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Proposition of a net-like model of snow

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ABSTRACT. Thin-section photographs show that snow consists of lumpy parts and connecting branches. The model proposed here agrees with this real state. This new model is derived from four packing forms of isometric spheres by shrinking the original spheres while maintaining and connecting points of contact as a column. The texture of the model can be varied by setting the packing form, the shrinking ratio and the thickness of connecting branches. When the density and strength of the material of the model are set to the values of polycrystalline ice, the model density and tensile strength agree with published data for dry compacted snow.

INTRODUCTION

The physical properties of snow have often been explained by use of models. This paper proposes a new model of the texture and tensile strength for dry compacted snow. This new model is derived from the perfect packing forms of isometric spheres. These original forms were introduced by Manegold and others (1931), and are distinguished by their coordination number N, the number of points of contact between spheres.

The cubic system is a basic packing form and its N value is 6. The other forms have N values equal to 8, 10 and 12. The form of $N = 12$ is the densest packing form. The model proposed here is derived from the four models mentioned above by the following procedure. The radius of each sphere shrinks from R to pR, and each of the original points of contact become the central axis of a column having a radius qR. The model can adopt manifold textures by varying the values R, N, p, and q. It is termed a "net-like" model because of its structure.

At first, we assume that the material of the model has the same density and tensile strength as polycrystalline ice. According to the studies of the author and others (Ishida and Shimizu, 1955; Watanabe, 1964), the value of R is directly decided by the snow density, though our experiments were limited to snow of density less than $0.45\,\mathrm{g\,cm^{-3}}$. The value of q is decided from experimental results on the tensile strength of snow, and the value of p is also calculated from the density, N and q. Thus, we can propose a net-like model whose density and strength agree with snow.

For compacted snow of density greater than $0.45\,\mathrm{g\,cm^{-3}}$, we must select a net-like model of $N = 6$, $p = 1$ and large q. In the case of $N = 6$ and $p = q = 1$, the air is enclosed in the ice and the density reaches $0.831\,\mathrm{g\,cm^{-3}}$ at maximum. Of course we can calculate the tensile strength of the high density model, and the result agrees well with the data of Keeler and Weeks (1968). As mentioned above, this model results in tensile strength values that are consistent with real snow.

NET-LIKE MODEL

The perfect packing models of isometric spheres

According to Manegold and others (1931), there are five varieties in the systematic packing forms of isometric spheres. These five forms are distinguished by their coordination number, N, which can have values of 4, 6, 8, 10 and 12.

The form of $N=6$
For $N = 6$, the isometric spheres are packed in the form of a cubic lattice (Fig. 1). The three coordinate axes are set as shown in the figure, and each layer of spheres is numbered. The radius of each sphere is set to R(cm).

The form of $N=8$
From the case of $N = 6$, we move the even-numbered layer of spheres perpendicular to the Z-axis along the X-axis by a distance R.

The form of $N=10$
As for the form of $N = 8$, we move the even numbered

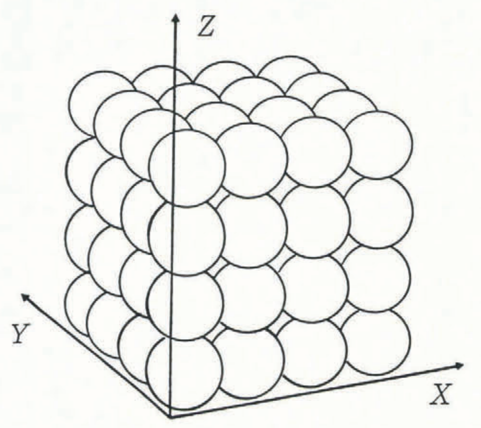

Fig. 1. A perfect packing form of isometric spheres (the cubic lattice, $N = 6$).

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layer of spheres vertical to the Y-axis by a distance R. Then, we can make the form of $N = 10$.

The form of N=12

In the case of $N = 10$, the even numbered layer of spheres vertical to the Y-axis is moved in the direction of the Z-axis by a distance R. This form is the densest packing form.

The form of N=4

This form is the Wurzite Structure: the centers of the spheres are located on the center of gravity and each apex of a regular tetrahedron.

