*Probability in the Engineering and Informational Sciences* (2023), **37**:4 1029–1034
doi:10.1017/S0269964822000407

CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

# A simple European option pricing formula with a skew Brownian motion

Puneet Pasricha[1] and Xin-Jiang He[2]

[1] Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India.
E-mail: pasrichapuneet5@gmail.com
[2] School of Economics, Zhejiang University of Technology, Hangzhou, China. E-mail: xinjiang@zjut.edu.cn.

**Keywords:** European options, Owen's $T$ function, Skew Brownian motion

**Abstract**
Zhu and He [(2018). A new closed-form formula for pricing European options under a skew Brownian motion. *The European Journal of Finance* 24(12): 1063–1074] provided an innovative closed-form solution by replacing the standard Brownian motion in the Black–Scholes framework using a particular skew Brownian motion. Their formula involves numerically integrating the product of the Guassian density and corresponding distribution function. Being different from their pricing formula, we derive a much simpler formula that only involves the Gaussian distribution function and Owen's $T$ function.

## 1. An alternative and a simple pricing formula

For the sake of completeness, we first summarize the modeling framework considered by Zhu and He [3]. An equivalent martingale measure $Q$ was constructed with the underlying price following

$$S(T) = S(t)e^{\sigma(X(T)-X(t))-\ell(|W_2(t)|)+(r-0.5\sigma^2)\tau}, \tag{1}$$

with $\tau = T - t$. $\ell(y)$ is defined in Section 3.1 of Zhu and He [3], given by

$$\ell(y) = \ln\left[N\left(\frac{y+\sigma\epsilon\tau}{\sqrt{\tau}}\right) + e^{-2\sigma\epsilon y}N\left(\frac{-y+\sigma\epsilon\tau}{\sqrt{\tau}}\right)\right],$$

and

$$X(s) = \epsilon|W_2(s)| + \sqrt{1-\epsilon^2}W_1(s), \quad \epsilon \in (-1,1). \tag{2}$$

Here, $W_1$ and $W_2$ are two Brownian motions independent of each other, with $N(\cdot)$ being a standard normal distribution function. Note that this particular dynamic yields $E^Q(e^{-r(T-t)}S(T) \mid \mathcal{F}_t) = S(t)$ [3], and for $\epsilon = 0$, it reduces to the standard Black–Scholes framework.

It should be remarked that the formula Zhu and He derived actually involves the integral of the product combining the Gaussian density with its corresponding distribution function, which is relatively complicated for numerical implementation. This has prompted us to try to find a simpler one. In fact, European call option prices have an expression of

$$P(S,t) = E^Q((S(T)-K)^+|\mathcal{F}_t)/e^{r\tau}, \tag{3}$$

where $\mathcal{F}_t = \sigma\{(W_1(u), W_2(u)); 0 \le u \le t\}$.

Let $W_1(t) = x$ and $R(t) = |W_2(t)| = y$. Then, $X(t) = \sqrt{1-\epsilon^2}x + \epsilon y$. Define

$$M = S(t)e^{-\sigma X(t) - \ell(y) + (r - 0.5\sigma^2)\tau},$$

so that the call option price has an alternative expression of

$$P(S,t) = \int_0^\infty \int_{-\infty}^\infty (Me^{\sigma\epsilon u + \sigma\sqrt{1-\epsilon^2}v} - K)^+ f(u,v)\, dv\, du / e^{r\tau}, \tag{4}$$

where $f(a_1, a_2) = f_{(|W_2(T)|, W_1(T))|(|W_2(t)|, W_1(t))}(a_1, a_2 \mid y, x)$ is a conditional probability density function, which is further given by

$$f(a_1, a_2) = f_1(a_1) \times f_2(a_2)$$

with

$$f_1(a_1) = f_{|W_2(T)|||W_2(t)|}(a_1 \mid y) = \frac{1}{\sqrt{2\pi\tau}}(e^{-(a_1-y)^2/2\tau} + e^{-(a_1+y)^2/2\tau}), \quad 0 \le a_1 < \infty,$$

$$f_2(a_2) = f_{W_1(T)|W_1(t)}(a_2 \mid x) = \frac{e^{-(a_2-x)^2/2\tau}}{\sqrt{2\pi\tau}}, \quad -\infty < a_2 < \infty.$$

The derivation of our simpler formula requires some fundamental results that need to be derived first, which are provided below.

