\theta P_n(\cos\theta, \sin\theta)$$

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are homogeneous polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. In the region

$$C = \{(r,\theta) : 1 + g(\theta)r^{n-1} > 0\}$$

the differential system (1.2) is equivalent to the differential equation

$$\frac{dr}{d\theta} = \frac{\lambda r + f(\theta)r^n}{1 + g(\theta)r^{n-1}}. \tag{1.3}$$

It is known that the periodic orbits surrounding the origin of system (1.2) do not intersect the curve $\dot{\theta} = 0$ (see the Appendix of [3]). Therefore, these periodic orbits are contained in the region C, and, consequently, they are also periodic orbits of equation (1.3). Moreover, these periodic orbits can be studied via the change of variables

$$\rho = \frac{r^{n-1}}{1 + g(\theta)r^{n-1}},$$

as in Cherkas [4], which in fact goes back to Liouville [9]. In terms of ρ, the differential equation (1.3) becomes

$$\frac{d\rho}{d\theta} = (n-1)g(\lambda g - f)\rho^3 + ((n-1)(f - 2\lambda g) - g')\rho^2 + (n-1)\lambda\rho. \tag{1.4}$$

We now summarize previous results on the existence of limit cycles for the polynomial differential systems (1.1), using the differential equations (1.3) or (1.4).

(i) If the trigonometric polynomial $f - \lambda g \not\equiv 0$ does not change sign, then system (1.1) has at most one limit cycle, and when one exists, it surrounds the origin (see [5, Theorem A]).

(ii) If the trigonometric polynomial $(n-1)g(\lambda g - f) \not\equiv 0$ does not change sign, then equation (1.1) has at most one limit cycle in the region $r > 0$ if n is even, and at most two limit cycles in the region $r > 0$ if n is odd, and when they exist, they surround the origin (see [2, Theorem 1.1(b)]).

(iii) If the trigonometric polynomial $(n-1)(f - 2\lambda g) - g' \not\equiv 0$ does not change sign, then equation (1.1) has at most two limit cycles, and when they exist, they surround the origin (see [6, Theorem C(a)]).

(iv) If either the trigonometric polynomial

$$(n-1)g(\lambda g - f) \equiv 0 \quad \text{or} \quad (n-1)(f - 2\lambda g) - g' \equiv 0,$$

then equation (1.1) has at most one limit cycle, and when it exists surrounds the origin (see [6, Theorem C(b)]).

We remark that all the previous results only provide information on the limit cycles surrounding the origin of the polynomial differential system (1.1).

We must mention that (i), which eventually resulted in (ii), was given by Pliss [12] and Lins Neto [8].

All results (i)–(iv) are about the existence of limit cycles for the polynomial differential systems (1.1). Now we provide results on the non-existence of limit cycles for systems (1.1) and a new result on their existence and uniqueness.

Theorem 1.1 *The polynomial differential system (1.1) with $n \geq 2$ has no limit cycles surrounding the origin if one of the following condition holds:*

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(i) $f = 0$;
(ii) $f - \lambda g = 0$;
(iii) $g = 0$;
(iv) $(2n-1)^2 f - \lambda(2n-3)^2 g = 0$;
(v) $(f - \lambda g)(n^2 f - \lambda(n-2)^2 g) \leq 0$ *for all* θ;
(vi) $(f - \lambda g)((2n-1)^2 f - \lambda(2n-3)^2 g) \leq 0$ *for all* θ.

The polynomial differential system (1.1) *has at most one limit cycle surrounding the origin if the following condition holds:*

(vii) $(f - \lambda g)((2n-3)^2 f - \lambda(2n-1)^2 g) \leq 0$ *for all* θ.

In some sense, Theorem 1.1 is an extension to any positive integer n of a similar result for $n = 2$ (see [10, Theorem 1]) and will be proved in Section 3.

2 Preliminary Results

In this section we recall some basic facts that we will need for the proof of Theorem 1.1.

The next two results correspond to Lloyd [11, Theorems 2 and 3].

