THE DEFINITION OF THE TERRESTRIAL COORDINATE FRAME BY LONG BASELINE INTERFEROMETRY

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ABSTRACT

This paper examines the question of the definition of the celestial and terrestrial coordinate frames by the technique of long baseline interferometry. It demonstrates how the celestial coordinate frame may be usefully defined in terms of basis 1-forms associated with the advancing phase fronts of the radiation fields from compact radio sources using only interferometer observables. The paper then proceeds to show how the terrestrial coordinate frame could be usefully defined, incorporating fully the effects of plate tectonics and secular motion of the observatories, by an application of the theory of continuum mechanics to interferometer observations.

If we consider a long baseline interferometer with baseline $\vec{B}$ observing radiation at angular frequency ω_o emanating from a source which lies in the direction of the unit vector $\hat{s}$, then the delay observable τ is a scalar quantity which can be expressed as $\tau = \frac{1}{c}\,\hat{s}\cdot\vec{B}$ where c is the velocity of light in m.sec^{-1} (Thomas 1972, Cannon 1978). The phase Φ of the interferometer is related to the delay τ by $\Phi = \omega_o\tau$. We may also express the interferometer phase as $\Phi = \nabla\phi\cdot\vec{B}$ where $\nabla\phi$ is the gradient of the phase ϕ of the radiation field from the source which lies in the direction $\hat{s}$. Associated with the gradient vector $\nabla\phi$ there is a 1-form $\tilde{\omega}$ such that the surfaces of the 1-form $\tilde{\omega}$ correspond physically to the constant phase fronts of the radiation from the source $\hat{s}$. This permits us to express the interferometer phase Φ and delay τ in terms of the 1-form $\tilde{\omega}$ as $\Phi = \langle\tilde{\omega},\vec{B}\rangle$ and $\tau = \frac{1}{\omega^o}\langle\tilde{\omega},\vec{B}\rangle$.

We may now introduce a space-fixed coordinate frame by choosing three arbitrary radio sources lying in the directions of the unit vectors $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ and defining these unit vectors as the basis vectors of the space fixed frame. In general this space fixed coordinate frame will not be orthogonal and we shall have

$$\hat{S}_i\cdot\hat{S}_j = G_{ij}$$

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where $G_{ij} = \cos \widehat{S_i S_j}$ are the covariant components of the metric tensor of the space fixed frame and where $\widehat{S_i S_j}$ indicates the angular distance between the radio sources lying in the directions $\hat{S}_i$ and $\hat{S}_j$. The basis vectors $\hat{S}_1$, $\hat{S}_2$, and $\hat{S}_3$ have associated with them a covariant set of basis 1-forms $\tilde{\Omega}_1$, $\tilde{\Omega}_2$, and $\tilde{\Omega}_3$ defined such that

$$\langle \tilde{\Omega}_i , \hat{S}_j \rangle = G_{ij} .$$

The phase $\Phi(i)$ of the interferometer which is measured when observing the radio sources in the directions $\hat{S}_i$ is a direct measure of the space fixed covariant components B_i of the baseline vector $\vec{B}$. This is seen as follows:

$$\Phi(i) = \langle \tilde{\Omega}_i , \vec{B} \rangle = \langle \tilde{\Omega}_i , B^1 \hat{S}_1 + B^2 \hat{S}_2 + B^3 \hat{S}_3 \rangle$$
$$= \langle \hat{\Omega}_i \cdot \hat{S}_1 \rangle B^1 + \langle \tilde{\Omega}_i , \hat{S}_2 \rangle B^2 + \langle \tilde{\Omega}_i , \hat{S}_3 \rangle B^3 = G_{ij} B^j = B_i .$$

The familiar contravariant components B^i follow from $B^i = G^{ij} B_j$ where G^{ij} is defined by $G^{ik} G_{kj} = \delta^i_j$.

This approach is ideally suited to geodetic applications of long baseline interferometry for it makes no appeal to traditional celestial or terrestrial coordinate frames and depends entirely on interferometer observables including the angles $S_i S_j$ which define the elements of G_{ij} and its matrix inverse G^{ij}.

In the case where the interferometer is not observing one of the sources $\hat{S}_1$, $\hat{S}_2$, or $\hat{S}_3$ which define the basis vectors of the space fixed frame but is instead observing some arbitrary radio source which is lying in the direction $\hat{s}$, we introduce a 1-form $\tilde{\omega}$ defined such that $\langle \tilde{\omega}, \hat{s} \rangle = 1$. The unit vector $\hat{s}$ can be expressed as a linear combination of the basis vectors, $\hat{s} = C^k \hat{S}_k$, where $C^k = \cos \widehat{s\, S_k}$ and the 1-form $\tilde{\omega}$ can be expressed as a linear combination of the basis 1-forms $\tilde{\omega} = A^k \tilde{\Omega}_k$.

