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ABSTRACT

We show that a virtually residually finite rationally solvable (RFRS) group G of type $\text{FP}_n(\mathbb{Q})$ virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{Q})$ if and only if the first n ℓ^2 -Betti numbers of G vanish, that is, $b_p^{(2)}(G) = 0$ for $0 \leq p \leq n$. This confirms a conjecture of Kielak. We also offer a variant of this result over other fields, in particular in positive characteristic. As an application of the main result, we show that amenable virtually RFRS groups of type $\text{FP}(\mathbb{Q})$ are virtually Abelian. It then follows that if G is a virtually RFRS group of type $\text{FP}(\mathbb{Q})$ such that $\mathbb{Z}G$ is Noetherian, then G is virtually Abelian. This confirms a conjecture of Baer for the class of virtually RFRS groups of type $\text{FP}(\mathbb{Q})$, which includes (for instance) the class of virtually compact special groups.

1. Introduction

A group G is *algebraically fibred* (or simply *fibred*) if it admits a homomorphism onto \mathbb{Z} with a finitely generated kernel. The interest in algebraic fibrings arose from the study of 3-manifolds fibring over the circle. Using the long exact sequence of homotopy groups associated to a fibration, one sees that a surface bundle $M \rightarrow S^1$, where M is a compact 3-manifold, induces an algebraic fibration $\pi_1(M) \rightarrow \mathbb{Z}$. A celebrated theorem of Stallings [Sta62] establishes the converse: if G is isomorphic to the fundamental group of a closed, compact 3-manifold M , then an algebraic fibration $G \rightarrow \mathbb{Z}$ is induced by a surface bundle $M \rightarrow S^1$.

Recently, Kielak characterised the virtual algebraic fibring of residually finite rationally solvable (RFRS) groups in terms of the vanishing of the first ℓ^2 -Betti number. More precisely, he showed the following.

THEOREM 1.1 (Kielak [Kie20c]). *Let G be an infinite finitely generated virtually RFRS group. Then G virtually algebraically fibres if and only if $b_1^{(2)}(G) = 0$.*

Virtually RFRS groups arise naturally in geometric group theory; for example, subgroups of right-angled Artin groups and right-angled Coxeter groups are virtually RFRS and, in particular, special groups (in the sense of Haglund and Wise) are RFRS. Moreover, the RFRS property passes to subgroups and is preserved by taking products and free products of RFRS groups. Kielak's theorem is an algebraic analogue of the following theorem of Agol, which was a key step in confirming Thurston's virtually fibred conjecture.

THEOREM 1.2 (Agol [Ago08]). *Every compact and irreducible 3-manifold M with $\chi(M) = 0$ and nontrivial RFRS fundamental group admits a finite covering that fibres over the circle.*

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Since algebraic fibrings induce topological fibrings of 3-manifolds, Kielak’s theorem generalises the above theorem of Agol by removing the assumption that G is the fundamental group of a 3-manifold. Note that [Lüc02, Theorem 4.1] states that if M is a compact, irreducible 3-manifold with no S^2 boundary components, then $b_1^{(2)}(\pi_1(M)) = -\chi(M)$. Thus, we interpret the condition $b_1^{(2)}(G) = 0$ in Kielak’s theorem as the algebraic analogue of the condition $\chi(M) = 0$ in Agol’s theorem.

In light of Kielak’s theorem, it is natural to ask whether the vanishing of higher ℓ^2 -Betti numbers of a group G controls the higher finiteness properties of the kernel of the virtual fibration. Indeed, Kielak conjectured that a virtually RFRS group of type $\text{FP}_n(\mathbb{Q})$ virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{Q})$ if and only if $b_p^{(2)}(G)$ vanishes for all $p \leq n$ (see [Kie20b, Conjecture 8]). The main result of this paper confirms Kielak’s conjecture, and gives another characterisation of RFRS groups virtually fibring with kernel of type $\text{FP}_n(\mathbb{Q})$. We also note that the equivalence of parts (2) and (3) in the following theorem generalises [JZ21, Corollary 1.5], where Jaikin-Zapirain proved the $n = 1$ case.

THEOREM A. *Let G be a virtually RFRS group of type $\text{FP}_n(\mathbb{Q})$. Then the following are equivalent:*

- (1) $b_p^{(2)}(G) = 0$ for all $p \leq n$;
- (2) there is a finite-index subgroup $H \leq G$ and a surjection $\varphi: H \rightarrow \mathbb{Z}$ such that $\ker \varphi$ is of type $\text{FP}_n(\mathbb{Q})$;
- (3) there is a finite-index subgroup $H' \leq G$ and a surjection $\varphi': H' \rightarrow \mathbb{Z}$ such that $b_p(\ker \varphi') < \infty$ for all $p \leq n$.

It should be emphasised that if $\ker \varphi'$ has finite Betti numbers in dimensions $\leq n$, it is not necessarily the case that $\ker \varphi'$ is of type $\text{FP}_n(\mathbb{Q})$. We prove a more general theorem that treats algebraic fibring with kernels of type $\text{FP}_n(\mathbb{F})$ for any skew-field \mathbb{F} , from which we obtain Theorem A as a special case. Before stating the result, we give some background. If \mathbb{F} is a skew-field and G is a locally indicable group, then under certain conditions the group ring $\mathbb{F}G$ embeds into a skew-field $\mathcal{D}_{\mathbb{F}G}$ called the *Hughes-free division ring* of $\mathbb{F}G$ (see Definition 2.3). If it exists, $\mathcal{D}_{\mathbb{F}G}$ is unique up to $\mathbb{F}G$ -algebra isomorphism [Hug70] and the $\mathcal{D}_{\mathbb{F}G}$ -homology of G in dimension p is defined to be $H_p^{\mathcal{D}_{\mathbb{F}G}}(G) := H_p(G; \mathcal{D}_{\mathbb{F}G})$. The p th $\mathcal{D}_{\mathbb{F}G}$ -Betti number is defined to be

$$b_p^{\mathcal{D}_{\mathbb{F}G}}(G) := \dim_{\mathcal{D}_{\mathbb{F}G}} H_p^{\mathcal{D}_{\mathbb{F}G}}(G).$$

In [JZ21, Corollary 1.3], Jaikin-Zapirain proved that if G is finitely generated and RFRS, then $\mathcal{D}_{\mathbb{F}G}$ exists for any skew-field \mathbb{F} and, if $\mathbb{F} = \mathbb{Q}$, it is isomorphic to the Linnell skew-field of G . For the purposes of this paper, it will not be necessary to know how the Linnell skew-field is defined, but that it can be used to define the ℓ^2 -homology and ℓ^2 -Betti numbers of a group G (see Definition 6.9). Importantly for us, when G is finitely generated and RFRS, we have $b_p^{\mathcal{D}_{\mathbb{Q}G}}(G) = b_p^{(2)}(G)$ for all p . We prove the following theorem, which reduces to Theorem A in the case $\mathbb{F} = \mathbb{Q}$.

