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ORDERED PROBABILISTIC METRIC SPACES

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Abstract

Probabilistic quasi-metric spaces are introduced and used to define ordered probabilistic metric spaces. The latter spaces arise naturally in the study of probability and statistics; they closely resemble the uniform ordered spaces of L. Nachbin. A procedure is described for constructing ordered probabilistic metric spaces from quasi-simple spaces, and a completion theory is developed simultaneously for probabilistic quasi-metric spaces and ordered probabilistic metric spaces.

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1. Introduction

A probabilistic metric space is a generalized metric space appropriate to the study of situations in which "distances" are measured in terms of distribution functions rather than non-negative real numbers. In many cases, a probabilistic metric space is endowed with an intrinsic order and questions naturally arise concerning the interaction between the order and the generalized metric, possible ways to extend the order to a completion space, etc..

An effective way to define compatibility between a uniform structure and an order structure on the same set S was introduced by L. Nachbin [2]. A *quasi-uniformity* (which Nachbin calls a "semi-uniform structure") is a filter $\mathscr{U}$ on

$S \times S$ which is required to satisfy all the properties of uniformity except the symmetry property $\mathscr{U} = \mathscr{U}^{-1}$. Nachbin defines a triple $(S, \mathscr{V}, \leq)$, where $\mathscr{V}$ is a uniformity and $\leq$ a partial order on S, to be a ***uniform ordered space*** if there is a quasi-uniformity $\mathscr{U}$ on S, such that $\mathscr{V} = \mathscr{U} \wedge \mathscr{U}^{-1}$ is the smallest uniformity containing $\mathscr{U}$, and $p \leq q$ if and only if $(p, q) \in U$ for all $U \in \mathscr{U}$. In a similar way, a (probabilistic) quasi-metric on S gives rise to an ordered (probabilistic) metric space.

Probabilistic quasi-metric spaces, ordered probabilistic metric spaces, and the relationships between them are the subject of Sections 2 and 3. Section 4 deals with quasi-simple spaces; these are probabilistic quasi-metric spaces derived from ordinary quasi-metrics. It is shown in Section 4 that every ordered probabilistic metric space is compatible with a quasi-simple space. It is also shown in this section that a probabilistic metric space with a partial order is an ordered probabilistic metric space if and only if the associated strong uniformity and the given order constitute a uniform ordered space. In Section 5, a completion theory, similar to that of Sherwood [4] for probabilistic metric spaces, is developed for probabilistic quasi-metric spaces and ordered probabilistic metric spaces.

2. Probabilistic quasi-metric spaces

A ***distance distribution function*** F is a non-decreasing function from $R^+ = [0, \infty]$ into $[0, 1]$, which is left continuous on $(0, \infty)$ and takes on the values $F(0) = 0$ and $F(\infty) = 1$. The set of all distance distribution functions, denoted by Δ^+, is equipped with the modified Lévy metric d_L (see page 45 of [4]). Convergence in the metric space (Δ^+, d_L) is characterized by: $(F_n) \to F$ if and only if $(F_n(x)) \to F(x)$ in R, whenever x is a point of continuity of F. The metric space (Δ^+, d_L) is compact, and Δ^+ is partially ordered by the usual (pointwise) order for real-valued functions.

Let ε_0 be the element of Δ^+ defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

A ***triangle function*** τ is defined to be a binary operation on Δ^+ which is commutative, associative, non-decreasing in each component, and has ε_0 as its identity. ***All triangle functions considered in this paper will, in addition, be assumed to be continuous*** with respect to the topology induced by metric d_L. Given $t > 0$, let $N_{\varepsilon_0}(t) = \{F \in \Delta^+ : d_L(F, \varepsilon_0) < t\}$ denote a basic neighborhood of ε_0 in this topology.

DEFINITION 2.1. Let S be a non-empty set, and let τ be a triangle function on Δ^+. A function $\mathscr{F}: S \times S \to \Delta^+$ is a *probabilistic quasi-metric* if the following conditions are satisfied for all p, q, r in S:

$(\text{PM})_1$ $\mathscr{F}(p,p) = \varepsilon_0$;

$(\text{PM})_2$ if $\mathscr{F}(p,q) = \varepsilon_0$ and $\mathscr{F}(q,p) = \varepsilon_0$, then $p = q$;

$(\text{PM})_3$ $\mathscr{F}(p,q) \geq \tau(\mathscr{F}(p,r), \mathscr{F}(r,q))$.

If $\mathscr{F}$ is a probabilistic quasi-metric, then the triple $(S, \mathscr{F}, \tau)$ is called a *probabilistic quasi-metric space* (abbreviated *PQM space*). A probabilistic quasi-metric $\mathscr{F}$ is called a *probabilistic metric* if it satisfies the symmetry condition

$(\text{PM})_4$ $\mathscr{F}(p,q) = \mathscr{F}(q,p)$ for all p, q in S.

In the latter case, $(S, \mathscr{F}, \tau)$ is a *probabilistic metric space* (abbreviated *PM space*).

We shall often adopt the common practice of writing F_{pq} in place of $\mathscr{F}(p,q)$.

In [1], P. S. Marcus gives an example of a PQM space based on stationary Markov chains which is not a PM space.