**Proposition 1.1.** *If $I(m,\sigma)$ denotes*

$$I(m,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty e^{ax} N(a_1 + a_2 w) e^{-(x-m)^2/2\sigma^2}\, dx$$

*where $a, a_1, a_2, m, \sigma \in \mathbb{R}$, then we have its simplified value shown in the following formula*

$$I(m,\sigma) = e^{\frac{1}{2}(a^2\sigma^2 + 2am)} \left( \frac{N(h_1) - N(h_2)}{2} - T(h_1, \tilde{a}_1) - T(h_2, \tilde{a}_2) \right), \tag{5}$$

*where*

$$h_1 = \frac{a_1 + a_2(m + a\sigma^2)}{\sqrt{1 + a_2^2\sigma^2}}, \quad h_2 = \frac{-m - a\sigma^2}{\sigma}, \quad \tilde{a}_1 = \frac{h_2 - rh_1}{h_1\sqrt{1-r^2}},$$

$$\tilde{a}_2 = \frac{h_1 - rh_2}{h_2\sqrt{1-r^2}}, \quad r = \frac{-a_2\sigma}{\sqrt{1 + a_2^2\sigma^2}},$$

*with the Owen's T function denoted by $T(h,a)$. The relationship connecting the Owen's T function with the bivariate normal probability is provided in the Appendix.*

*Proof.* We can re-write

$$\begin{aligned} I(m,\sigma) &= \frac{e^{\frac{1}{2}(a^2\sigma^2 + 2am)}}{\sigma\sqrt{2\pi}} \int_0^\infty N(a_1 + a_2 x) e^{-(x-(m+a\sigma^2))^2/2\sigma^2}\, dx \\ &= \frac{e^{\frac{1}{2}(a^2\sigma^2 + 2am)}}{\sigma\sqrt{2\pi}} \int_0^\infty \int_{-\infty}^{a_1 + a_2 x} N(u) e^{-(x-(m+a\sigma^2))^2/2\sigma^2}\, du\, dx, \end{aligned} \tag{6}$$

which implies

$$I(m,\sigma) = e^{\frac{1}{2}(a^2\sigma^2+2am)}P(Y \le a_1 + a_2X, X > 0). \tag{7}$$

Here, the joint distribution of $X$ and $Y$ gives a bivariate Gaussian one, the covariance matrix of which is $\left(\begin{smallmatrix}\sigma^2 & 0\\ 0 & 1\end{smallmatrix}\right)$ along with its mean as $(m + a\sigma^2, 0)$. By further denoting $Z = Y - a_1 - a_2X$, one would be able to obtain $E(Z) = -a_1 - a_2(m + a\sigma^2)$, variance $\mathrm{Var}(Z) = 1 + a_2^2\sigma^2$ and covariance with $X$ being $\mathrm{Cov}(Z,X) = -a_2\sigma^2$. As a result, the unknown probability involved in Eq. (7) can then be simplified through

$$\begin{aligned}
I(m,\sigma) &= e^{\frac{1}{2}(a^2\sigma^2+2am)}P(Z \le 0, X > 0)\\
&= e^{\frac{1}{2}(a^2\sigma^2+2am)}(P(X > 0) - P(Z > 0, X > 0))\\
&= e^{\frac{1}{2}(a^2\sigma^2+2am)}\left(P(X > 0) - P\left(Z_1 > \frac{a_1 + a_2(m + a\sigma^2)}{\sqrt{1 + a_2^2\sigma^2}}, Z_2 > \frac{-m - a\sigma^2}{\sigma}\right)\right),
\end{aligned}$$

where both $Z_1$ and $Z_2$ follow a standard normal distribution and their correlation is captured with a parameter $r$. Thus, considering how the Owen's $T$ function is related to the bivariate normal CDF, one can obtain

$$\begin{aligned}
I(m,\sigma) &= e^{\frac{1}{2}(a^2\sigma^2+2am)}\left(N\left(\frac{m + a\sigma^2}{\sigma}\right) - P(Z_1 > h_1, Z_2 > h_2)\right)\\
&= e^{\frac{1}{2}(a^2\sigma^2+2am)}\left(N(-h_2) - \left(1 - \frac{N(h_1) + N(h_2)}{2} - T(h_1,\tilde{a}_1) - T(h_2,\tilde{a}_2)\right)\right)\\
&= e^{\frac{1}{2}(a^2\sigma^2+2am)}\left(\frac{N(h_1) - N(h_2)}{2} - T(h_1,\tilde{a}_1) - T(h_2,\tilde{a}_2)\right).
\end{aligned} \tag{8}$$

This completes the proof. □

From Equation (4), we obtain

$$P(S,t) = e^{-r(T-t)}\int_0^\infty I(u)f_1(u)\,du \tag{9}$$

where

$$I(u) = \int_{-\infty}^\infty (Me^{\sigma\epsilon u+\sigma\sqrt{1-\epsilon^2}v} - K)^+ f_2(v)\,dv. \tag{10}$$