Lemma 2.1 *We have a differential system in polar coordinates*

$$\dot{r} = F(r,\theta), \quad \dot{\theta} = G(r,\theta) \tag{2.1}$$

defined in a simply connected open set U *containing the origin, where* F *and* G *are* C^1 2π*-periodic functions such that* $F(0,\theta) = 0$ *for all* θ*, and* $G(r,\theta) > 0$ *in* U*. Then, in* U*, the differential system* (2.1) *is equivalent to the differential equation*

$$\frac{dr}{d\theta} = \frac{F(r,\theta)}{G(r,\theta)} = S(r,\theta). \tag{2.2}$$

Therefore, if

$$\frac{\partial S}{\partial r} \not\equiv 0 \tag{2.3}$$

and either

$$\frac{\partial S}{\partial r} \leq 0, \quad \text{or} \quad \frac{\partial S}{\partial r} \geq 0 \tag{2.4}$$

in U*, then the differential system* (2.1) *has no limit cycles in* U*.*

Remark 2.2 We note that in [11] the inequalities (2.4) appear without the equality, but checking the proof of [11, Theorem 2] we see that it also works under the conditions (2.3) and (2.4).

Lemma 2.3 *Consider the differential system* (2.1) *defined in an annular region* $\mathcal{A}$ *that encircles the origin and where* $G(r,\theta) > 0$*. Then in* $\mathcal{A}$*, the differential system* (2.1) *is equivalent to the differential equation* (2.2)*. If* (2.3) *and* (2.4) *hold in* $\mathcal{A}$*, then the differential system* (2.1) *has at most one limit cycle in* $\mathcal{A}$*.*

Remark 2.2 applies to Lemma 2.3 using the proof of [11, Theorem 3].

Lemma 2.4 *Under the assumptions of Lemma 2.3 if $\partial^3 S/\partial r^3 \geq 0$ in $\mathcal{A}$, then the differential system (2.1) has at most three limit cycles in $\mathcal{A}$.*

Again Remark 2.2 applies to Lemma 2.4 using the proof of [11, Theorem 8].

3 Proof of Theorem 1.1

We prove Theorem 1.1 statement by statement. The proof of the first two statements of Theorem 1.1 are essentially the same as that for the particular case when $n = 2$ in [10], but since our proofs are shorter and easier, we provide them here for completeness.

Proof of Theorem 1.1(i) Since $f = 0$, $dr/d\theta$ does not change sign. If $\lambda \neq 0$ the solution $r(\theta)$ of (1.3) increases or decreases, so these solutions cannot be periodic in the region C, and consequently the polynomial differential system (1.1) has no limit cycles surrounding the origin.

If $\lambda = 0$, then $dr/d\theta \equiv 0$ and all the solutions in the region C are periodic and circular (except the equilibrium point at the origin), so the system has no isolated periodic orbits surrounding the origin, *i.e.*, no limit cycles surrounding the origin. So statement (i) is proved. ∎

Proof of Theorem 1.1(ii) Since $f - \lambda g = 0$, we have that $dr/d\theta = \lambda r$. Now the proof ends following the same arguments used in the proof of statement (i). ∎

Proof of Theorem 1.1(iii) If $g = 0$, then the differential equation (1.3) becomes

$$\frac{dr}{d\theta} = \lambda r + f(\theta) r^n. \tag{3.1}$$

Its general solution $r(\theta)$ satisfying that $r(0) = r_0 > 0$ is

$$r(\theta) = \left(e^{(1-n)\theta\lambda} r_0^{-n} \left(r_0 + (1-n) \left(\int_0^\theta e^{(n-1)s\lambda} f(s)\, ds \right) r_0^n \right) \right)^{\frac{1}{1-n}}.$$

There are at most two values of r_0 such that $r(2\pi) = r_0$, namely $r_0 = 0$ and

$$r_0 = \left(\frac{e^{2(1-n)\pi\lambda} (n-1) \int_0^{2\pi} e^{(n-1)s\lambda} f(s)\, ds}{e^{2(1-n)\pi\lambda} - 1} \right)^{\frac{1}{1-n}},$$

if this last expression is real. But the differential equation (3.1) is defined on the cylinder $\{(r,\theta) \in \mathbb{R} \times \mathbb{S}^1\}$ and it is invariant under the symmetry $(r,\theta) \to (-r, \theta + \pi)$, so if we have a periodic solution with $r_0 > 0$, we also have a periodic solution with $r_0 < 0$. But this is in contradiction with the fact that the differential equation has at most a unique periodic solution with $r_0 \neq 0$. This completes the proof of statement (iii). ∎

Proof of Theorem 1.1(iv) From $(2n-1)^2 f - \lambda(2n-3)^2 g = 0$, we have

$$f = \lambda(2n-3)^2 g/(2n-1)^2.$$

Substituting f into the differential equation (1.3) we get

$$\frac{dr}{d\theta} = \lambda \frac{r + (2n-3)^2 g r^n/(2n-1)^2}{1 + g r^{n-1}} = S(r,\theta), \tag{3.2}$$

defined in the simply connected region C.