Net-like model

Introduction of the net-like model

The net-like model is derived from the forms of $N = 6, 8, 10$ and 12 by the following procedure. First, the radius of each sphere shrinks from R to $pR\,(0 < q \le 1)$, and then a line connecting each original point of contact becomes the central axis of a column having a radius $qR\,(0 <\le p \le 1)$. A part of this transformation is shown in Figure 2, where the stippled part is the net-like model. The spheres are connected by a column that may simply be called a "branch".

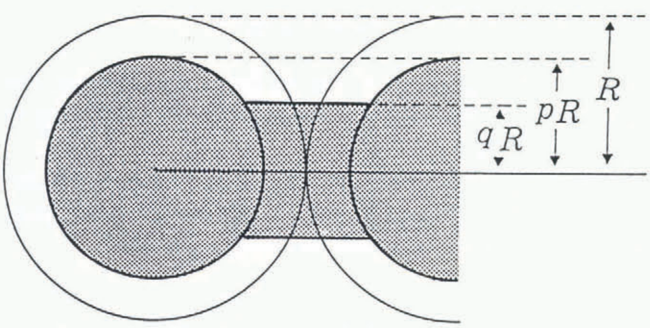

Fig. 2. A part of the net-like model (stippled part).

The crossing state of the branches

As the value of q increases, the branches become thicker and ultimately intersect. Let the angle made by two adjacent branches be θ (radian) as shown by Figure 3. The condition for the branches not to intersect or overlap is $\sin(\theta/2) \le q/p$. θ is $\pi/2$ in the case of $N = 6$, and is equal to $\pi/3$ in the case of $N = 8, 10$ and 12. Then, this equation is expressed by the following forms,

$$q \le (1/\sqrt{2})\, p \quad \cdots\cdots \quad (N = 6)$$
$$q \le (1/2)\, p \quad \cdots\cdots \quad (N = 8,\ 10 \text{ and } 12)\,. \qquad (1)$$

The structure of the model made by uncrossed branches

The net-like model is introduced by shrinking of the spheres of the original form, and therefore the material volume is less than the original. In the case where the model is composed of uncrossed branches, we define the material volume limited to that inside the original sphere as $V_u(\mathrm{cm}^3)$. For calculation of the volume, each branch in V_u is divided by a plane tangent to the original sphere. V_u is then given by the next equation (Watanabe, 1980).

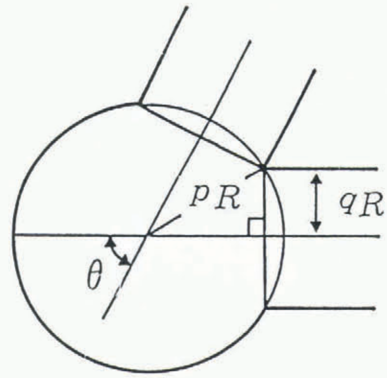

Fig. 3. The state of two branches in contact (θ is the angle made by two branches).

$$V_u = (4/3)\pi R^3[P^3 + (N/4)\{3q^2 - 2p^3 + 2(p^2 - q^2)^{3/2}\}]\,. \qquad (2)$$

Define n as the number per unit length of the original sphere along a line. Then n is equal to $1/(2R)$. The number per unit volume of the shrunken spheres, n', is expressed for each coordination number as

$$N = 6 \quad \cdots\cdots \quad n' = n^3$$
$$N = 8 \quad \cdots\cdots \quad n' = (2/\sqrt{3})n^3$$
$$N = 10 \quad \cdots\cdots \quad n' = (4/3)n^3 \qquad (3)$$
$$N = 12 \quad \cdots\cdots \quad n' = \sqrt{2}n^3\,.$$

If the model is made of a material whose density is $\rho_0\,(\mathrm{g\,cm^{-3}})$, the density of the model, ρ, is expressed

$$\rho = \rho_0\ n'\ V_u\,. \qquad (4)$$

In order to bring the model close to snow, the actual value of ρ_0 is fixed at $0.917\,\mathrm{g\,cm^{-3}}$.

The border model (in the case of N=6 and p=1)

According to Watanabe (1989), the solid angle made by an ice bond in snow increases with the increase in snow density. Therefore, N of the high-density model must be 6.