An excellent treatment of PM spaces is given in the book co-authored by B. Schweizer and A. Sklar [4]. One familiar concept associated with a PM space $(S, \mathscr{F}, \tau)$ (recall that τ is always assumed to be continuous) is the *strong uniformity* $\mathscr{U}$ on S. For each $t > 0$, let $U(t) = \{(p,q) \in S \times S\colon F_{pq} \in N_{\varepsilon_0}(t)\}$; the sets $\{U(t)\colon t > 0\}$ constitute a filter base for $\mathscr{U}$. If $(S, \mathscr{F}, \tau)$ is a PQM space, it is easy to verify that the filter $\mathscr{U}$ constructed in this way is a quasi-uniformity, which we shall call the *strong quasi-uniformity* determined by $(S, \mathscr{F}, \tau)$. From the strong quasi-uniformity, we obtain the *strong topology* σ associated with $\mathscr{F}$ which has for its neighborhood base at $p \in S$ the collection $\{U_p(t)\colon U \in \mathscr{U}, t > 0\}$, where $U_p(t) = \{q \in S\colon (p,q) \in U(t)\}$.

We can also associate with any PQM space $(S, \mathscr{F}, \tau)$ an "inverse PQM space" $(S, \mathscr{F}', \tau)$, where $\mathscr{F}'(p,q) = \mathscr{F}(q,p)$ for all p, q in S. If $\mathscr{U}$ is the strong quasi-uniformity for $\mathscr{F}$, then $\mathscr{U}^{-1}$ is the strong quasi-uniformity for $\mathscr{F}'$. Also associated with $\mathscr{F}$ is the uniformity $\hat{\mathscr{U}} = \mathscr{U} \vee \mathscr{U}^{-1}$. From $\mathscr{U}, \mathscr{U}^{-1}$, and $\hat{\mathscr{U}}$ we derive in the natural way the topologies $\sigma, \sigma^1, \hat{\sigma}$. In the next section, we define a probabilistic metric $\hat{\mathscr{F}}$ associated with $\mathscr{F}$, such that $\hat{\mathscr{U}}$ (respectively, $\hat{\sigma}$) is the strong uniformity (respectively, strong topology) for $\hat{\mathscr{F}}$. Some properties of the topologies σ, σ^1, and $\hat{\sigma}$ are given in the next two propositions; the straightforward proofs are omitted. The symbols $\wedge$ and $\vee$ are used throughout this work to denote the greatest lower bound and least upper bound, respectively.

PROPOSITION 2.1. *Let $(S, \mathscr{F}, \tau)$ be a PQM space, let (p_n) be a sequence in S, and $p \in S$.*

(1) *$(p_n) \to p$ in (S, σ) if and only if $F_{pp_n} \to \varepsilon_0$ in (Δ^+, d_L).*

(2) *$(p_n) \to p$ in (S, σ') if and only if $F_{p_n p} \to \varepsilon_0$ in (Δ^+, d_L).*

(3) $(p_n) \to p$ in $(S,\hat{\sigma})$ if and only if $F_{pp_n} \wedge F_{p_np} \to \varepsilon_0$ in (Δ^+, d_L) or, equivalently if and only if (1) and (2) hold.

Given a non-empty subset A of S, let $F_{pA} = \sup\{F_{pq}: q \in A\}$ and $F_{Ap} = \sup\{F_{qp}: q \in A\}$; note that F_{pA} and F_{Ap} are both elements of Δ^+.

PROPOSITION 2.2. *Let $(S, \mathscr{F}, \tau)$ be a PQM space. A non-empty subset A of S is σ-closed (respectively, σ'-closed) if and only if $F_{pA} \neq \varepsilon_0$ (respectively, $F_{Ap} \neq \varepsilon_0$) whenever $p \notin A$.*

In addition to generating certain quasi-uniformities and topologies on S, a probabilistic quasi-metric $\mathscr{F}$ also determines an order on S according to the rule: $p \leq q$ if and only if $F_{pq} = \varepsilon_0$. This is also the order induced by the strong quasi-uniformity associated with $\mathscr{F}$. If $\mathscr{F}$ is a probabilistic metric, the induced order is the discrete (or trivial) order: $p \leq q$ if and only if $p = q$.

3. Ordered probabilistic metric spaces

DEFINITION 3.1. A quadruple $(S, \mathscr{G}, \tau, \leq)$ is an *ordered probabilistic metric space* (abbreviated *OPM space*) if $(S, \mathscr{G}, \tau)$ is a PM space, and there is a PQM space $(S, \mathscr{F}, \tau')$ such that the following conditions are satisfied:

(i) If $\mathscr{U}$ is the strong quasi-uniformity for $\mathscr{F}$, then $\hat{\mathscr{U}} = \mathscr{U} \vee \mathscr{U}^{-1}$ is the strong uniformity for $\mathscr{G}$.

(ii) $(S, \leq)$ is a poset with partial order determined by $\mathscr{F}$.

A PQM space $(S, \mathscr{F}, \tau')$ and an OPM space $(S, \mathscr{G}, \tau, \leq)$ which are related in the manner specified in Definition 3.1 are said to be *compatible*. Any PM space $(S, \mathscr{G}, \tau)$ with the discrete order can be regarded as an OPM space, where a compatible PQM space is $(S, \mathscr{F}, \tau) = (S, \mathscr{G}, \tau)$. If $(S, \mathscr{G}, \tau, \leq)$ is an OPM space, then $(S, \hat{\mathscr{U}}, \leq)$ is a uniform ordered space. Two OPM spaces are *uniformly equivalent* if they induce the same uniform ordered space; uniformly equivalent OPM spaces must have the same underlying set, but their triangle functions may be different.