THEOREM B. *Let \mathbb{F} be a skew-field and let G be a virtually RFRS group of type $\text{FP}_n(\mathbb{F})$. Then the following are equivalent:*

- (1) $b_p^{\mathcal{D}_{\mathbb{F}G}} = 0$ for all $p \leq n$;
- (2) there is a finite-index subgroup $H \leq G$ and a surjection $\varphi: H \rightarrow \mathbb{Z}$ such that $\ker \varphi$ is of type $\text{FP}_n(\mathbb{F})$;

- (3) *there is a finite-index subgroup $H' \leq G$ and a surjection $\varphi' : H' \rightarrow \mathbb{Z}$ such that $b_p(\ker \varphi'; \mathbb{F}) < \infty$ for all $p \leq n$.*

We highlight the following corollary, which implies, in particular, that if \mathbb{F} and \mathbb{F}' are skew-fields of the same characteristic, then a RFRS group G is $\mathcal{D}_{\mathbb{F}G}$ -acyclic in dimensions $\leq n$ if and only if it is $\mathcal{D}_{\mathbb{F}'G}$ -acyclic in dimensions $\leq n$. Moreover, it also implies that if G is $\mathcal{D}_{\mathbb{F}_pG}$ acyclic in dimensions $\leq n$ for some prime p , then it is also ℓ^2 -acyclic in dimensions $\leq n$.

COROLLARY C (Corollary 6.15). *Let G be a virtually RFRS group and let $n \in \mathbb{N}$.*

- (1) *If \mathbb{F} and \mathbb{F}' are skew-fields of the same characteristic, then G virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F})$ if and only if it virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F}')$.*
- (2) *If p is a prime such that G virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F}_p)$, then it virtually fibres with kernel of type $\text{FP}_n(\mathbb{Q})$.*

The final section of the paper is devoted to some applications of the main theorems. An interesting question is to determine general conditions under which amenable groups are elementary amenable. There are many examples of amenable groups that are not elementary amenable, for instance, Grigorchuk’s group of intermediate growth, but it is not known whether there are examples of amenable groups of finite cohomological dimension that are not elementary amenable. Moreover, elementary amenable groups of finite cohomological dimension are virtually solvable by [Hil91, Lemma 2] and [HL92, Corollary 1]. This leads us to the following question, which first appeared in the work of Degrijse.

QUESTION 1.3 (Degrijse [Deg16]). *Are amenable groups of finite cohomological dimension over \mathbb{Z} virtually solvable?*

We obtain the following as an application of Theorem A, which provides evidence of a positive answer for virtually RFRS groups.

THEOREM D (Theorem 7.4). *If G is a virtually (amenable RFRS group of type $\text{FP}(\mathbb{Q})$), then G is virtually Abelian.*

We think of this result as being in the same vein as the well-known fact that nilpotent RFRS groups must be virtually Abelian (see, e.g., [KS20, Proposition 2.5]). As a corollary, we confirm the following conjecture of Baer for virtually RFRS groups of type $\text{FP}(\mathbb{Q})$. Note that this class of groups includes all virtually compact special groups.

CONJECTURE 1.4. *If G is a group such that the group ring $\mathbb{Z}G$ is Noetherian, then G is polycyclic-by-finite.*

Hall showed that polycyclic-by-finite groups have Noetherian group rings [Hal59, Theorem 4], but it is still unknown whether the converse holds. Some progress was made recently by Kropholler and Lorenzen, who showed that if RG is right Noetherian and R is a domain, then G is amenable and all of its subgroups are finitely generated [KL19, Corollary B]. This result provides evidence for the conjecture, as the only known amenable groups in which every subgroup is finitely generated are polycyclic-by-finite. We obtain the following as a consequence of Theorem D and Kielak’s appendix to [Bar19].

COROLLARY E (Corollary 7.6). *Let G be a virtually RFRS group of type $\text{FP}(\mathbb{Q})$ such that $\mathbb{Z}G$ is Noetherian. Then G is virtually Abelian.*

Finally, we mention that Llosa-Isenrich, Martelli and Py remarked in [LMP21] that an easy consequence of Theorem A, Agol’s theorem [Ago13], and the work of Agol and Bergeron, Haglund and Wise [BHW11] is the existence of hyperbolic groups containing type $\text{FP}_{n-1}(\mathbb{Q})$ subgroups that are not of type $\text{FP}_n(\mathbb{Q})$ for all $n \in \mathbb{N}$. We also remark that, by the same argument, such lattices in $\text{PO}(2n + 1, 1)$ algebraically fibre with kernel of type $\text{FP}(\mathbb{Q})$.

PROPOSITION 1.5 [LMP21, Proposition 19]. *Let $\Gamma \in \text{PO}(m, 1)$ be a hyperbolic arithmetic lattice of the simplest type. If $m = 2n$, then Γ virtually fibres with kernel of type $\text{FP}_{n-1}(\mathbb{Q})$ but not of type $\text{FP}_n(\mathbb{Q})$. If $m = 2n + 1$, then Γ virtually fibres with kernel of type $\text{FP}(\mathbb{Q})$.*

Our methods do not allow us to say anything about algebraic fibring with kernels of type $\text{FP}_n(\mathbb{F}_p)$ for p prime (nor about algebraic fibring with stronger finiteness properties of the kernel) since the $\mathcal{D}_{\mathbb{F}_p G}$ -Betti numbers of simplest type hyperbolic arithmetic lattices are not known to vanish. In a subsequent paper [LP24], Llosa-Isenrich and Py showed that there are hyperbolic arithmetic lattices of the simplest type in $\text{PU}(n, 1)$ that virtually fibre with kernel of type F_{n-1} but not of type $\text{FP}_n(\mathbb{Q})$, answering a question of Brady about the existence of subgroups of hyperbolic groups with exotic finiteness properties.

1.1 Structure of the paper

In § 2 we introduce some of the main tools and objects that will be used throughout the paper. In particular, we define finiteness properties of groups, Hughes-free division rings, RFRS groups and Ore localisation.

Section 3 recalls what we will need from the theory of valuations on free resolutions developed by Bieri and Renz in [BR88]. The results in [BR88] are stated for free resolutions over group rings with coefficients in \mathbb{Z} , however we will need them for coefficients in an arbitrary associative, unital ring R . There is no essential dependence on the ring, so our proofs are similar to those of Bieri and Renz after replacing \mathbb{Z} with R .