The phase of the interferometer while observing this source is $\Phi(\hat{s}) = \langle \tilde{\omega}, \vec{B} \rangle$. This reduces as follows:

$$\Phi(\hat{s}) = \langle \tilde{\omega}, \vec{B} \rangle = \langle A^k \tilde{\Omega}_k , B^j \hat{S}_j \rangle = A^k B^j \langle \tilde{\Omega}_k , \hat{S}_j \rangle$$
$$= A^k B^j G_{kj} = A^k B_k .$$

To determine the space fixed covariant components B_k of the baseline vector requires observations on three sources $\hat{s}_i$ and the solution of the equations $\Phi(\hat{s}_i) = A^k_i B_k$, $i = 1,2,3$. The values of the elements A^k_j follow from the property that $\langle \tilde{\omega}_j , \hat{s}_j \rangle = 1$. This reduces as follows:

$$\langle \tilde{\omega}_j , \hat{s}_j \rangle = \langle A^k_j \tilde{\Omega}_k , C^\ell_j \hat{S}_\ell \rangle = A^k_j C^\ell_j \langle \tilde{\Omega}_k , \hat{S}_\ell \rangle$$
$$= A^k_j C^\ell_j G_{k\ell} = 1 .$$

From this it follows that

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$$A_j^k = [\cos(\widehat{s_j S_1}) G_{k1} + \cos(\widehat{s_j S_2}) G_{k2} + \cos(\widehat{s_j S_3}) G_{k3}]^{-1} .$$

Once again the procedure is independent of any traditional celestial or terrestrial coordinate frame and depends entirely on interferometer observables namely the angles $\widehat{S_k S_\ell}$ which constitute the elements of $G_{k\ell}$ as well as the angles $\widehat{s_j S_\ell}$ appearing in the above formula.

The rotating body fixed frame of the earth has traditionally been spanned by an orthogonal set of body fixed basis vectors $\hat{e}_i^o$ which are defined by the mean axis of figure of the earth, the mean equator of figure of the earth, and the mean prime meridian of longitude or the "mean observatory". A space fixed coordinate frame has traditionally been spanned by an orthogonal set of space fixed basis vectors $\hat{E}_i$ defined dynamically (inertially) by the mean pole of the ecliptic, the mean plane of the ecliptic and the mean equator of a given epoch. The time dependent rotation of the set $\hat{e}_i$ relative to the set $\hat{E}_j$ can be expressed as $\hat{e}_i = P_{ij}(t)\ N_{jk}(t)\ S_{k\ell}(t)\ W_{\ell m}(t)\ \hat{E}_m$ when the orthogonal matrices $P_{ij}, N_{ij}, S_{ij}, W_{ij}$ represent the effects of earth precession, nutation, rotation (spin) and polar motion (wobble) respectively. (Woolard & Clemence 1966, Mueller 1969).

In so far as the earth is rigid and of a known figure the elements of all matrices can be predicted in advance on the basis of known astronomical forcing functions. The finite strength of the earth and the interaction between the earth's solid and fluid portions has complicated this transformation somewhat (Rochester 1973) but its continued use in the present day implicitly carries the assumption that an observer who remains locally fixed relative to the material of the solid earth will, on average, also remain globally fixed relative to the material of the solid earth.

The discovery that an observer who is fixed locally relative to the material of the solid earth will not, on average, remain fixed globally relative to the material of the solid earth but will, according to the theory of plate tectonics, exhibit secular motion relative to the global distribution of earth material at rates varying from 1 to 10 cm yr^{-1} has rendered this assumption invalid. This fact together with the development of modern space techniques of geodesy such as long baseline interferometry, which are expected to yield intercontinental baseline measurements with precisions of a few centimeters, have made it necessary to re-examine the question of the definition of the body fixed frame and to develop rigorous operational definitions and computational procedures which are capable of incorporating secular deformability and continuing fracture of the earth.