It is obvious from Definition 3.1, that every OPM space is compatible with a PQM space. We next describe a procedure for constructing an OPM space from a given PQM space. Starting with a PQM space $(S, \mathscr{F}, \tau)$, recall that $\mathscr{F}'$ is the probabilistic quasi-metric inverse to $\mathscr{F}$.

Let τ^* denote any (continuous) triangle function which dominates τ (see [4], page 209), and define $\mathscr{F}^*(p,q) = \tau^*(\mathscr{F}(p,q), \mathscr{F}'(p,q))$. The triangle function defined by $\hat{\tau}(F,G) = F \wedge G$ dominates τ and note that $\mathscr{F}^*$ reduces to $\mathscr{F} \wedge \mathscr{F}'$ in this case. Define $\hat{\mathscr{F}} = \mathscr{F} \wedge \mathscr{F}'$. Moreover, $\tau \gg \tau$ ([4], page 210).

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PROPOSITION 3.2. *If $(S, \mathscr{F}, \tau)$ is a PQM space and $\leq$ the partial order determined by $\mathscr{F}$, then $(S, \mathscr{F}^*, \tau, \leq)$ is an OPM space compatible with $(S, \mathscr{F}, \tau)$. In particular, $(S, \hat{\mathscr{F}}, \tau, \leq)$ and $(S, \mathscr{F}, \tau)$ are compatible.*

PROOF. It must be shown that $(S, \mathscr{F}^*, \tau)$ is a PM space and $\mathscr{V} = \mathscr{U} \vee \mathscr{U}^{-1}$, where $\mathscr{V}$ is the strong uniformity for $\mathscr{F}^*$ and $\mathscr{U}$ is the strong quasi-uniformity for $\mathscr{F}$.

First, note that the symmetry of $\mathscr{F}^*$ is obvious and that $F^*_{pp} = \tau^*(F_{pp}, F_{pp}) = \varepsilon_0$. Next, if $F^*_{pq} = \varepsilon_0$, then $F_{pq} \wedge F_{qp} \geq \tau^*(F_{pq}, F_{qp}) = \varepsilon_0$, which implies that $F_{pq} = F_{qp} = \varepsilon_0$ and thus $p = q$. Let us use the definition of $\tau^* \gg \tau$ to prove the triangle inequality. We have $\tau(F^*_{pr}, F^*_{rq}) = \tau(\tau^*(F_{pr}, F_{rp}),\ \tau^*(F_{rq}, F_{qr})) \leq \tau^*(\tau(F_{pr}, F_{rq}),\ \tau(F_{rp}, F_{qr})) \leq \tau^*(F_{pq}, F_{qp}) = F^*_{pq}$. Hence, $(S, \mathscr{F}^*, \tau)$ is a PM space.

Note $\hat{\tau} \gg \tau^* \gg \tau$. Using the continuity of τ at $(\varepsilon_0, \varepsilon_0)$, it is straightforward to verify that the strong uniformities for $\tau^* = \tau$ and $\tau^* = \hat{\tau}$ coincide. Moreover, for $\tau^* = \hat{\tau}$, $\mathscr{V} = \mathscr{U} \vee \mathscr{U}^{-1}$ and thus $(S, \mathscr{F}, \tau)$ and $(S, \mathscr{F}^*, \tau, \leq)$ are compatible whenever $\tau^* \gg \tau$.

EXAMPLES 3.3. To illustrate the above, we show that the set of all equivalence classes of random variables defined on a given probability space and equipped with partial order less than or equal almost surely is an OPM space. The probabilistic metric on this space induces convergence in probability.

Let (Ω, A, P) denote a probability space and denote by $[X]$ the set of all random variables defined on (Ω, A, P) which are equal to X almost surely. Suppose that S is the set of all such equivalence classes.

For each $p = [X]$, $q = [Y]$, define F_{pq} to be the distribution function of the random variable $Z = (X - Y).\ 1_{\{X \geq Y\}}$, where

$$\underset{\{X \geq Y\}}{1(w)} = \begin{cases} 1, & \text{if } X(w) \geq Y(w) \\ 0, & \text{otherwise,} \end{cases}$$

$w \in \Omega$. Since Z is a non-negative random variable, then $F_{pq} \varepsilon \Delta^+$. Clearly, $F_{pp} = \varepsilon_0$. Next, suppose that $F_{pq} = F_{qp} = \varepsilon_0$. Then $Z = 0$ almost surely and thus $X \leq Y$ almost surely. Similarly, $Y \leq X$ almost surely, and hence $p = q$.

Let $I = [0, 1]$ and define $W\colon I \times I \to I$ by $W(a, b) = (a + b - 1) \vee 0$ and let $\tau_W(F, G)(x) = \sup\{W(F(y), F(z)) \mid y, z \in I, y + z = x\}$, where $F, G \in \Delta^+$ and $x \geq 0$. The proof that $F_{pq} \geq \tau_W(F_{pr}, F_{rq})$ is given in Theorem 9.1.2 of [4] and thus $(S, \mathscr{F}, \tau_W)$ is a PQM space. Recall that $p = [X] \leq q = [Y]$ if and only if $F_{pq} = \varepsilon_0$. Again, this happens if and only if $X \leq Y$ almost surely, and thus, this is the partial order induced by the PQM space $(S, \mathscr{F}, \tau_W)$.