In § 4, we introduce the complex of horochains associated to a free resolution equipped with a valuation.

Section 5 begins with the definition of the higher Σ -invariants $\Sigma_R^n(G; M)$ for a group G , a unital, associative ring R , and an RG -module M . Again, these were introduced in the case $R = \mathbb{Z}$ in [BR88], and reduce to the usual Bieri–Neumann–Strebel (BNS) invariant when $n = 1$, $R = \mathbb{Z}$ and $M = \mathbb{Z}$ is the trivial $\mathbb{Z}G$ -module. The rest of the section is devoted to the proof of Theorem 5.3, which gives equivalent characterisations of the invariants $\Sigma_R^n(G; M)$. The most important characterisation for our purposes is the following: $[\chi] \in \Sigma_R^n(G; M)$ if and only if $\text{Tor}_i^{RG}(\widehat{RG}^\chi, M) = 0$ for all $i \leq n$, where \widehat{RG}^χ is the Novikov ring. When $n = 1$, this result is Sikorav’s theorem [Sik87] and is a key ingredient in Kielak’s proof of Theorem 1.1. The proof follows arguments given in [BR88] and Schweitzer’s appendix to [Bie07]. Theorem 5.3 is the main technical result that will be used in the proof of Theorem 6.7.

In § 6 we introduce $\mathcal{D}_{\mathbb{F}_p G}$ -homology and prove properties of $\mathcal{D}_{\mathbb{F}_p G}$ -Betti numbers analogous to those of ℓ^2 -Betti numbers. Namely, we prove that

$$[G : H] \cdot b_p^{\mathcal{D}_{\mathbb{F}_p G}}(G) = b_p^{\mathcal{D}_{\mathbb{F}_p H}}(H)$$

(Lemma 6.3) whenever $\mathcal{D}_{\mathbb{F}_p G}$ exists and H is a finite index subgroup of G . In Theorem 6.4, we show that if $b_p^{\mathcal{D}_{\mathbb{F}_p K}}(K) < \infty$ and G fits into a short exact sequence $1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$, then $b_p^{\mathcal{D}_{\mathbb{F}_p G}}(G) = 0$. This should be thought of as an analogue of [Lüc02, Theorem 7.2] for $\mathcal{D}_{\mathbb{F}_p G}$ -Betti numbers. We then prove the main result, Theorem 6.7, and obtain Theorem 6.11 as a special case. In Theorem 6.14, we show that virtually fibring with kernel of type

$\text{FP}_n(\mathbb{Q})$ is equivalent to having a virtual map to \mathbb{Z} whose kernel has finite Betti numbers in dimensions $\leq n$.

We conclude with the proofs of Theorem 7.4 and Corollary 7.6 in §7, and mention related work of Llosa-Isenrich, Martelli, and Py.

2. Preliminaries

Remark. In the following, all rings will be associative and unital with $1 \neq 0$.

2.1 Finiteness properties

DEFINITION 2.1 (Type FP_n). Let R be a ring and M be a left R -module. We say that M is of type FP_n , and write $M \in \text{FP}_n$, if M has a projective resolution

$$\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

by left R -modules, where P_j is finitely generated for $j \leq n$. If we want to specify the ring, we say that M is of type FP_n over R and write $M \in \text{FP}_n(R)$. If $P_j = 0$ for $j > n$, then we write $M \in \text{FP}(R)$.

A group G is of type FP_n over R if the trivial RG -module R is of type $\text{FP}_n(RG)$; in this case, we write $G \in \text{FP}_n(R)$. Similarly, if the trivial RG -module R is of type $\text{FP}(RG)$, then we write $G \in \text{FP}(R)$.

We will often use the fact that an R -module M is of type FP_n if and only if there is a free resolution $\cdots \rightarrow F_{n+1} \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$ with F_j finitely generated for $j \leq n$ (see [Bro94, Proposition VIII.4.3]). Note that the analogous fact does not hold for type FP .

The following definition will not be needed until §7.

DEFINITION 2.2 (cohomological dimension). A resolution $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of an R -module M has length n if $P_n \neq 0$ and $P_m = 0$ for $m > n$. A group G has cohomological dimension n over R if n is the shortest length of any projective resolution of the trivial RG -module R . In this case we will write $\text{cd}_R(G) = n$. If R has no finite-length projective resolution, then $\text{cd}_R(G) = \infty$.

Note that $G \in \text{FP}(R)$ implies that G has finite cohomological dimension over R , but not conversely.

2.2 Hughes-free division rings

We are following Jaikin-Zapirain’s exposition of this material; in particular, Definitions 2.3 and 2.4 are taken from [JZ21].

DEFINITION 2.3 (Hughes-free division rings). Let \mathbb{F} and \mathcal{D} be skew-fields, let G be a locally indicable group, and let $\varphi: \mathbb{F}G \rightarrow \mathcal{D}$ be a ring homomorphism. Then the pair (\mathcal{D}, φ) is Hughes-free if:

- (1) \mathcal{D} is the skew-field generated by $\varphi(\mathbb{F}G)$, i.e. if $\mathcal{E} \subseteq \mathcal{D}$ is a skew-field such that $\varphi(\mathbb{F}G) \subseteq \mathcal{E}$, then $\mathcal{E} = \mathcal{D}$;
- (2) for every nontrivial finitely generated subgroup $H \leq G$, every normal subgroup $N \triangleleft H$ such that $H/N \cong \mathbb{Z}$, and every set of elements $h_1, \dots, h_n \in H$ lying in pairwise distinct cosets of N , the sum

$$\langle \varphi(\mathbb{F}N) \rangle \cdot \varphi(h_1) + \cdots + \langle \varphi(\mathbb{F}N) \rangle \cdot \varphi(h_n)$$

is direct, where $\langle \varphi(\mathbb{F}N) \rangle$ is the sub-skew-field of \mathcal{D} generated by $\varphi(\mathbb{F}N)$.