In the case of $N = 6$, if $p = 1$ and $q = 1/\sqrt{2}$, the model has the maximum density possible with uncrossed branches, and we name this model the border model. V_u of the border model is expressed by substituting the above mentioned values into Equation (2); hence,

$$V_u = (1/3)\,\pi\,R^3\,(3\sqrt{2} + 1) = 5.490\,R^3\,. \qquad (5)$$

The density, ρ, is calculated by substituting this V_u and n' of Equation (3) into Equation (4)

$$\rho = 0.917 \times (8\,R^3)^{-1} \times 5.490\,R^3 = 0.629\,(\mathrm{g\,cm^{-3}})\,.$$

The model made by crossed branches

Where the model density is greater than $0.629\,\mathrm{g\,cm^{-3}}$ it is composed of crossing branches and the following relations are valid: $N = 6$, $p = 1$ and $(1/\sqrt{2}) < q$. A sectional view of this model is shown in Figure 4; this section is perpendicular to the Z-axis and point 0 is the center of a sphere. The stippled part is the branch of the border model, and V_u of this model must add the volume of parts A and B to the volume of the border model.

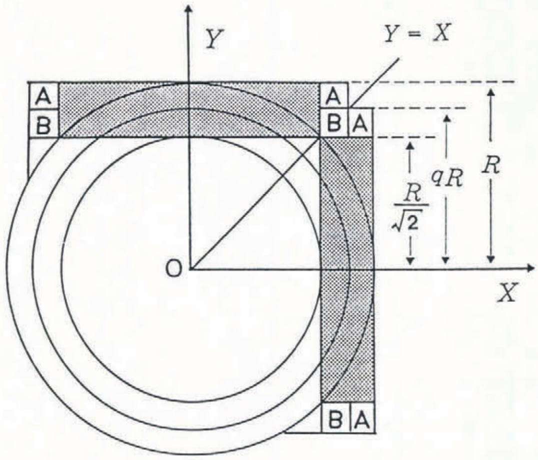

Fig. 4. A sectional view of the model which is composed of crossing branches; the stippled part is the branch of the border model.

Let V_A and V_B stand for these additional volumes per branch. V_A and V_B can be found from Figure 5 which is an expanded view of Figure 4, where a(cm), b(cm) and θ' are defined as $a = qR$, $b = R/\sqrt{2}$ and $\cos^{-1}\theta' = b/a$, respectively. The next equations are obtained by using V_w and are

$$V_A = \pi\,(a^2 - b^2)(R - a), \qquad V_B = \pi\,(a^2 - b^2)(a - b) - 2\,V_w\,, \tag{6}$$

where the part of B on a branch has four overlapping parts, one of which is shown in Figure 5 by stippling. The

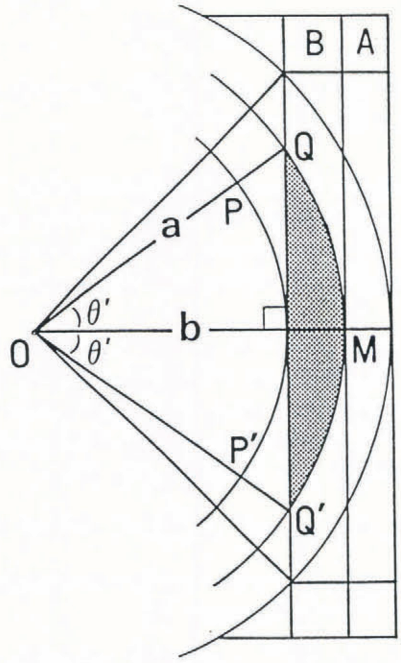

Fig. 5. The expanded view of Fig. 4 in the direction of X-axis.

volume of this part is expressed by V_w. The whole volume of part B is equal to $\pi(a^2 - b^2)(a - b)$, and this volume multiplied by $(2\theta'/2\pi)$ is the volume which is shown by the part of PP′Q′MQP in Figure 5. Then we assume that the next approximate equation is valid, hence

$$V_w = \pi(a^2 - b^2)(a - b)(\theta'/\pi) \times \frac{\text{(the area of the segment QQ′M)}}{\text{(the area of the figure PP′Q′MQP)}}\,.$$