Let us characterize convergence in $(S, \hat{\sigma})$. Denote by $p_n = [X_n]$, $p = [X]$, and $Z_n = (X_n - X) \cdot 1_{\{X_n \geq X\}}$. Then, for $t > 0$, $F_{p_n p}(t) = P\{Z_n < t\} \to 1$ if and only

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if $P\{X_n - X < t\} \to 1$. Similarly, $F_{pp_n}(t) \to 1$ if and only if $P\{X - X_n < t\} \to 1$, and thus, $p_n \to p$ in $(S, \hat{\sigma})$ if and only if $X_n \to X$ in probability.

Let us conclude this section with another probabilistic quasi-metric which induces the same order on S above. Define, for each $p = [X], q = [Y], G_{pq}$ to be the distribution function of the random variable $1_{\{X>Y\}}$. It can by shown that $(S, \mathscr{G}, \tau_W)$ is a PQM space. Moreover, note that $p \leq q$ if and only if $P\{1_{\{X>Y\}} < t\} = 1$ for each $t > 0$. Hence $p \leq q$ if and only if $X \leq Y$ almost surely, which agrees with the order induced in the example above. However, if $p_n = [X_n], p = [X]$, then $p_n \to p$ in $(S, \hat{\sigma})$ if and only if $P\{1_{\{X_n>X\}} < t\} \to 1$ and $P\{1_{\{X>X_n\}} < t\} \to 1$, for each $t > 0$. This implies that $p_n \to p$ in $(S, \hat{\sigma})$ if and only if $P\{X_n = X\} \to 1$. The topology here is strictly finer than that of the example above even though the partial orders agree.

4. Quasi-simple spaces

A quasi-simple space is a PQM space derived (as in [4, Section 8.4] from a quasi-metric δ on S. It turns out that in the definition of OPM space (Definition 3.1), there is no loss of generality if the PQM space $(S, \mathscr{F}, \tau')$ is assumed to be a quasi-simple space.

Let $\tau_{\mathscr{M}}$ be the triangle function defined as follows: $\tau_{\mathscr{M}}(F, G)(x) = \sup\{F(y) \wedge G(z)\colon\ y \geq 0, z \geq 0, y + z = x\}$. A *quasi-metric* δ on S is a function from $S \times S$ into $[0, \infty)$, which is required to satisfy the following proper;ties: (1) $\delta(p,p) = 0$ for each $p \in S$; (2) if $\delta(p,q) = \delta(q,p) = 0$, then $p = q$; and (3) $\delta(p,q) \leq \delta(p,r) + \delta(r,q)$ for each p, q, r in S. Given a quasi-metric space (S, δ) and an element G of Δ^+ which is distinct from ε_0; define $\mathscr{F}_\delta(p,q)(x) = G(x/\delta(p,q))$ for each $p, q, \varepsilon S$ and $x \geq 0$. A PQM space $(S, \mathscr{F}, \tau)$ is said to be a *quasi-simple space* if $\tau = \tau_M$ and there is a quasi-metric δ on S such that $\mathscr{F} = \mathscr{F}_\delta$. It is shown in [4, Theorem 8.4.2] that $\mathscr{F}_\delta$ is a probabilistic metric relative to the triangle function τ_M whenever δ is a metric on S. This theorem generalizes without difficulty to quasi-metrics. It should be noted that $\mathscr{F}_\delta$ depends on G as well as δ.

PROPOSITION 4.1. *Let (S, δ) be a quasi-metric space and let G be an element of Δ^+ which is distinct from ε_0. Then $(S, \mathscr{F}_\delta, \tau_M)$ is a PQM space.*

With each quasi-metric δ on S, we associate the metric $\hat{\delta}$ defined by $\hat{\delta}(p,q) = \delta(p,q) \vee \delta(q,p)$ for all p, q in S; we also associate the quasi-uniformity $\mathscr{V}_\delta$ generated by sets of the form $V(t) = \{(p,q)\colon\ \delta(p,q) < t\}$ for all $t > 0$, and the uniformity $\hat{\mathscr{V}_\delta} = \mathscr{V}_\delta \vee \mathscr{V}_\delta^{-1}$ determined by $\hat{\delta}$. For a given $G \in \Delta^+$, the simple spaces generated by δ and $\hat{\delta}$ are related in the expected way.

PROPOSITION 4.2. *Let δ be a quasi-metric on S and let $G \in \Delta^+$ be such that $G \neq \varepsilon_0$; let $\mathscr{F}(p,q)(x) = G(x/\delta(p,q))$ for all $x \in [0,\infty]$. Then $(S, \hat{\mathscr{F}}, \tau_{\mathscr{M}})$ is the simple space determined by G and $\hat{\delta}$; in other words, $\hat{\mathscr{F}}(p,q)(x) = G(x/\hat{\delta}(p,q))$, for all $x \in [0,\infty]$.*

PROOF. Note that $\hat{F}_{pq}(x) = F_{pq}(x) \wedge F_{qp}(x) = G(x/\delta(p,q)) \wedge G(x/\delta(q,p)) = G(x/\hat{\delta}(p,q))$, for all $x \in [0,\infty]$.