In our treatment we shall be guided by the well known procedures of continuum mechanics (cf. Prager 1973) in which the deformation field imposed on a continuous medium is described by a differentiable vector function of position $\vec{u}(\vec{r})$. The components of the relative displacement $\vec{\Delta}$ of two mass elements at positions $\vec{r}_1$, $\vec{r}_2$ separated, before deformation,

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by the infinitesimal separation vector $\vec{dx}$ is given by

$$\Delta_i = e_{ij}\, dx_j + \Omega_{ij}\, dx_j$$

where

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] \quad \text{and} \quad \Omega_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

are the strain tensor and rotation tensor respectively. The strain tensor has the property that the variation in the squared distance $|\vec{dx}|^2$ between the mass elements accompanying deformation is given by $\delta|\vec{dx}|^2 = 2e_{ij}dx_i dx_j$. The rotation tensor has the property that the components Ω_i of rigid body rotation $\vec{\Omega}$ imparted to the separation vector dx by the deformation field are given by $\Omega_i = -\frac{1}{2}\varepsilon_{ijk}\Omega_{jk}$ where ε_{ijk} is the alternating tensor.

We proceed by considering a long baseline interferometer baseline $\vec{B}(t) = b_1(t)\hat{e}_1^o + b_2(t)\hat{e}_2^o + b_3(t)\hat{e}_3^o$. The procedure for determining the traditional body fixed components $b_i(t)$ involves: (i) a determination of the contravariant space fixed components $\vec{B}(t) = B^1(t)\hat{S}_1 + B^2(t)\hat{S}_2 + B^3(t)\hat{S}_3$ given with respect to the space fixed basis vectors $\hat{S}_i(i=1,2,3)$, (ii) a transformation using the celestial coordinates (right ascension and declination) of the radio sources $\hat{s}_i(i=1,2,3)$ to express the baseline components in terms of the space fixed basis vectors $\hat{E}_i(i=1,2,3)$, or $\vec{B}(t) = \beta_1(t)\hat{E}_1 + \beta_2(t)\hat{E}_2 + \beta_3(t)\hat{E}_3$, (iii) a transformation $b_i(t) = W^T_{ij}(t)S^T_{jk}(t)N^T_{k\ell}(t)P^T_{\ell m}(t)\beta_m(t)$ where the superscipt "T" denotes a matrix transpose.

Repeated measurements of the baseline $\vec{B}(t)$ at times ... t_{m-1}, t_m, t_{m+1} ... yield observatory relative displacement vectors $\vec{D}(t_m) = \vec{B}(t_m) - \vec{B}(t_{m-1})$ with body fixed components $d_i(t_m) = b_i(t_m) - b_i(t_{m-1})$. We introduce the quantities

$$C_{ij}(t_m) = \frac{d_i(t_m)b_j(t_m)}{b_\ell(t_m)b_\ell(t_m)}$$

to serve as analogs to the tensor $\frac{\partial u_i}{\partial x_j}$ of continuum mechanics. We may decompse C_{ij} into symmetric and antisymmetric parts as $C_{ij} = E_{ij} + \Theta_{ij}$ where $E_{ij} = \frac{1}{2}[C_{ij}+C_{ji}]$ and $\Theta_{ij} = \frac{1}{2}[C_{ij}-C_{ji}]$. It can be readily shown that the body fixed components of the relative displacement vector can be written as $d_i = E_{ij}b_j + \Theta_{ij}b_j$ which is the analog of the expression $\Delta_i = e_{ij}dx_j + \Omega_{ij}dx_j$ from the theory of continuum mechanics.

Further similarities to the theory of continuum mechanics follow. It may be shown that the variation $\delta|s^2|$ in the squared distance s^2 between the observatories which occurs as a result of the relative displacement $\vec{D}$ is given by $\delta|s^2| = 2E_{ij}b_i b_j$ which is the analog of the expression $\delta|\vec{dx}|^2 = 2e_{ij}dx_i dx_j$ from the theory of continuum mechanics. It may also be shown that the body fixed components of the rigid body rotation $\vec{\theta} = \theta_1\hat{e}_1^o + \theta_2\hat{e}_2^o + \theta_3\hat{e}_3^o$ imparted to the interferometer baseline by the relative displacement of the observatories D are given by $\theta_i = -\frac{1}{2}\varepsilon_{ijk}\theta_{jk}$ which is the analog of the expression $\Omega_i = -\frac{1}{2}\varepsilon_{ijk}\Omega_{jk}$ from the theory of continuum mechanics.