The area of the segment QQ′M is $a^2\theta' - b(a^2 - b^2)^{1/2}$ and the area of the figure PP′Q′MQP is equal to $(a^2 - b^2)\theta'$, thus V_w is shown by

$$V_w = (a - b)(a^2\theta' - b(a^2 - b^2)^{1/2})\,. \tag{7}$$

V_u of the crossed-branches model is equal to the volume that adds $6(V_A + V_B)$ to V_u of the border model, and is expressed as

$$V_u = 5.490R^3 + 6\{(\pi\,(a^2 - b^2)(R - b) - 2(a - b)(a^2\theta' - b(a^2 - b^2)^{1/2})\}\,, \tag{8}$$

where $5.490R^3$ is the volume of the border model as shown in Equation (5). Substituting $a = qR$ and $b = R/\sqrt{2}$ into Equation (8), it follows that

$$V = R^3(5.490 + 5.521(q^2 - 1/2) - 12(q - 1/\sqrt{2})\{q^2\cos^{-1}(1/\sqrt{2}q) - (1/\sqrt{2})(q^2 - 1/2)^{1/2}\})\,. \tag{9}$$

Substituting Equation (9) for V_u into Equation (4), the density of the model is calculated from the value of q as shown in Table 1.

In this table, the first line is the value of q and ρ for the border model. In the case of $q = 1$, the model corresponds with snow and ice in which the air is enclosed.

CORRESPONDENCE BETWEEN THE MODEL AND SNOW

On the size of the unit sphere

Ishida and Shimizu (1955) measured the air flow resistance through snow layers. They considered that the snow was a set of isometric ice spheres, and found that the radii of the spheres were within a certain range according to snow density.

Watanabe (1964) measured the height of capillary rise of water in compacted snow whose density ρ (g cm^{-3}) was in the range of $0.15 \simeq 0.45$ g cm^{-3}, and expressed the radii of the isometric spheres R(cm) by the approximate equation

$$R = 0.15\,\rho\,, \tag{10}$$

Table 1. q and density, ρ, of the model

q	0.707	0.75	0.80	0.85	0.90	0.95	1.00
ρ (g cm^{-3})	0.629	0.668	0.712	0.752	0.787	0.813	0.831

where R is in the range pointed out by Ishida and Shimizu (1955) and almost agrees with the value obtained from the projection of untied snow particles. Therefore, we make the assumption that R of the net-like model is described by Equation (10).

On the coordination number

Thin-section photographs of snow block taken at equal distances were analyzed to determine separate branches, and the solid angle of branches was measured by using spherical geometry (Watanabe, 1989). Figure 6 shows the distribution chart of the solid angle of compacted snow with density of $0.39\,\mathrm{g\,cm^{-3}}$.

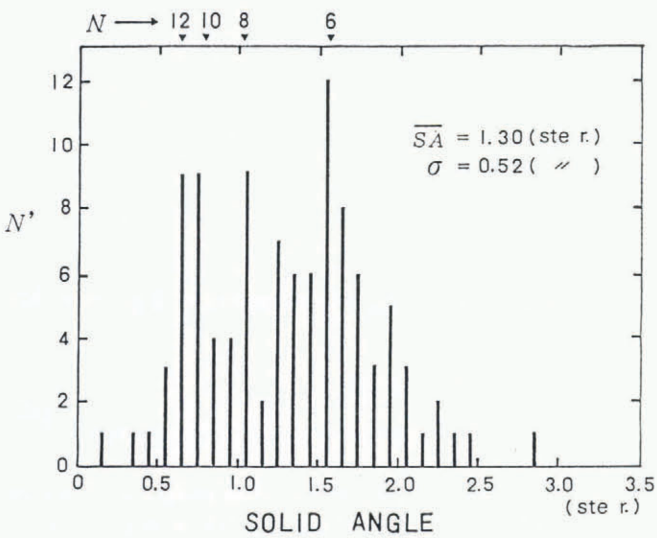

Fig. 6. The distribution of the solid angle of ice bonds (snow density is $0.39\,g\,cm^{-3}$).

The mean value is equal to 1.30 (ste r.) and the standard deviation is 0.52. The upper scale N in this figure is the coordination number of the net-like model, and for N equal 12, 10, 8 and 6 corresponds to $\pi/5$, $\pi/4$, $\pi/3$ and $\pi/2$ (ste r.), respectively. The observed frequency of solid angle at $N = 12$, 10, 8 and 6 was high and we can say that the model predicts well the structure of real snow.