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We have

$$\frac{\partial S}{\partial r} = \frac{\lambda\big((2n-3)^2 g^2 r^{2n-2} + 2gr^{n-1} + (2n-1)^2\big)}{(2n-1)^2(1+gr^{n-1})^2}.$$

Then, since $4g^2 - 4(2n-1)^2(2n-3)^2 g^2 = -g^2(n-1)^2(1+2n(n-2)) \leq 0$, we have that $\partial S/\partial r$ satisfies (2.3) and (2.4) in the region S, and consequently, we can apply Lemma 2.1 to the differential equation (3.2), and the proof of statement (iv) is done. ∎

Proof of Theorem 1.1(v) Consider the differential equation (1.3):

$$\frac{dr}{d\theta} = \frac{\lambda r + f r^n}{1 + gr^{n-1}} = S(r,\theta), \tag{3.3}$$

defined in the simply connected region C.

We have

$$\frac{\partial S}{\partial r} = \frac{fgr^{2n-2} + (nf - (n-2)\lambda g)r^{n-1} + \lambda}{(1+gr^{n-1})^2}.$$

Then, if $(nf-(n-2)\lambda g)^2 - 4\lambda f g = (f-\lambda g)(n^2 f - \lambda(n-2)^2 g) \leq 0$, we have that $\partial S/\partial r$ satisfies (2.3) and (2.4) in the region S, and consequently, we can apply Lemma 2.1 to the differential equation (3.3), and the proof of statement (v) is done. ∎

Proof of Theorem 1.1(vi) Introducing the change of variables $R = \sqrt{r}$ in the region C, the differential equation (1.3) becomes

$$\frac{dR}{d\theta} = \frac{\lambda R + f R^{2n-1}}{2(1 + gR^{2n-2})} = S(R,\theta). \tag{3.4}$$

Clearly the image of the simply connected region C under the map $r \to \sqrt{r} = R$ is another simply connected region S containing the origin $R = 0$.

We have

$$\frac{\partial S}{\partial R} = \frac{fgR^{4n-4} + ((2n-1)f + \lambda(3-2n)g)R^{2n-2} + \lambda}{2(1+gR^{2n-2})^2}.$$

Then, if $((2n-1)f + \lambda(3-2n)g)^2 - 4fg\lambda = (f-\lambda g)((2n-1)^2 f - \lambda(2n-3)^2 g) \leq 0$, we have that $\partial S/\partial R$ satisfies (2.3) and (2.4) in the region S, and consequently, we can apply Lemma 2.1 to the differential equation (3.4), and the proof of statement (vi) follows. ∎

Proof Theorem 1.1(vii) With the change of variables $R = 1/\sqrt{r}$ in the region C, the differential equation (1.3) becomes

$$\frac{dR}{d\theta} = \frac{-fR - \lambda R^{2n-1}}{2(g + R^{2n-2})} = S(R,\theta). \tag{3.5}$$

Now the image of the region C by the map $r \to 1/\sqrt{r} = R$ is an annular region $\mathcal{A}$, and one of the boundaries of this annulus is the infinity.

We get

$$\frac{\partial S}{\partial R} = -\frac{\lambda R^{4n-4} + ((3-2n)f + \lambda(2n-1)g)R^{2n-2} + fg}{2(g+R^{2n-2})^2}.$$

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Then, clearly, if

$$((3-2n)f+\lambda(2n-1)g)^2-4\lambda fg=(f-\lambda g)\big((2n-3)^2f-\lambda(2n-1)^2g\big)\leq 0,$$

we have that $\partial S/\partial R$ satisfies conditions (2.3) and (2.4) in the annular region $\mathcal{A}$, and consequently we can apply Lemma 2.3 to the differential equation (3.5), and this completes the proof of statement (vii). ∎

It was proved in [10] that the system

$$\dot{x}=\lambda x-y+\frac{1}{5\lambda}(5\lambda x-y)^2,\quad \dot{y}=x+\lambda y+\frac{1}{5\lambda}(5\lambda x-y)(x+5\lambda y)$$

has a unique limit cycle surrounding the origin. We note that this system satisfies Theorem 1.1(vii) for $n=2$.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
e-mail: jllibre@mat.uab.cat

Department of Mathematics, MOE–LSC, Shanghai Jiao tong University, Shanghai, 200240, P. R. China
e-mail: xzhang@sjtu.edu.cn