Using the standard Black–Scholes formula yields

$$I(u) = e^{\ln(M)+\sigma\epsilon u+\sigma\sqrt{1-\epsilon^2}x+\frac{1}{2}\sigma^2(1-\epsilon^2)(T-t)}N(d_1) - KN(d_2), \tag{11}$$

where

$$\begin{aligned}
d_1 &= -\frac{\ln(K) - (\ln(M) + \sigma\epsilon u + \sigma\sqrt{1-\epsilon^2}x + \sigma^2(1-\epsilon^2)\tau))}{\sigma\sqrt{1-\epsilon^2}\sqrt{\tau}}\\
d_2 &= d_1 - \sigma\sqrt{1-\epsilon^2}\sqrt{\tau}.
\end{aligned}$$

Therefore, Eq. (9) further leads to

$$P(S,t) = \int_0^\infty (e^{\ln(M)+\sigma\epsilon u+\sigma\sqrt{1-\epsilon^2}x+\frac{1}{2}\sigma^2(1-\epsilon^2)\tau}N(c_1 + c_2u) - KN(c_3 + c_2u))f_1(u)\,du/e^{r\tau} \tag{12}$$

https://doi.org/10.1017/S0269964822000407 Published online by Cambridge University Press

where

$$c_1 = -\frac{\ln(K) - (\ln(M) + \sigma\sqrt{1-\epsilon^2}x + \sigma^2(1-\epsilon^2)\tau))}{\sigma\sqrt{1-\epsilon^2}\sqrt{\tau}},$$
$$c_2 = +\frac{\epsilon}{\sqrt{1-\epsilon^2}\sqrt{\tau}},$$
$$c_3 = c_1 - \sigma\sqrt{1-\epsilon^2}\sqrt{\tau}.$$

Using Proposition 1.1 leads to

$$P(S,t) = (e^{\ln(M)+\sigma\sqrt{1-\epsilon^2}x+\frac{1}{2}\sigma^2(1-\epsilon^2)\tau}J_1 - KJ_2)/e^{r\tau} \tag{13}$$

with

$$J_1 = \int_0^\infty e^{\sigma\epsilon u}N(c_1+c_2u)f_1(u)\,du = I_1(y,\lambda,c_1,c_2) + I_1(-y,\lambda,c_1,c_2)$$
$$J_2 = \int_0^\infty N(c_3+c_2u)f_1(u)\,du = I_1(y,0,c_1,c_2) + I_1(-y,0,c_1,c_2),$$

where $\lambda = \epsilon\sigma$ is introduced for notation ease. If we apply Proposition 1.1 once again, we obtain

$$I_1(y,\lambda,c_1,c_2) = \int_0^\infty e^{\lambda u}N(c_1+c_2u)\frac{e^{-(u-y)^2/2\tau}}{\sqrt{2\pi\tau}}\,du$$
$$= e^{\frac{1}{2}(\lambda^2\tau+2\lambda y)}\left(\frac{N(H_1)-N(H_2)}{2} - T(H_1,A_1) - T(H_2,A_2)\right),$$

where

$$H_1 = \frac{c_1+c_2(y+\lambda\tau)}{\sqrt{1+c_2^2\tau}},\ H_2 = \frac{-y-\lambda\tau}{\sqrt{\tau}},\ A_1 = \frac{H_2-R_1H_1}{H_1\sqrt{1-R_1^2}},$$
$$A_2 = \frac{H_1-R_1H_2}{H_2\sqrt{1-R_1^2}},\ R_1 = \frac{-c_2\sqrt{\tau}}{\sqrt{1+c_2^2\tau}}.$$

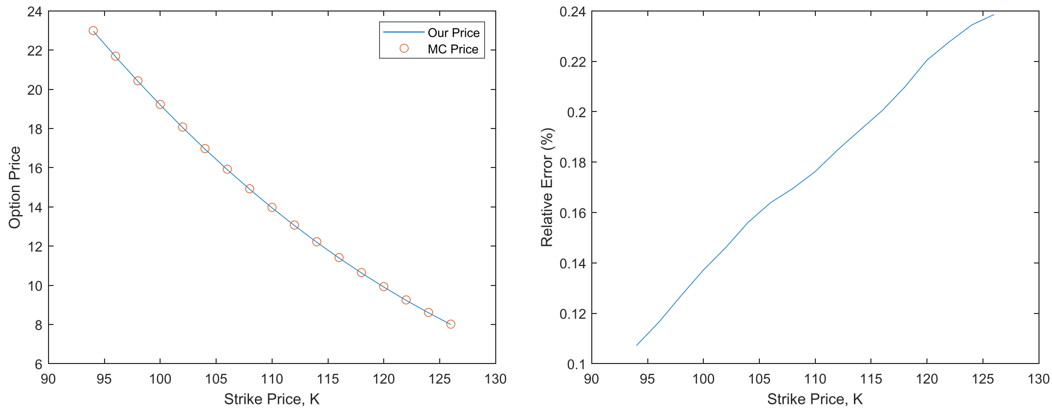
This clearly shows that the pricing formula presented in Eq. (13) is fully analytical now.