Hughes showed that if such a pair (\mathcal{D}, φ) exists, then \mathcal{D} is unique up to $\mathbb{F}G$ -algebra isomorphism [Hug70]. Thus, we denote \mathcal{D} by $\mathcal{D}_{\mathbb{F}G}$. Let $H \leq G$ be a subgroup and suppose that $\mathcal{D}_{\mathbb{F}G}$ exists. Then $\mathcal{D}_{\mathbb{F}H}$ exists as well and is equal to $\langle \varphi(\mathbb{F}H) \rangle \subseteq \mathcal{D}_{\mathbb{F}G}$. Hence, we will view $\mathcal{D}_{\mathbb{F}H}$ as a subset of $\mathcal{D}_{\mathbb{F}G}$ whenever H is a subgroup of G . Moreover, Gräter showed that $\mathcal{D}_{\mathbb{F}G}$ is *strongly Hughes-free* whenever it exists [Grä20, Corollary 8.3], which is to say that condition (2) above can be replaced with the following:

- (2') for every nontrivial subgroup $H \leq G$, every normal subgroup $N \triangleleft H$, and every set of elements $h_1, \dots, h_n \in H$ lying in pairwise distinct cosets of N , the sum

$$\langle \varphi(\mathbb{F}N) \rangle \cdot \varphi(h_1) + \dots + \langle \varphi(\mathbb{F}N) \rangle \cdot \varphi(h_n)$$

is direct.

DEFINITION 2.4 (Crossed products). A ring R is *G-graded* if its underlying Abelian group is isomorphic to a direct sum $\bigoplus_{g \in G} R_g$ of Abelian groups R_g and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If R_g contains a unit for each $g \in G$, then we say that R is a *crossed product* of R_e and G , and we write $R = R_e * G$.

Strong Hughes-freeness implies the following useful properties, which are stated in [JZ21].

PROPOSITION 2.5. *Let G be a locally indicable group such that $\mathcal{D}_{\mathbb{F}G}$ exists, let φ be as in Definition 2.3 and let $N \triangleleft G$ be a normal subgroup. If R is the subring of $\mathcal{D}_{\mathbb{F}G}$ generated by $\mathcal{D}_{\mathbb{F}N}$ and $\varphi(\mathbb{F}G)$, then:*

- (1) $R \cong \mathcal{D}_{\mathbb{F}N} * (G/N)$;
- (2) if $[G : N] < \infty$, then $\mathcal{D}_{\mathbb{F}G} = R$.

Proof. Starting with property (1), let $\{t_i\}_{i \in I}$ be a transversal for N in G . We claim that every element of R can be written as a finite sum $\sum_i \alpha_i \varphi(t_i)$, where $\alpha_i \in \mathcal{D}_{\mathbb{F}N}$ for each $i \in I$. This will conclude the proof of property (1) by strong Hughes-freeness. Since every element of G is of the form nt_i for some $n \in N$ and $i \in I$, it is enough to show that $\varphi(t_i) a \varphi(t_i)^{-1} \in \mathcal{D}_{\mathbb{F}N}$ for every $i \in I$ and every $a \in \mathcal{D}_{\mathbb{F}N}$. Note that $\mathcal{D}_{\mathbb{F}N} = \bigcup_{j \geq 0} \mathcal{D}_j$ where $\mathcal{D}_0 = \varphi(\mathbb{F}N)$ and \mathcal{D}_{j+1} is generated by \mathcal{D}_j and the set $\{x^{-1} : x \in \mathcal{D}_j \setminus \{0\}\}$. Because N is normal in G , it is clear that $\varphi(\mathbb{F}N)$ is closed under conjugation by $\varphi(G)$. By induction, we see that each \mathcal{D}_j is closed under conjugation by $\varphi(G)$.

To prove property (2), we recall the fact that a finite-dimensional algebra over a skew-field with no zero-divisors is a skew-field. Hence, if $[G : N] < \infty$, then $R \cong \mathcal{D}_{\mathbb{F}N} * (G/N)$ is finite-dimensional over $\mathcal{D}_{\mathbb{F}N}$ and is therefore a skew-field. However, $\mathcal{D}_{\mathbb{F}G}$ is the smallest skew-field containing $\varphi(\mathbb{F}G)$, so $\mathcal{D}_{\mathbb{F}G} = R$. □

2.3 RFRS groups

RFRS groups were defined by Agol in [Ago08] in order to show that certain hyperbolic 3-manifolds virtually fibre over the circle. Let G be a group and let $G^{\text{ab}} := G/[G, G]$ be its Abelianisation. Since G^{ab} is Abelian, it is canonically a \mathbb{Z} -module, so we can form the tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} G^{\text{ab}}$, and there is a group homomorphism $G \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} G^{\text{ab}}$ sending $g \in G$ to $1 \otimes g[G, G]$.

DEFINITION 2.6. A group G is *RFRS* if:

- (1) there is a chain $G = G_0 \geq G_1 \geq G_2 \geq \dots$ of finite index normal subgroups of G such that $\bigcap_{i=0}^{\infty} G_i = \{1\}$;
- (2) $\ker(G_i \rightarrow \mathbb{Q} \otimes_{\mathbb{Z}} G_i^{\text{ab}}) \leq G_{i+1}$ for every $i \geq 0$.

The following fact that will be used in the proof of Theorem 6.7.

PROPOSITION 2.7. *Let G be a finitely generated RFRS group. Then:*

- (1) G is residually poly- \mathbb{Z} ;
- (2) G virtually retracts onto all of its finitely generated Abelian subgroups;
- (3) G is locally indicable.

Proof. Item (1) is proven in [JZ21, Proposition 4.4]. We now prove item (2). By [Min21, Proposition 1.5], it suffices to show that G virtually retracts onto its cyclic subgroups. Let $G = G_0 \geq G_1 \geq \dots$ be a RFRS chain and let $\langle a \rangle$ be a cyclic subgroup of G . Let i be the unique integer such that $a \in G_i \setminus G_{i+1}$. It follows that a is not in the kernel of the free Abelianisation map $G_i \rightarrow G_i^{\text{fab}}$. Let \bar{a} denote the image of a in G_i^{fab} . There is a finite index subgroup $H' \leq G_i^{\text{fab}}$ and a retraction $H' \rightarrow \langle \bar{a} \rangle$. Let $H \leq G_i$ be the preimage of H' . The composition $H \rightarrow H' \rightarrow \langle \bar{a} \rangle \rightarrow \langle a \rangle$ is the desired retraction. Item (3) is an immediate consequence of item (1). □

2.4 Ore localisation

Ore localisation is an analogue of usual localisation for noncommutative rings. Let R be a ring and let S be its set of non-zero-divisors. Then R satisfies the *Ore condition* if for every $r \in R$ and $s \in S$ there are elements $p, p' \in R$ and $q, q' \in S$ such that

$$qr = ps \quad \text{and} \quad rq' = sq'.$$

If $S = R \setminus \{0\}$ and R satisfies the Ore condition, then it is called an *Ore domain*, and we can form its *Ore localisation* $\text{Ore}(R)$ as follows. Define an equivalence relation \sim_R on $R \times S$ by declaring that $(r, s) \sim_R (r', s')$ if and only if there are elements $a, b \in S$ such that

$$ra = r'b \quad \text{and} \quad sa = s'b.$$

The equivalence class of (r, s) under \sim_R is denoted r/s and called a *right fraction*. Then $\text{Ore}(R)$ is defined to be the set of right fractions. We can similarly define an equivalence relation \sim_L and define $\text{Ore}(R)$ as a set of left fractions. The Ore condition ensures that these two constructions are isomorphic and indicates how to convert right fractions into left fractions and vice versa. For a detailed construction of $\text{Ore}(R)$ and the definition of addition and multiplication making this set into a ring, we refer the reader to §4.4 of Passman’s book [Pas77].