These procedures may be applied to the problem of the definition of the terrestrial frame in the presence of arbitrary earth deformation fields producing arbitrary relative motions of the observatories. We presuppose a global network of N nonredundant interferometer baselines $\vec{B}^\alpha$ (α = 1, 2, 3 ... N) and their measured strain tensor analogs E^α_{ij} and rotation tensor analogs Θ^α_{ij} α = 1, 2, 3 ... N. We may define a unique global strain tensor analog Γ_{ij} as being that which minimizes in a weighted least squares sense the total departure between the observed baseline length variations δs^{α^2} given by

$$\delta s^{\alpha^2} = 2E^\alpha_{ij}\, b^\alpha_i\, b^\alpha_j$$

and the predicted baseline length variations $\overline{\delta s}^{\alpha^2}$ given by

$$\overline{\delta s}^{\alpha^2} = 2\Gamma_{ij}\, b^\alpha_i\, b^\alpha_j\ .$$

We then choose to minimize P^2 where

$$P^2 = \sum_{\alpha=1}^{N} w^\alpha [\delta s^{\alpha^2} - \overline{\delta s}^{\alpha^2}\ ]^2$$

or

$$P^2 = 4\sum_{\alpha=1}^{N} w^\alpha [(E^\alpha_{ij} b^\alpha_i b^\alpha_j)^2 - 2E_{ij}\Gamma_{\ell m} b^\alpha_i b^\alpha_j b^\alpha_\ell b^\alpha_m + (\Gamma_{\ell m} b^\alpha_\ell b^\alpha_m)^2]$$

where w^α are appropriate weights. A choice of $\Gamma_{\ell m}$ which minimizes P^2 is given by

$$\frac{\partial P^2}{\partial \Gamma_{pq}} = 0$$

which leads to

$$4\sum_{\alpha=1}^{N} w^\alpha \left[-2E^\alpha_{ij} + 2\Gamma_{ij}\right] b^\alpha_i b^\alpha_j b^\alpha_p b^\alpha_q = 0.$$

Since this condition should be fulfilled independently of the particular choice of baseline network we conclude that we shall in general require

$$\Gamma_{ij} = \frac{1}{\sum\limits_{\alpha=1}^{N} w^\alpha} \sum_{\alpha=1}^{N} w^\alpha\ E^\alpha_{ij}\ .$$

We may also define a unique global rotation tensor analog Λ_{ij} as being that which minimizes in a weighted least squares sense the vector norm of the net departures between the observed baseline rotation $\vec{\theta}^\alpha$ whose components are given by

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$$\Theta_i^\alpha = -\frac{1}{2}\varepsilon_{ijk}\,\Theta_{jk}^\alpha$$

and the predicted baseline rotation $\vec{\Lambda}$ whose components are given by

$$\Lambda_i = -\frac{1}{2}\varepsilon_{ijk}\,\Lambda_{jk}\ .$$

We then choose to minimize

$$Q^2 = \sum_{\alpha=1}^{N} w^\alpha |\vec{\Theta}^\alpha - \vec{\Lambda}|^2$$

or

$$Q^2 = \frac{1}{2}\sum_{\alpha=1}^{N} w^\alpha [\Theta_{\ell m}^{\alpha 2} - 2\Lambda_{\ell m}\,\Theta_{\ell m}^\alpha + \Lambda_{\ell m}^2]$$

where w are appropriate weights. A choice of Λ_{pq} which minimizes Q^2 is given by

$$\frac{\partial Q^2}{\partial \Lambda_{pq}} = 0$$

which leads to

$$\sum_{\alpha=1}^{N} w^\alpha [-2\Theta_{pq}^\alpha + 2\Lambda_{pq}] = 0$$

which has the solution

$$\Lambda_{ij} = \frac{1}{\sum_{\alpha=1}^{N} w^\alpha}\sum_{\alpha=1}^{N} w^\alpha\,\Theta_{ij}^\alpha .$$

The tensor analogs E_{ij}^α and Θ_{ij}^α for each interferometer baseline can be expressed as the sum of the global quantities Γ_{ij} and Λ_{ij} plus a "local" residual ε_{ij}^α and ω_{ij}^α respectively.

$$E_{ij}^\alpha = \Gamma_{ij} + \varepsilon_{ij}^\alpha$$

$$\Theta_{ij}^\alpha = \Lambda_{ij} + \omega_{ij}^\alpha$$

where, by definition, the weighted global means of the local residuals vanish

$$\frac{1}{\sum_{\alpha=1}^{N} w^\alpha}\sum_{\alpha=1}^{N} w^\alpha\,\varepsilon_{ij}^\alpha = 0$$

$$\frac{1}{\sum_{\alpha=1}^{N} w^\alpha}\sum_{\alpha=1}^{N} w^\alpha\,\omega_{ij}^\alpha = 0.$$

The "local" residuals $\varepsilon_{ij}^{\alpha}$, ω_{ij}^{α} $\alpha = 1, 2, 3 \ldots N$, given relative to the traditional body fixed basis vectors $\hat{e}_1^o$ $\hat{e}_2^o$ $\hat{e}_3^o$, represent strains and rotations of individual baselines relative to the network as a whole and will presumably contain information about regional tectonics including inter- and intraplate geologic processes.