The relation between the density and the mean value of the solid angle $\overline{SA}$ for five snow samples is shown in

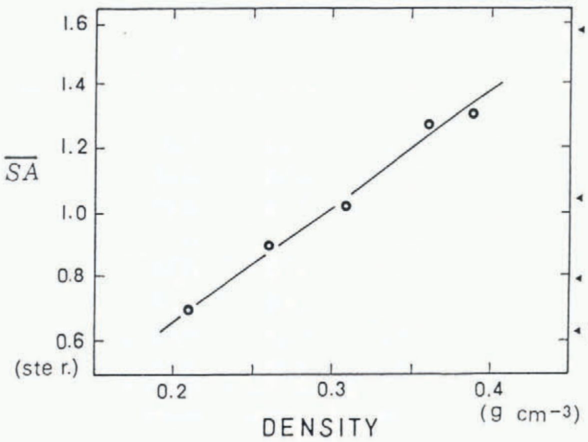

Fig. 7. The mean value of solid angle $\overline{SA}$ versus snow density.

Figure 7. N on the right side in the figure is the coordination number. $\overline{SA}$ increases with ρ, and is roughly expressed as

$$\overline{SA} = 3.6\rho - 0.06\,. \tag{11}$$

Thus, N must be a small value for snow density greater than $0.3\,\mathrm{g\,cm^{-3}}$.

On the tensile strength

The tensile strength of the model

When a tension or pressure F ($\mathrm{g\,W\,cm^{-2}}$) is applied to one of the spheres in the model in the direction of one of the axes shown in Figure 1, the force is carried in one of three ways depending on the coordination number (Fig. 8).

In the case of $N = 8$, if F is applied in the direction of the Z-axis, it is separated into two forces as shown in Figure 8b. If this separate force is f, f/F is equal to $1/\sqrt{3}$ (Watanabe, 1991). When $N = 12$ (Fig. 8c), F is applied in the direction of the Y-axis, and f/F is equal to $1/\sqrt{6}$. Table 2 shows carrying states of force (Watanabe, 1991).

In this table f/F is the proportion of force which is applied to only one branch, and f/F multiplied by the number of carrying branches of F and the number of spheres per unit area equals the actual number of branches per unit area. As shown in the table, the actual number of branches depends only on N, not on the axis.

If the strength of the material of model is represented by σ_0 ($\mathrm{g\text{-}W\,cm^{-2}}$), then the strength of the branch is expressed by $\pi(qR)^2\sigma_0$. In that case, the tensile strength or compressive strength, σ, of bulk snow can be expressed

$$\sigma = \pi(qR)^2\sigma_0 \times \text{(the actual number of branches)}. \tag{12}$$

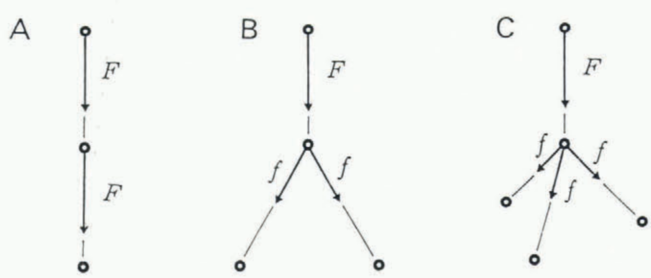

Fig. 8. The carried forms of force in the model.

In the case of the coordination numbers of 6, 8, 10 and 12, the actual number of branches are n^2, $2/\sqrt{3}n^2$, $4/3n^2$ and $\sqrt{2}n^2$, respectively (Table 2).

The tensile strength of snow

Measured values of the strength of snow show large scatter. The author measured the tensile strength of snow at about −3 to −5°C using a centrifugal separator, and developed the equation (Watanabe, 1974)

$$\log\sigma_t = 3.24\log\rho + 4.28\,, \tag{13}$$

where ρ is the density of compacted snow and σ_t is the tensile strength. Application of this equation is limited to $0.1 \le \rho \le 0.45$ because of the snow samples used in deriving the equation.

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Table 2. Carrying state of force in the model

A (N)	6			8			10			12		
B (axis)	X	Y	Z	X	Y	Z	X	Y	Z	X	Y	Z
C ($\times n$)	1	1	1	1	1	$2/\sqrt{3}$	1	$2/\sqrt{3}$	$2/\sqrt{3}$	1	$3/\sqrt{6}$	$2/\sqrt{3}$
D ($\times n^2$)	1	1	1	$2/\sqrt{3}$	$2/\sqrt{3}$	1	4/3	$2/\sqrt{3}$	$2/\sqrt{3}$	$\sqrt{2}$	$2/\sqrt{3}$	$\sqrt{6}/2$
E_1	1	1	1	1	1	2	1	2	2	1	3	2
E_2	1	1	1	1	1	$1/\sqrt{3}$	1	$1/\sqrt{3}$	$1/\sqrt{3}$	1	$1/\sqrt{6}$	$1/\sqrt{3}$
F ($\times n^2$)	1	1	1	$2/\sqrt{3}$	1	$1/\sqrt{3}$	4/3	$1/\sqrt{3}$	$1/\sqrt{3}$	$\sqrt{2}$	$1/\sqrt{6}$	$1/\sqrt{3}$