The facts about Ore localisation that we will use are summarised in the following proposition.

PROPOSITION 2.8. *Let R be an Ore domain. Then:*

- (1) $\text{Ore}(R)$ is a skew-field;
- (2) for every $r \in R$, we have $r/1 = 1 \setminus r$ and the map $R \rightarrow \text{Ore}(R), r \mapsto r/1 = 1 \setminus r$ is an injective ring homomorphism;
- (3) (see [JZ21, Proposition 2.2(2)]) if G is a group and \mathbb{F} is a skew-field such that $\mathcal{D}_{\mathbb{F}G}$ exists and K is a normal subgroup such that $G/K \cong \mathbb{Z}$, then $\mathcal{D}_{\mathbb{F}G} \cong \text{Ore}(\mathcal{D}_{\mathbb{F}K} * (G/K))$.

3. Valuations on free resolutions

In this section, we introduce valuations on free resolutions over a group ring. We very closely follow Bieri and Renz [BR88] where the theory is developed in the case where the ring is \mathbb{Z} . Their proofs go through without change when \mathbb{Z} is replaced by an arbitrary ring R .

Let R be a ring, G a group, and M a left RG -module. A *free resolution* of M is an exact sequence

$$\dots \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \rightarrow 0$$

of left RG -modules, where F_i is free for all $i \geq 0$. We will usually omit the subscripts on the boundary maps ∂_n and denote the free resolution by $F_\bullet \rightarrow M \rightarrow 0$. Let F be the free RG -module $\bigoplus_{i=0}^\infty F_i$, and define the n -skeleton of F to be $F^{(n)} := \bigoplus_{i=0}^n F_i$. The elements of F are called *chains*, so a chain is not necessarily an element of F_i for any i in our context. Fixing a basis X_i for each F_i , we note that $X := \bigcup_{i=0}^\infty X_i$ is a basis for F and $X^{(n)} := \bigcup_{i=0}^n X_i$ is a basis for $F^{(n)}$. The resolution $F_\bullet \rightarrow M \rightarrow 0$ is *admissible with respect to X* if $\partial x \neq 0$ for every $x \in X$. We will always assume that our free resolutions are admissible with respect to the basis we are working with. This is not a strong requirement, since if all boundary maps are nonzero, then F has a basis with respect to which $F_\bullet \rightarrow M \rightarrow 0$ is admissible; otherwise, we can truncate the resolution and choose a basis to obtain an admissible resolution of finite length. We also define the *support* of a chain $c \in F$ (with respect to X), denoted $\text{supp}_X(c)$, as follows: every chain $c \in F$ can be written uniquely as $\sum_{g \in G, x \in X} r_{g,x} gx$, where $r_{g,x} \in R$. Then $\text{supp}_X(c) := \{gx : r_{g,x} \neq 0\}$; we will usually drop the subscript X when the basis is understood.

Let $\chi: G \rightarrow \mathbb{R}$ be a nontrivial *character*, that is, a nonzero group homomorphism from G to the additive group \mathbb{R} . This provides the elements of G with a notion of height, which we now extend to the chains of F . Let $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$, where ∞ is an element such that $t < \infty$ for every $t \in \mathbb{R}$. We construct a function $v_X: F \rightarrow \mathbb{R}_\infty$ via the following inductive procedure. For an element $c \in F_0$, define $v_X(c) = \inf\{\chi(g) : gx \in \text{supp}(c)\}$. Let $n > 0$ and assume that we have defined v_X on F_{n-1} . For $x \in X_n$, let $v_X(x) := v_X(\partial x)$. For $c \in F_n$, set $v_X(c) = \inf\{\chi(g) + v_X(x) : gx \in \text{supp}(c)\}$. For an arbitrary $c \in F$, write $c = \sum_i c_i$, where $c_i \in F_i$, and define $v = \inf_i\{v_X(c_i)\}$. We are assuming the convention $v_X(0) = \infty$, since $\text{supp}(0) = \emptyset$. The function v_X is called the *valuation extending χ with respect to X* . It is clear from the definition that $v_X(c) = \inf\{\chi(g) + v_X(x) : gx \in \text{supp}(c)\}$ for any chain $c \in F$. Again, we will usually drop the X in the subscript when the basis is understood.

PROPOSITION 3.1. *For the valuation $v_X = v: F \rightarrow \mathbb{R}_\infty$ defined above and for any $c, c' \in F$ and $g \in G$, we have:*

- (1) $v(c + c') \geq \min\{v(c), v(c')\}$;
- (2) $v(c) \leq v(rc)$ for all $r \in R$, and $v(c) = v(rc)$ if r is not a zero-divisor;
- (3) if $v(c) \neq v(c')$, then $v(c + c') = \min\{v(c), v(c')\}$;
- (4) $v(gc) = \chi(g) + v(c)$;
- (5) $v(c) = \infty$ if and only if $c = 0$;
- (6) if $c \in \bigoplus_{i \geq 1} F_i$, then $v(\partial c) \geq v(c)$.

Proof. Property (1) follows from the fact that $\text{supp}(c + c') \subseteq \text{supp}(c) \cup \text{supp}(c')$. The first part of property (2) follows from the fact that $\text{supp}(c) \supseteq \text{supp}(rc)$. If r is not a zero-divisor, then $\text{supp}(c) = \text{supp}(rc)$, which yields the second statement of property (2).

To prove property (3), assume without loss of generality that $v(c) < v(c')$. Then,

$$v(c) = v((c + c') - c') \geq \min\{v(c + c'), v(-c')\} = \min\{v(c + c'), v(c')\}.$$

Since we assumed that $v(c) < v(c')$, the previous line implies that $\min\{v(c + c'), v(c')\} = v(c + c')$; hence, $v(c) = \min\{v(c), v(c')\} \geq v(c + c')$. However, $v(c + c') \geq \min\{v(c), v(c')\}$ by property (1), so we obtain property (3).