It is also true that the quasi-uniformity determined by a quasi-metric coincides with the strong quasi-uniformity determined by the quasi-simple space arising from the same quasi-metric.

PROPOSITION 4.3. *Let δ, G, and $\mathscr{F}$ be as specified in Proposition 4.2. Assume that G is an element in Δ^+ which is distinct from ε_0 and such that $\lim_{y\to\infty} G(y) = 1$, and let $\mathscr{U}$ be the strong quasi-uniformity for $(S, \mathscr{F}, \tau_{\mathscr{M}})$. Then $\mathscr{U} = \mathscr{V}_\delta$.*

PROOF. Let $U(t) = \{(p,q)\colon F_{pq} \in N_{\varepsilon_0}(t)\}$ be a basic set for $\mathscr{U}$, and let $s > 0$ be chosen so that $G(t/s) > 1 - t$; this choice for s is possible because $\lim_{y\to\infty} G(y) = 1$. Then if $(p,q) \in V(s) = \{(p,q)\colon \delta(p,q) < s\}$, it follows that $F_{pq}(t) = G(t/\delta(p,q)) \geq G(t/s) > 1 - t$, and so $(p,q) \in U(t)$. Thus, $\mathscr{U} \subseteq \mathscr{V}$.

Conversely, let s be given, where $0 < s < 1$. Since $G \neq \varepsilon_0$, we can assume without loss of generality that $G(0+) < 1 - s$. Denote $G'(1-s) = \sup\{x\colon G(x) < 1 - s\}$. Note that $G'(1-s) > 0$, and let $t = sG'(1-s) \wedge s$. We will show that $U(t) \subseteq V(s)$. If $(p,q) \notin V(s)$, then $\delta(p,q) \geq s$, and so $F_{pq}(t) = G(t/\delta(p,q)) \leq G(sG'(1-s)/\delta(p,q)) \leq G(G'(1-s)) \leq 1 - s \leq 1 - t$. Thus, $(p,q) \notin U(t)$. It follows that $\mathscr{U} = \mathscr{V}$.

THEOREM 4.4. *Let $(S, \mathscr{G}, \tau)$ be a PM space with strong uniformity $\mathscr{W}$, and let $\leq$ be a partial order on S. Then $(S, \mathscr{G}, \tau, \leq)$ is OPM space if and only if $(S, \mathscr{W}, \leq)$ is a uniform ordered space.*

PROOF. If $(S, \mathscr{G}, \tau, \leq)$ is an OPM space compatible with the PQM space $(S, \mathscr{F}, \tau')$ and $\mathscr{U}$ the strong quasi-uniformity for $(S, \mathscr{F}, \tau')$, then $(S, \mathscr{W}, \leq)$ is the uniform ordered space determined by $\mathscr{U}$.

Conversely, assume $(S, \mathscr{W}, \leq)$ is a uniform ordered space, and let $\mathscr{U}$ be a quasi-uniformity on S, such that $\mathscr{W} = \mathscr{U} \vee \mathscr{U}^{-1}$ and $p \leq q$ if and only if $(p,q) \in U$, for all $U \in \mathscr{U}$. Since $\mathscr{W}$ has a countable base, we may assume that $\mathscr{U}$ has a countable base. It follows from [2, Theorem 8] that there is a quasi-metric δ on S, such that $\mathscr{U} = \mathscr{V}_\delta$ and $p \leq q$ if and only if $\delta(p,q) = 0$.

Let G be any element of Δ^+, such that $G \notin \varepsilon_0$ and $\lim_{y\to\infty} G(y) = 1$, and define $\mathscr{F} = \mathscr{F}_\delta$ as in Proposition 4.1; then $(S, \mathscr{F}, \tau_{\mathscr{M}})$ is a quasi-simple space.

By Proposition 4.3, the strong quasi-uniformity for $(S, \mathscr{F}, \tau_{\mathscr{M}})$ is $\mathscr{U} = \mathscr{V}_\delta$, and so $(S, \mathscr{F}, \tau_{\mathscr{M}})$ generates both the order and the strong uniformity for $(S, \mathscr{G}, \tau, \leq)$. In other words, $(S, \mathscr{G}, \tau, \leq)$ is an OPM space.

The next corollary follows immediately from the proof of Theorem 4.4.

COROLLARY 4.5. *Every OPM space is compatible with a quasi-simple space.*

A distance distribution function, abbreviated *ddf*, that is continuous and strictly increasing on $[0, \infty]$ is said to be a *strict ddf* [4, page 48]. Let (S, δ) be a metric space, T a strict t norm (see [4], pages 65, 66) and define $\tau_T(F, G)(z) = \sup\{T(F(x), G(y)) \mid x + y = z\}$. It is proved in [5, Theorem 2] that if $\alpha > 1$, there exists a strict *ddf* G such that $\mathscr{F}(p,q)(x) = G(x/(\delta(p,q))^\alpha)$ defines a probabilistic metric with respect to the triangle function τ_T. The verification of condition $(\mathrm{PM})_3$ did not require symmetry of δ. Hence, if (S, δ) is only a quasi-metric space, it follows that $\mathscr{F}$, defined above, is a PQM space with respect to the triangle function τ_T. In this case $(S, \mathscr{F}, \tau_T)$ is called an *α-quasi-simple space.*

It is straightforward to modify the proof of Proposition 4.3 to show that the strong uniformity of an α-quasi-simple space coincides with the quasi-uniformity determined by the quasi-metric. This fact, combined with the proof of Theorem 4.4, gives the following.