The global quantities Γ_{ij} and Λ_{ij} represent global strain and global rigid rotation of the network as a whole relative to the traditional body fixed basis vectors $\hat{e}_1^o$ $\hat{e}_2^o$ $\hat{e}_3^o$. The global network of interferometers and the global quantities Γ_{ij}, Λ_{ij} can be used to redefine the body fixed coordinate frame. In general we may express the new body fixed basis vectors $\hat{e}_i$ in terms of the traditional body fixed basis vectors $\hat{e}_i^o$ by a transformation of the form

$$\hat{e}_i = [\delta_{ij} + \sigma_{ij} + \alpha_{ij}]\hat{e}_j^o$$

where σ_{ij} and α_{ij} are small quantities and where:

(i) $\sigma_{ij} = \sigma_{ji}$ is the symmetric part of the transformation,

(ii) $\alpha_{ij} = - \alpha_{ji}$ is the antisymmetric part of the transformation.

The symmetric and antisymmetric parts of the transformation are independent of each other and must be determined separately. In general the antisymmetric part of the transformation, given by α_{ij}, represents a rigid rotation of the basis vectors $\hat{e}_i$ relative to the basis vectors $\hat{e}_i^o$ while the symmetric part of the transformation, given by σ_{ij}, represents a deformation of the basis vectors $\hat{e}_i$ relative to the basis vectors $\hat{e}_i^o$.

We shall define the body fixed frame to be spanned by new body fixed basis vectors $\hat{e}_i$ such that if the material of the earth were subjected to a uniform strain Γ_{ij} and a uniform rotation Λ_{ij} relative to the basis vectors $\hat{e}_i^o$ then the body fixed coordinates of the mass elements of the earth would remain constant when referred to the basis vectors $\hat{e}_i$. Accomplishing this in the presence of the global rotation Λ_{ij} fixes the antisymmetric part of the transformation and accomplishing this in the presence of the global strain Γ_{ij} fixes the symmetric part of the transformation.

It is clear that the antisymmetric part of the transformation requires

$$\alpha_{ij} = \Lambda_{ij}$$

while it can be shown (cf. Brillouin, 1964, pp. 287 ff.) that the symmetric part of the transformation can be deduced from the requirement that to preserve the coordinates of the mass elements in the presence of the deformation Γ_{ij} we require a nonorthogonal set of basis vectors $\hat{e}_i$ which have the property that

$$\hat{e}_i \cdot \hat{e}_j = g_{ij}$$

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where

$$g_{ij} = \delta_{ij} + 2\Gamma_{ij}$$

is the metric tensor of the body fixed coordinate frame. This leads to the result that

$$\sigma_{ij} = \Gamma_{ij} .$$

This gives

$$\hat{e}_i = [\delta_{ij} + \Gamma_{ij} + \Lambda_{ij}]e_j^o$$

and

$$\hat{e}_i = [\delta_{ij} + \Gamma_{ij} + \Lambda_{ij}] P_{jk} N_{k\ell} S_{\ell m} W_{mn} \hat{E}_n$$

as the transformation from the dynamical space fixed frame to the body fixed frame of the interferometer network.

REFERENCES

Brillouin, L.: 1964, Tensors in Mechanics and Elasticity, Academic Press.

Cannon, W. H.: 1978, Geophys. J. R. astr. Soc., 53, 503.

Mueller, I. I.: 1969, Spherical and Practical Astronomy as Applied to Geodesy, F. Ungar Publ. Co.

Prager, W.: 1973, Introduction to Mechanics of Continua, Dover Publications.

Rochester, M. G.: 1973, E⊕S, Transactions of the American Geophysical Union, 54, 769.

Thomas, J. B.: 1972, An Analysis of Long Baseline Radio Interferometry, Deep Space Network Progress Report, Jet Propulsion Laboratory, Pasadena, California Technical Report, 32-1526, Vol. VII.

Woolard, E. W., and Clemence, G. M.: 1966, Spherical Astronomy, Academic Press.