For property (4), we have

$$\begin{aligned} v(gc) &= \inf\{\chi(h) + v(x) : hx \in \text{supp}(gc)\} \\ &= \inf\{\chi(g(g^{-1}h)) + v(x) : hx \in g \cdot \text{supp } c\} \\ &= \chi(g) + \inf\{\chi(g^{-1}h) + v(x) : (g^{-1}h)x \in \text{supp } c\} \\ &= \chi(g) + v(c). \end{aligned}$$

For property (5), we first show that if $c \in F_n \setminus \{0\}$, then $v(c) < \infty$ by induction on n . This is true for $n = 0$ since $\chi(G) \subseteq \mathbb{R}$. Now let $n > 0$. Since $c \neq 0$, there is some element gx in its support, where $g \in G$ and $x \in X$. Then

$$v(c) \leq v(gx) = \chi(g) + v(x) = \chi(g) + v(\partial x) < \infty$$

by the inductive hypothesis and by admissibility of $F_\bullet \rightarrow M \rightarrow 0$ with respect to X . For a general nonzero element $c \in F$, write $c = \sum_i c_i$ with $c_i \in F_i$. Then $v(c) = \inf_i \{v(c_i)\} < \infty$ since at least one of the chains c_i is nonzero. Conversely, if $c = 0$, then $v(c) = \infty$, since the infimum of the empty set is ∞ .

For property (6), let $c = \sum_{g \in G, x \in X} r_{g,x}gx \in \bigoplus_{i \geq 1} F_i$. Then

$$\begin{aligned} v(\partial c) &= v\left(\partial\left(\sum_{g \in G, x \in X} r_{g,x}gx\right)\right) \\ &= v\left(\sum_{g \in G, x \in X} r_{g,x}g\partial x\right) \\ &\geq \inf\{v(r_{g,x}g\partial x) : gx \in \text{supp}(c)\} && \text{(by (1))} \\ &\geq \inf\{v(g\partial x) : gx \in \text{supp}(c)\} && \text{(by (2))} \\ &= \inf\{\chi(g) + v(\partial x) : gx \in \text{supp}(c)\} \\ &= \inf\{\chi(g) + v(x) : gx \in \text{supp}(c)\} \\ &= v(c). \end{aligned} \quad \square$$

DEFINITION 3.2 (Valuation subcomplex and essential acyclicity). Given an admissible free resolution $F_\bullet \rightarrow M \rightarrow 0$ over RG (with respect to some fixed basis X), a nontrivial character $\chi: G \rightarrow \mathbb{R}$, and the valuation $v: F \rightarrow \mathbb{R}_\infty$ extending χ , define the *valuation subcomplex* of F with respect to v to be the chain complex $\cdots \rightarrow F_n^v \rightarrow \cdots \rightarrow F_0^v \rightarrow M \rightarrow 0$, where $F_n^v = \{c \in F_n : v(c) \geq 0\}$. We denote the valuation subcomplex by $F_\bullet^v \rightarrow M \rightarrow 0$ and let $F^v := \bigoplus_{i=0}^\infty F_i^v$. Proposition 3.1(6) ensures that $F_\bullet^v \rightarrow M \rightarrow 0$ is a chain complex of left RG_χ -modules, where G_χ is the monoid $\{g \in G : \chi(g) \geq 0\}$. It is not hard to show that each F_i^v is a free RG_χ -module and has an RG_χ -basis of cardinality $|X_i|$, where X_i is an RG -basis for F_i .

The chain complex $F_\bullet^v \rightarrow M \rightarrow 0$ is *essentially acyclic in dimension n* if there is a real number $D \geq 0$ such that for every cycle $z \in F_n^v$ there is a $c \in F_{n+1}^v$ with $\partial c = z$ and $D \geq v(z) - v(c)$. We extend the definition of essential acyclicity to dimension -1 by declaring that $v(m) = 0$ for all $m \in M \setminus \{0\}$.

The definition of essential acyclicity in dimension n is equivalent to the following seemingly weaker condition: for every cycle $z \in F_n^v$, there is a $c \in F_{n+1}^v$ such that $\partial c = z$ and $v(c) \geq -D$. To see this, let $z \in F_n^v$ be a cycle. It is easily shown that $v(F) \subseteq \chi(G) \cup \{\infty\}$, so there is a $g \in G$ such that $\chi(g) = v(z)$. Since $g^{-1}z$ is also in F_n^v with $v(g^{-1}z) = 0$, there is some $c \in F_{n+1}^v$ such that $\partial c = g^{-1}z$ and $v(c) \geq -D$. Thus, $\partial(gc) = z$, and $D \geq v(z) - v(gc)$.

4. Horochains

DEFINITION 4.1 (Complex of horochains and horo-acyclicity). Let $F_\bullet \rightarrow M \rightarrow 0$ be an admissible free resolution with respect to some basis X , let $\chi: G \rightarrow \mathbb{R}$ be a nontrivial character, and let $v: F \rightarrow \mathbb{R}_\infty$ be the valuation extending χ . Define \widehat{F} to be the left RG -module of chains that are finitely supported below every height. More precisely, \widehat{F} is the RG -module of formal sums $\sum_{g \in G, x \in X} r_{g,x} gx$ such that $\{gx : v(gx) \leq t, r_{g,x} \neq 0\}$ is finite for every $t \in \mathbb{R}$. The elements of \widehat{F} are called *horochains*. If $\hat{c} \in \widehat{F}$, then its *support* is $\text{supp}_X(\hat{c}) := \{gx : r_{g,x} \neq 0\}$. Let $\widehat{F}_i \subseteq \widehat{F}$ be the subset of chains with support in F_i and let $\widehat{F}^{(n)} := \bigoplus_{i=0}^n \widehat{F}_i$. Proposition 3.1(6) guarantees that $\partial: F_n \rightarrow F_{n-1}$ extends to a map $\partial: \widehat{F}_n \rightarrow \widehat{F}_{n-1}$ in the obvious way, so we get a complex $\cdots \rightarrow \widehat{F}_n \rightarrow \cdots \rightarrow \widehat{F}_0 \rightarrow 0$. Note that \widehat{F} is not equal to $\bigoplus_{i=0}^\infty \widehat{F}_i$ since the support of a horochain might intersect infinitely many of the modules \widehat{F}_i . A cycle in the chain complex \widehat{F} is called a *horocycle*. We say that $F_\bullet \rightarrow M \rightarrow 0$ is *horo-acyclic* in dimensions $n \geq 0$ with respect to v if the chain complex $\cdots \rightarrow \widehat{F}_1 \rightarrow \widehat{F}_0 \rightarrow 0$ is acyclic in dimension n .