COROLLARY 4.6. *Every OPM space $(S, \mathscr{F}, \tau_T, \leq)$, where T is a strict τ-norm, is compatible with an α-quasi-simple space, for each $\alpha > 1$.*

A characterization as to when a uniform space with a partial order is in fact a uniform ordered space is given in [2, Theorem 10]. In particular, $(S, \mathscr{W}, \leq)$ is a uniform ordered space when it is a sup-lattice such that the map $(p,q) \to p \vee q$ is uniformly continuous [2, Proposition 11].

5. Completions of OPM spaces

A completion theory for PM spaces has been developed by Sherwood (see [3], [4]). Completing an OPM space involves extending the order of the original space to its Sherwood completion, and this requires completing a compatible PQM space.

DEFINITION 5.1. Let $(S, \mathscr{G}, \tau, \leq)$ be an OPM space. Then $(S^*, \mathscr{G}^*, \tau, \leq^*)$ is an OPM completion of $(S, \mathscr{G}, \tau, \leq)$ if $(S^*, \mathscr{G}^*, \tau)$ is a probabilistic metric completion of $(S, \mathscr{G}, \tau)$ (see Section 12.5 of [4]) and the embedding of S into S^* is an order isomorphism.

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We shall show that every PQM space has a PQM completion which induces an OPM completion of any compatible OPM space. Let $g: S_1 \times S_2 \to T$ be any function and let A_i be a subset of $S_i, i = 1, 2$. Then $g(A_1 \times A_2)$ denotes the subset $\{g(a_1, a_2) \mid a_i \in A_i\}$ and, moreover, if Φ_i is a filter on S_i, then $g(\Phi_1 \times \Phi_2)$ denotes the filter $\{A \mid g(A_1 \times A_2) \subseteq A \subseteq T$, for some $A_i \in \Phi_i\}$, $i = 1, 2$. Recall that a filter converges to an element in a topological space provided it contains each neighborhood of this element.

Let $(S, \mathscr{F}, \tau)$ be a PQM space. A filter Φ on S is defined to be a *$\mathscr{F}$-Cauchy filter* if and only if $\mathscr{F}(\Phi \times \Phi) \to \varepsilon_0$ in (Δ^+, d_L). Equivalently, Φ is a $\mathscr{F}$-Cauchy filter if and only if $\mathscr{U} \subseteq \Phi \times \Phi$, where $\mathscr{U}$ is the strong quasi-uniformity for $\mathscr{F}$. Note that the statements $\mathscr{U} \subseteq \Phi \times \Phi$, $\mathscr{U}^{-1} \subseteq \Phi \times \Phi$, and $\hat{\mathscr{U}} \subseteq \Phi \times \Phi$ are all equivalent. Thus there is no difference between $\mathscr{F}$-Cauchy filters, $\mathscr{F}'$-Cauchy filters, and $\hat{\mathscr{F}}$-Cauchy filters. Furthermore, if $(S, \mathscr{G}, \tau', \leq)$ is any OPM space with which $(S, \mathscr{F}, \tau)$ is compatible, then the $\mathscr{F}$-Cauchy filters are the same as the $\mathscr{G}$-Cauchy filters since $\hat{\mathscr{U}}$ is the strong uniformity for $\mathscr{G}$. For simplicity, we shall henceforth refer to the "$\mathscr{F}$-Cauchy filters" simply as "Cauchy filters."

Two Cauchy filters Φ and Ψ are *equivalent* if and only if $\Phi \cap \Psi$ is also a Cauchy filter. Let $[\Phi]$ be the equivalence class of all Cauchy filters equivalent to Φ, and let $S^* = \{[\Phi] : \Phi \text{ a Cauchy filter}\}$. For $p \in S$, let $\dot{p}$ denote the fixed ultrafilter generated by $\{p\}$. The canonical map $j: S \to S^*$ is given by $j(p) = [\dot{p}]$. Before defining a probabilistic quasi-metric $\mathscr{F}^*$ on S^*, we need the following lemma.

LEMMA 5.2. *Let $(S, \mathscr{F}, \tau)$ be a PQM space. If Φ, Ψ are Cauchy filters, then $\mathscr{F}(\Phi \times \Psi)$ is a Cauchy filter relative to (Δ^+, d_L).*

PROOF. It must be shown that $d_L[\mathscr{F}(\Phi \times \Psi) \times \mathscr{F}(\Phi \times \Psi)] \to 0$ in R. Let $\delta > 0$ be given. By [4, Lemma 12.2.1], there exists $\eta > 0$ such that, if $F, G, H \in \Delta^+$, with $F \geq \tau(H, G)$, $G \geq \tau(H, F)$, and $d_L(H, \varepsilon_0) < \eta$, then $d_L(F, G) < \delta$.