A: coordinate number
B: axis (direction of force)
C: number of spheres per unit length
D: number of spheres per unit area perpendicular to the axis
E_1: number of branches for F (number)
E_2: number of branches for F (f/F)
F: actual number of branches per unit area
(where $n = 1/2R$: R is the radius of original sphere)

Next, the author measured the tensile strength of a polycrystalline ice, σ_{it}, at approximately −3°C. The average value from more than 100 measurements (Watanabe, 1991) is

$$\sigma_{it} = 21.9\ (\text{kg W cm}^{-2}) \tag{14}$$

where the standard deviation is 4.4.

The net-like model of snow

The model for density less than 0.45 g cm⁻³

We try to find a model whose density and strength match snow. The tensile strength of snow, calculated by Equation (13), is shown in the second column in Table 3. We assume that the strength of the material in the model (σ_0) is equal to σ_{it} in Equation (14). Equation (12) is then rewritten as

$$\sigma = \pi(qR)^2 \times 21.9 \times 10^3 \times (\text{the actual number of branches})\,, \tag{15}$$

where the actual number of branches contains the factor $n^2 = (1/4R^2)$. Thus R in Equation (15) is eliminated.

The values of q in Table 3 are obtained by substituting σ_t as calculated from Equation (13) for σ in Equation (15), and then solving for q. Next, let us find the value of p. n' from Equation (3) and V_u from Equation (2) are substituted into Equation (4), where q is shown as Table 3. Equation (4) then has only two factors of ρ and p, thereby giving the values of p in Table 3. Thus, Table 3

Table 3. q and p of the model where strength agrees with snow

ρ	σ_t	Coordinate number 6		8		10		12	
		q	p	q	p	q	p	q	p
0.45	1369	0.28	0.97	0.26	0.92	0.24	0.86	0.24	0.83
0.40	935	0.23	0.93	0.22	0.88	0.20	0.83	0.20	0.80
0.35	606	0.19	0.89	0.17	0.84	0.16	0.80	0.16	0.77
0.30	368	0.15	0.85	0.14	0.80	0.13	0.76	0.12	0.74
0.25	204	0.11	0.80	0.10	0.76	0.09	0.72	0.09	0.70
0.20	99	0.08	0.74	0.07	0.71	0.07	0.67	0.06	0.66
0.15	40	0.05	0.68	0.05	0.64	0.04	0.61	0.04	0.60
0.10	11	0.03	0.59	0.02	0.56	0.02	0.54	0.02	0.52

ρ (in g cm^{-3}): density
σ_t (in g W cm^{-2}): measured value of tensile strength

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shows a model whose density and strength match snow. R is decided by Equation (10).

The coordinate number of snow decreases with the increase of snow density as before, so the area in Table 3 bounded by the thick line is the portion of the table applicable to real snow.

The model for density between 0.45 and 0.63 g cm^{-3}.
$0.45\,g\,cm^{-3}$ is the maximum snow density in the measurement set and $0.63\,g\,cm^{-3}$ is the density of the border model. Where snow density is greater than $0.45\,g\,cm^{-3}$, considering the value of q in Table 3, $p = 1$ is satisfactory. The tensile strength of the model of $p = 1$ is shown in the second column of Table 4.

Table 4. Comparison of tensile strength of the model with calculated values (density between 0.45 and 0.63)

ρ (g cm^{-3})	σ_t (kg W cm^{-2}) Model	Calculated
0.45	1.37	3.42
0.50	3.22	5.14
0.55	6.01	7.17
0.60	7.77	9.47
0.629	8.60	10.91

The strength of snow, σ (kg W cm^{-2}), was derived theoretically by Ballard and Feld (1966) to be

$$\sigma = k \exp\left(\frac{-2\lambda}{1-\lambda}\right), \tag{16}$$

where k is a constant and λ is the porosity of snow. Of course, λ is equal to $(1-1.09\rho)$. Later, Keeler and Weeks (1968) measured the strength of snow and their measured value of tensile strength nearly agreed with the values of Equation (16) by letting $k = 27$. They expressed σ_t by the equation

$$\sigma_t = 27 \exp\left(\frac{-2\lambda}{1-\lambda}\right). \tag{17}$$

Butkovich (1956) also measured the tensile strength of snow. He tested only high-density snow ($0.4 < \rho$), but his measured values almost agree with the values given by Equation (17).