We can extend the definition of v to \widehat{F} by defining $v(\hat{c}) := \inf\{v(gx) : \text{supp}(\hat{c})\}$ for any horochain \hat{c} . If $\hat{c} \neq 0$, then $\{v(gx) : \text{supp}(\hat{c})\}$ is nonempty and attains a minimum because chains are finitely supported below any given height. Properties (1)–(5) of Proposition 3.1 hold in this setting with the same proofs.

A version of Proposition 3.1(6) holds for horochains, namely we have $v(\partial\hat{c}) \geq v(\hat{c})$ for all horochains \hat{c} , but we need to modify the proof: If $\hat{c} = 0$, then the claim is clear. Otherwise, let $\hat{c} \neq 0$ be a horochain, and let $gx \in \text{supp}(\hat{c})$ be such that $v(gx) = v(\hat{c})$. By the finite version of property (6), we have that $v(\partial g'x') \geq v(g'x') \geq v(gx)$ for every $g'x' \in \text{supp}(\hat{c})$. Since every $g''x'' \in \text{supp}(\partial\hat{c})$ is contained in $\text{supp}(\partial g'x')$ for some $g'x' \in \text{supp}(\hat{c})$, we have that $v(g''x'') \geq v(\partial g'x') \geq v(gx)$ for every $g''x'' \in \text{supp}(\partial\hat{c})$. Thus, $v(\partial\hat{c}) \geq v(\hat{c})$.

The following lemma will be used in the proof of Theorem 5.3.

LEMMA 4.2. Let $F_\bullet \rightarrow M \rightarrow 0$ (respectively, $F'_\bullet \rightarrow M \rightarrow 0$) be a free resolution over RG admissible with respect to a basis X (respectively, X'), and let v (respectively, v') be the valuation extending a nontrivial character $\chi: G \rightarrow \mathbb{R}$. Suppose that $F^{(n)}$ is finitely generated, and that $\varphi: F \rightarrow F'$ is a homomorphism of RG -modules. Then:

- (1) φ induces a homomorphism of left RG -modules given by

$$\widehat{\varphi}: \widehat{F}^{(n)} \rightarrow \widehat{F}', \quad \sum r_{g,x} gx \mapsto \sum r_{g,x} g\varphi(x);$$

- (2) $v'(\widehat{\varphi}(\hat{c})) \geq v(\hat{c}) + \min_{x \in X^{(n)}} \{v'(\varphi(x)) - v(x)\}$ for every $\hat{c} \in F^{(n)}$.

Proof. For part (1), we need to show that $\widehat{\varphi}(\hat{c})$ is a horochain for any horochain $\hat{c} \in \widehat{F}^{(n)}$. To this end, let $\hat{c} = \sum r_{g,x} gx$, and note that there are only finitely many elements $x \in X$ such that $gx \in \text{supp}_X(\hat{c})$. If $\widehat{\varphi}(\hat{c})$ is not a horochain, then the set $\{gx \in \text{supp}_X(\hat{c}) : v'(g\varphi(x)) \leq t\}$ is infinite for some $t \in \mathbb{R}$. Since $F^{(n)}$ is finitely generated, there is some fixed $y \in X$ such that $v'(g\varphi(y)) \leq t$ and $gy \in \text{supp}_X(\hat{c})$ for infinitely many values of $g \in G$. But then

$$\begin{aligned} v(gy) &= \chi(g) + v(y) \\ &= \chi(g) + v'(\varphi(y)) + v(y) - v'(\varphi(y)) \\ &= v'(g\varphi(y)) + v(y) - v'(\varphi(y)) \\ &\leq t + v(y) - v'(\varphi(y)) \end{aligned}$$

for infinitely many $gy \in \text{supp}_X(\hat{c})$, but \hat{c} is a horochain.

For part (2), write $\hat{c} = \sum_{x \in X^{(n)}} \hat{c}_x$, where $\hat{c}_x = \sum_{g \in G} r_{g,x} g x$. Then

$$\begin{aligned} v'(\widehat{\varphi}(\hat{c})) &\geq \min_{x \in X^{(n)}} \{v'(\widehat{\varphi}(\hat{c}_x))\} \\ &\geq \min_{x \in X^{(n)}} \{\inf\{v'(g\varphi(x)) : gx \in \text{supp } \hat{c}_x\}\} \\ &= \min_{x \in X^{(n)}} \{\inf\{v(gx) : gx \in \text{supp } \hat{c}_x\} + v'(\varphi(x)) - v(x)\} \\ &= \min_{x \in X^{(n)}} \{v(\hat{c}_x) + v'(\varphi(x)) - v(x)\} \\ &\geq \min_{x \in X^{(n)}} \{v(\hat{c}_x)\} + \min_{x \in X^{(n)}} \{v'(\varphi(x)) - v(x)\} \\ &= v(\hat{c}) + \min_{x \in X^{(n)}} \{v'(\varphi(x)) - v(x)\}. \end{aligned} \quad \square$$

Note that Lemma 4.2(2) applies to chains in F since these are just finite horo-chains. We will use this in the proof Theorem 5.3.

5. Characterisations of the Σ -invariant

We introduce the invariants $\Sigma_R^n(G; M)$, which are generalisations of the classical BNS invariant [BNS87] and its higher-dimensional analogues [BR88]. The only difference is that we work over a general ring R , whereas the higher BNS invariants are defined over \mathbb{Z} .

Let G be a group. We declare two characters $\chi, \chi' : G \rightarrow \mathbb{R}$ to be *equivalent* if $\chi = \alpha \cdot \chi'$ for some $\alpha > 0$ and let $S(G)$ denote the set of equivalence classes of nonzero characters. We call $S(G)$ the *character sphere* of G , because it can be given the topology of a sphere when G is finitely generated.

DEFINITION 5.1 (Σ -invariants). Let M be an RG -module. Then define

$$\Sigma_R^n(G; M) = \{[\chi] \in S(G) : M \in \text{FP}_n(RG_\chi)\},$$

where $G_\chi = \{g \in G : \chi(g) \geq 0\}$. Note that $G_\chi = G_{\chi'}$ if $[\chi] = [\chi']$, so $\Sigma_R^n(G; M)$ is well-defined.

DEFINITION 5.2 (Novikov ring). Let G be a group, let R be a ring and let $\chi : G \rightarrow \mathbb{R}$ be a character. Then the *Novikov ring* \widehat{RG}^χ is the set of formal sums

$$\sum_{g \in G} r_g g$$

such that $\{g \in G : r_g \neq 0 \text{ and } \varphi(g) \leq t\}$ is finite for every $t \in \mathbb{R}$. We give \widehat{RG}^χ a ring structure by defining $rg + r'g := (r + r')g$ and $rg \cdot r'g' := rr'gg'$ for $r, r' \in R, g, g' \in G$, and extending multiplication to all of \widehat{RG}^χ in the obvious way.