Since $\mathscr{F}(\Phi \times \Phi) \to \varepsilon_0$, $\mathscr{F}(\Psi \times \Psi) \to \varepsilon_0$, and τ is continuous at $(\varepsilon_0, \varepsilon_0)$, we can choose $A \in \Phi$, $B \in \Psi$, such that $\tau[\hat{\mathscr{F}}(A \times A) \times \hat{\mathscr{F}}(B \times B)] \subseteq N_{\varepsilon_0}(\eta)$. Let $p, p' \in A$ and $q, q' \in B$. Then

$$\begin{aligned} F_{p'q'} &\geq \tau(F_{p'q}, F_{qq'}) \geq \tau[\tau(F_{p'p}, F_{pq}), F_{qq'}] = \tau[\tau(F_{p'p}, F_{qq'}), F_{pq}] \\ &\geq \tau[\tau(F_{p'p} \wedge F_{pp'}, F_{q'q} \wedge F_{qq'}), F_{qp}]. \end{aligned}$$

Similarly, $F_{pq} \geq \tau[\tau(F_{pp'} \wedge F_{p'p}, F_{qq'} \wedge F_{q'q}), F_{p'q'}]$. Since $\tau[\hat{\mathscr{F}}(A \times A) \times \hat{\mathscr{F}}(B \times B)] \subseteq N_{\varepsilon_0}(\eta)$, it follows that $\tau(F_{pp'} \wedge F_{p'p}, F_{qq'} \wedge F_{q'q}) \in N_{\varepsilon_0}(\eta)$. Let $F = F_{p'q'}$, $G = F_{pq}$, $H = \tau(F_{pp'} \wedge F_{p'p}, F_{qq'} \wedge F_{q'q})$; then by [4, Lemma 12.2.1] $d_L\ (F_{p'q'}, F_{pq}) < \delta$. This implies that $d_L(F(A \times B)) \times \mathscr{F}(A \times B)) \subseteq [0, \delta]$, and thus $\mathscr{F}(\Phi \times \Psi)$ is a Cauchy filter on (Δ^+, d_L).

Returning to the construction of a PQM completion of a PQM space $(S, \mathscr{F}, \tau)$, we define $\mathscr{F}^*: S^* \times S^* \to \Delta^+$ as follows: $\mathscr{F}^*([\Phi], [\Psi]) = \lim \mathscr{F}(\Phi \times \Psi)$ in

(Δ^+, d_L). The preceding lemma implies that this limit exists for any Cauchy filters in the given equivalence classes; furthermore, this limit is well defined because each equivalence class contains a smallest (coarsest) filter. Note that $\mathscr{F}^*([\dot{p}],[\dot{q}]) = \mathscr{F}(p,q)$. It will be convenient to write $F^*_{[\Phi][\Psi]}$ in place of $\mathscr{F}^*([\Phi],[\Psi])$.

PROPOSITION 5.3. *If $(S, \mathscr{F}, \tau)$ is a PQM space, then $(S^*, \mathscr{F}^*, \tau)$ is also a PQM space.*

PROOF. All parts are straightforward to verify except the triangle inequality. Let $[\Phi], [\Psi], [\Gamma]$ belong to S^*; we shall show that $F^*_{[\Phi][\Psi]} \geq \tau(F^*_{[\Phi][\Gamma]}, F^*_{[\Gamma][\Psi]})$. It is sufficient to show that $F^*_{[\Phi][\Psi]}(x) \geq \tau(F^*_{[\Phi][\Gamma]}, F^*_{[\Gamma][\Psi]})(x)$ when x is a point of continuity of $F^*_{[\Phi][\Psi]}$ and $\tau(F^*_{[\Phi][\Gamma]}, F^*_{[\Gamma][\Psi]})$. Since we are dealing with Cauchy filters for a metrizable uniform space $(S, \hat{\mathscr{U}})$, we can formulate the argument in terms of sequences. Let (p_n) be a sequence in S, such that $\langle p_n \rangle \in [\Phi]$, where $\langle p_n \rangle$ is the filter generated by (p_n); similarly, let $\langle q_n \rangle \in [\Psi]$, and $\langle r_n \rangle \in [\Gamma]$. Then $F_{p_n q_n} \geq \tau(F_{p_n r_n}, F_{r_n q_n})$, and so $F^*_{[\Phi][\Psi]}(x) \geq \tau(F^*_{[\Phi][\Gamma]}, F^*_{[\Gamma][\Psi]})(x)$. Thus, $(S^*, \mathscr{F}^*, \tau)$ is a PQM space.

Since, as we have noted, $\mathscr{F}$ and $\hat{\mathscr{F}} = \mathscr{F} \wedge \mathscr{F}'$ have the same Cauchy filters, the completion sets for $(S, \mathscr{F}, \tau)$ and $(S, \hat{\mathscr{F}}, \tau)$ are the same set S^* described above. Note that Lemma 5.2 is valid for $(S, \hat{\mathscr{F}}, \tau)$ as well as for $(S, \mathscr{F}, \tau)$; indeed, for $(S, \hat{\mathscr{F}}, \tau)$ the conclusion follows by [4, Theorem 12.2.2]. If the construction which was applied to $\mathscr{F}$ in order to obtain $\mathscr{F}^*$ is applied instead to $\hat{\mathscr{F}}$, we obtain a PM space $(S^*, (\hat{\mathscr{F}})^*, \tau)$, which is indeed the Sherwood completion of $(S, \hat{\mathscr{F}}, \tau)$.