## 2. Accuracy tests

This section is devoted to checking the correctness of the simple option pricing formula derived in the above section using the Monte Carlo benchmark. The certain parameter values we select are $r = 0.1$, $S(t) = 110$, $\sigma = \sqrt{0.4}$. It should be pointed out that at current time $t$, we observe the current stock price $S(t)$ in the market, but are unable to observe the current (starting) values of the Brownian motions, $W_i(t),\ i = 1,2$. Therefore, in practical applications, $W_i(t),\ i = 1,2$ are actually treated as model parameters and can be calibrated together with other model parameters (e.g., $\sigma$ and $\delta$) with real data. For illustration purposes, we fix $W_i(t),\ i = 1,2$ as 0.02 and −0.01, respectively. We also select 0.25 as the value of the time to maturity, and the skewness parameter $\epsilon$ is assumed to be equal to 0.5.

To be sure that we did not make any mistakes when deriving the pricing formula, we need to address its correctness. We accomplish this task by benchmarking our results using the Monte Carlo simulation, with the parameters kept as the same. Figure 1(a) displays the comparison results, which obviously show the point-wise agreement of both prices, and the relative error displayed in Figure 1(b) remains below 0.24%. These are actually a verification of the formula.

https://doi.org/10.1017/S0269964822000407 Published online by Cambridge University Press




(a) The two prices. (b) Relative difference.

***Figure 1.*** *A demonstration of the accuracy. (a) The two prices. (b) Relative difference.*

## 3. Conclusion

In this article, we price European options with the geometric skew Brownian motion considered in Zhu and He [3]. A simpler pricing formula is presented using the standard normal distribution function and Owen's $T$ function. The accuracy of the formula is also numerically verified.

**Funding.** This work is supported by the National Natural Science Foundation of China (No. 12101554; No. 72002201), the Fundamental Research Funds for Zhejiang Provincial Universities (No. GB202103001), and the Advanced Research Funds of Zhejiang University of Technology (No. SKY-ZX-20220212).

**Competing interest.** The authors declare that they have no conflict of interest.

## References

[1] Owen, D.B. (1956). Tables for computing bivariate normal probabilities. *The Annals of Mathematical Statistics* 27(4): 1075–1090.
[2] Sowden, R. & Ashford, J. (1969). Computation of the bi-variate normal integral. *Journal of the Royal Statistical Society: Series C (Applied Statistics)* 18(2): 169–180.
[3] Zhu, S.-P. & He, X.-J. (2018). A new closed-form formula for pricing European options under a skew Brownian motion. *The European Journal of Finance* 24(12): 1063–1074.

## Appendix

### Relationship between the Owen's $T$ function and bivariate normal density [1,2]

Let a pair of bivariate standard normal variables $(X_1, X_2)$ be correlated with each other with a parameter $r$. We have the following relation

$$F(u_1, u_2, r) = P(X_1 > u_1, X_2 > u_2) = 1 - \frac{N(u_1) + N(u_2)}{2} - T(u_1, b_1) - T(u_2, b_2), \tag{A.1}$$

where

$$b_1 = \frac{u_2 - ru_1}{u_1\sqrt{1-r^2}}, \quad b_2 = \frac{u_1 - ru_2}{u_2\sqrt{1-r^2}}, \tag{A.2}$$

and $T(u, b)$ is Owen's $T$ function formulated as

$$T(u, b) = \frac{1}{2\pi}\int_0^b \frac{e^{-\frac{1}{2}u^2(1+x^2)}}{1+x^2}\,dx. \tag{A.3}$$

https://doi.org/10.1017/S0269964822000407 Published online by Cambridge University Press

Furthermore, it can be shown that

$$\begin{aligned}F(-u_1, u_2, r) &= -F(u_1, u_2, -r) + N(-u_2),\\ F(u_1, -u_2, r) &= -F(u_1, u_2, -r) + N(-u_1),\\ F(-u_1, -u_2, r) &= F(u_1, u_2, r) + 1 - N(-u_1) - N(-u_2).\end{aligned}$$

As a result, is suffices to only cope with $u_1$ and $u_2$ when they are not negative.

**Cite this article:** Pasricha P and He X-J (2023). A simple European option pricing formula with a skew Brownian motion. *Probability in the Engineering and Informational Sciences* **37**, 1029–1034. https://doi.org/10.1017/S0269964822000407