The model of density greater than 0.63
The density of the border model is $0.63\,g\,cm^{-3}$ as before. In the case of the density greater than $0.63\,g\,cm^{-3}$, it is clear from the previous discussion that $N = 6$, $p = 1$ and $1/\sqrt{2} < q$. The first column in Table 5 is the density of the model, the second column is the strength which is decided from Equations (4) and (9), and the third column is the value calculated using Equation (17).

The values of the second columns of Tables 4 and 5

Table 5. Comparison of tensile strength of the model with calculated values (density greater than 0.63)

ρ (g cm^{-3})	σ_t (kg W cm^{-2}) Model	Calculated
0.65	9.12	12.00
0.70	10.65	14.66
0.75	12.34	17.47
0.80	14.72	20.13
0.831	17.20	22.11

are shown in Figure 9. Calculated and model strengths do not always agree, but the measured values of snow strength are generally scattered over a wide range as were the measured values of Keeler and Weeks (1968). Thus, we can say that the net-like model of snow has the same tensile strength of snow.

CONCLUSION

This paper proposes a new "net-like" model, derived from the systematic packing forms of isometric spheres by shrinking the spheres and connecting them with a cylindrical column. The texture of the model can be varied according to the setting of the packing form, the shrinking ratio and the radius of the connecting branch. In order to approximate the model to snow, the density

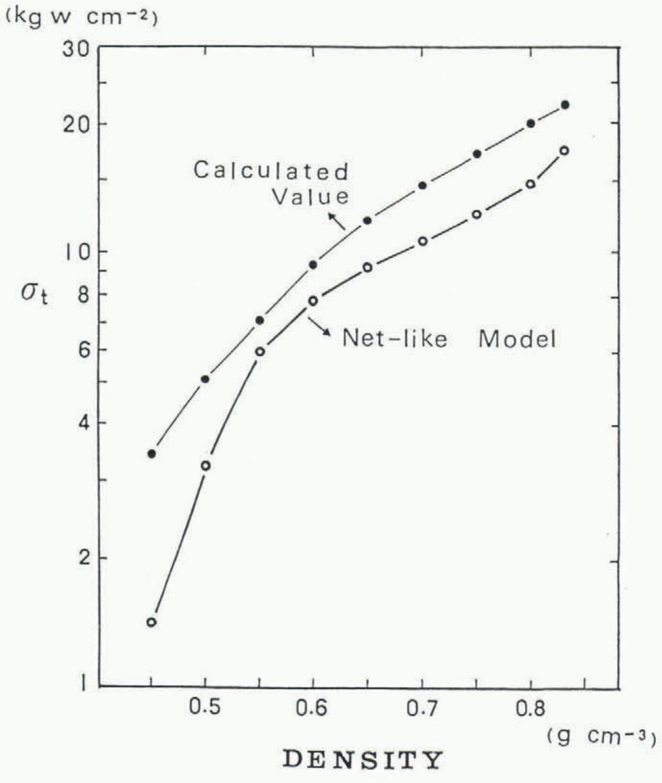

Fig. 9. Comparison of the tensile strength of model with the calculated values for high-density snow.

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and tensile strength of the modeled material are set to correspond to polycrystalline ice.

The author measured the tensile strength of dry compacted snow with density less than 0.45(g cm^{-3}), and has been able to make the net-like model agree in density and tensile strength. In the case of the model for density greater than 0.45 g cm^{-3}, the shrinking ratio of original spheres is equal to 1, and the density and strength are decided by the thickness of the connecting branches. Then, we can calculate the strength of model for each density range respectively.

These calculated values agree well with the data of Butkovich, Keeler and others. For the higher densities, the model corresponds to the border state of snow and ice. The density was 0.831 g cm^{-3} and the tensile strength was 17.2 (kg W cm^{-2}).

Watanabe (1974) measured the specific surface of snow whose density was less than 0.4; the specific surface of the model can be calculated, and the values are almost equivalent.

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The accuracy of references in the text and in this list is the responsibility of the author, to whom queries should be addressed.