PROPOSITION 5.4. *For any PQM space $(S, \mathscr{F}, \tau)$, $(\hat{\mathscr{F}})^* = (\hat{\mathscr{F}^*})$.*

PROOF. We must show that, for arbitrary $[\Phi], [\Psi] \in S^*$, $(\hat{F})^*_{[\Phi][\Psi]} = F^*_{[\Phi][\Psi]} \wedge F^*_{[\Psi][\Phi]}$. The verification is straightforward using $\langle p_n \rangle \in [\Phi]$ and $\langle q_n \rangle \in [\Psi]$.

COROLLARY 5.5. *If $(S, \mathscr{F}, \tau)$ is a PQM space, then $(S^*, \mathscr{F}^*, \tau)$ is a PQM completion relative to the embedding map $j: S \to S^*$.*

PROOF. The map j is clearly an isometry (see paragraph preceding Proposition 5.3) of $(S, \mathscr{F}, \tau)$ into the PQM space $(S^*, \mathscr{F}^*, \tau)$. Since $(\hat{\mathscr{F}})^*$ is complete, it follows by Proposition 5.4 that $(\hat{\mathscr{F}^*})$ is complete. Since $(\hat{\mathscr{F}^*})$ and $\mathscr{F}^*$ have the same Cauchy filters and the strong topology for $\mathscr{F}^*$ is coarser than that for $(\hat{\mathscr{F}^*})$, it follows that $(S^*, \mathscr{F}^*, \tau)$ is also complete.

THEOREM 5.6. *Every OPM space $(S, \mathscr{G}, \tau, \leq)$ has OPM completion.*

PROOF. Let $(S, \mathscr{F}, \tau')$ be a PQM space compatible with $(S, \mathscr{G}, \tau, \leq)$. If $\mathscr{U}$ is the strong quasi-uniformity for $\mathscr{F}$, then $\hat{\mathscr{U}} = \mathscr{U} \vee \mathscr{U}^{-1}$ is the strong uniformity for $\mathscr{G}$. Let $(S^*, \mathscr{G}^*, \tau)$ be the PM completion of $(S, \mathscr{G}, \tau)$; by our previous remarks, we can assume that the same set S^* is the underlying set for the PQM completion $(S^*, \mathscr{F}^*, \tau')$ of $(S, \mathscr{F}, \tau')$. Furthermore, the strong uniformity for $(S^*, \mathscr{G}^*, \tau)$ is $(\hat{\mathscr{U}})^*$, where $(S^*, (\hat{\mathscr{U}})^*)$ is the uniform completion of $(S, \hat{\mathscr{U}})$.

Let $\mathscr{U}^*$ be the strong quasi-uniformity for $(S^*, \mathscr{F}^*, \tau')$. By Proposition 5.4, $(\hat{\mathscr{U}})^* = \mathscr{U}^* \vee (\mathscr{U}^*)^{-1}$. Thus $\mathscr{G}^*$ is compatible with $\mathscr{F}^*$. Let $\leq^*$ be the order on S^* determined by $\mathscr{F}^*$. Since $\mathscr{F}(p, q) = \varepsilon_0$ if and only if $\mathscr{F}^*([\dot{p}], [\dot{q}]) = \varepsilon_0$, the embedding map $j\colon (S, \mathscr{G}, \tau, \leq) \to (S^*, \mathscr{G}^*, \tau, \leq^*)$ is an order-preserving isometry. Thus $(S^*, \mathscr{G}^*, \tau, \leq^*)$ is an OPM completion of $(S, \mathscr{G}, \tau, \leq)$.

A function $f\colon (S, \mathscr{U}) \to (T, \mathscr{V})$ between two quasi-uniform spaces is called uniformly continuous when $f^{-1}(V) \in \mathscr{U}$ for each $V \in \mathscr{V}$. The following is an extension result to the completion space for an increasing, uniformly continuous function.

PROPOSITION 5.7. *Let f be an increasing function from the OPM space $(S, \mathscr{G}, \tau, \leq)$ into the complete OPM space $(T, \mathscr{K}, \tau_1, \leq_1)$. Suppose that $(S, \mathscr{F}, \tau')$ and $(T, \mathscr{H}, \tau_1')$ are compatible PQM spaces with corresponding strong quasi-uniformities $\mathscr{U}$ and $\mathscr{V}$, respectively. If $f\colon (S, \mathscr{U}) \to (T, \mathscr{V})$ is uniformly continuous, then the uniformly continuous extension to the completion $(S^*, \mathscr{G}^*, \tau^*, \leq^*)$ is also increasing.*

PROOF. Suppose that $[\Phi] \leq^* [\Psi]$; then $F^*_{[\Phi][\Psi]} = \lim \mathscr{F}(\Phi \times \Psi) = \varepsilon_0$ and this implies that $\mathscr{U} \subseteq \Phi \times \Psi$. Since f is uniformly continuous, $\mathscr{V} \subseteq f\Phi \times f\Psi$ and thus $\mathscr{H}(\lim f\Phi, \lim f\Psi) = \lim \mathscr{H}(f\Phi \times f\Psi) = \varepsilon_0$. It follows that the uniformly continuous extension of f to $(S^*, \mathscr{G}^*, \tau^*, \leq^*)$ is increasing.

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