Theorem 5.3 gives several characterisations of the Σ -invariants and is the main technical tool we will need to prove Theorem 6.7. More specifically, we will need the characterisation of $\Sigma_R^n(G; M)$ in terms of the vanishing of Novikov homology; this is the equivalence of parts (1) and (5) in the following theorem, which should be thought of as a higher-dimensional version of Sikorav’s theorem [Sik87].

THEOREM 5.3. *Let R be a ring, let M be a left RG -module of type FP_n and let $\chi : G \rightarrow \mathbb{R}$ be a nontrivial character. Let $F_\bullet \rightarrow M \rightarrow 0$ be a free resolution admissible with respect to a*

basis $X = \bigcup_{i=0}^{\infty} X_i$ and with finitely generated n -skeleton $F^{(n)}$. Let $v: F \rightarrow \mathbb{R}_{\infty}$ be the valuation extending χ with respect to X . The following are equivalent:

- (1) $[\chi] \in \Sigma_R^n(G; M)$;
- (2) $F_{\bullet}^v \rightarrow M \rightarrow 0$ is essentially acyclic in dimensions $-1, \dots, n-1$;
- (3) there is a chain map $\varphi: F \rightarrow F$ lifting the identity id_M such that $v(\varphi(c)) > v(c)$ for every $c \in F^{(n)}$;
- (4) $F_{\bullet} \rightarrow M \rightarrow 0$ is horo-acyclic in dimensions $0, \dots, n$ with respect to v ;
- (5) $\text{Tor}_i^{RG}(\widehat{RG}^X, M) = 0$ for all $0 \leq i \leq n$.

The strategy of the proof will be as follows: we begin by proving $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2)$. This is done by Schweitzer in the appendix of [Bie07] in the case $R = \mathbb{Z}$. Once this is done, we prove the equivalence of parts (4) and (5), again following Schweitzer. Finally, we prove the equivalence of parts (1) and (2) following the appendix to Theorem 3.2 in [BR88], where again this is done in the case $R = \mathbb{Z}$. The proofs below are essentially the same as those given in the references just cited; there is no crucial dependence on the coefficient ring R .

Proof of (2) \Rightarrow (3). Assume that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is essentially acyclic in dimensions $\leq n-1$ and let $D > 0$ be a constant such that for each $k < n$ and every cycle $z \in F_k$, there is a chain $c \in F_{k+1}$ with $\partial c = z$ and $D \geq v(z) - v(c)$. We will construct a chain map $\varphi: F \rightarrow F$ lifting id_M such that $v(\varphi(c)) > v(c) + (n-k)D$ for every $c \in F^{(k)}$, which implies part (3).

We define φ on $F^{(k)}$ by induction on k . For the base case, let $x \in X_0$ be arbitrary and fix some $g \in G$ such that $\chi(g) > (n+1)D$. The element $g^{-1}\partial x \in M$ is a cycle, so there is some $c_x \in F_0$ such that $\partial c_x = g^{-1}\partial x$ and $D \geq v(g^{-1}\partial x) - v(c_x) = -v(c_x)$, since $v|_{M \setminus \{0\}} = 0$. Define φ on $F^{(0)}$ by setting $\varphi(x) = gc_x$ for each $x \in X_0$. It is clear that $\text{id}_M \partial = \partial \varphi$ on $F^{(0)}$. By Lemma 4.2(2),

$$v(\varphi(c)) \geq v(c) + \min_{x \in X_0} \{v(\varphi(x)) - v(x)\} > v(c) + nD$$

for every $c \in F^{(0)}$.

Let $k > 0$ and suppose φ is defined on $F^{(k-1)}$ such that it lifts id_M and $v(\varphi(c)) > v(c) + (n-k+1)D$ for all $c \in F^{(k-1)}$. Let $x \in X_k$ and note that $\varphi(\partial x)$ is a cycle. By essential acyclicity, there is a chain $d_x \in F_k$ such that $\partial d_x = \varphi(\partial x)$ and $D \geq v(\varphi(\partial x)) - v(d_x)$. Define φ on $F^{(k)}$ by setting $\varphi(x) = d_x$. Then $\varphi \partial = \partial \varphi$ by construction, and for every $x \in X_k$ we have

$$\begin{aligned} v(\varphi(x)) - v(x) &= v(d_x) - v(x) \\ &\geq v(\varphi(\partial x)) - v(x) - D \\ &= v(\varphi(\partial x)) - v(\partial x) - D \\ &> (n-k)D \end{aligned}$$

by induction. By Lemma 4.2(2), we have

$$v(\varphi(c)) \geq v(c) + \min_{x \in X_k} \{v(\varphi(x)) - v(x)\} > v(c) + (n-k)D. \quad \square$$

We pause here to prove a lemma that will immediately imply $(3) \Rightarrow (4)$ and will be useful in the proofs of $(4) \Rightarrow (2)$ and $(4) \Rightarrow (5)$. We recall that the maps \widehat{H} and $\widehat{\varphi}$ that appear in the statement of the lemma below are defined in Lemma 4.2.

LEMMA 5.4. *With the assumptions of Theorem 5.3, let $\varphi: F \rightarrow F$ be a chain map lifting id_M such that $v(\varphi(c)) > v(c)$ for all $c \in F^{(n)}$ and let $H: F \rightarrow F$ be a chain homotopy such that $\partial H + H \partial = \text{id}_F - \varphi$. Let $\hat{z} \in \widehat{F}^{(n)}$ be a horocycle and define $\hat{c}_z := \sum_{i=0}^{\infty} \widehat{H} \widehat{\varphi}^i(\hat{z})$. Then \hat{c}_z is a horochain and $\partial \hat{c}_z = \hat{z}$.*

Proof. By Lemma 4.2(2) there are constants α and β such that

$$v(\widehat{\varphi}(\hat{c})) \geq v(\hat{c}) + \alpha \quad \text{and} \quad v(\widehat{H}(\hat{c})) \geq v(\hat{c}) + \beta$$

for every horochain $\hat{c} \in \widehat{F}^{(n)}$. Moreover, $\alpha > 0$ since $v(\varphi(f)) > v(f)$ for every $f \in F^{(n)}$. To see that \hat{c} is a horochain, by induction we have $v(\widehat{H}\widehat{\varphi}^i(\hat{z})) \geq v(\hat{z}) + i\alpha + \beta$, so for all $t \in \mathbb{R}$ there are only finitely many integers $i \geq 0$ such that $v(\widehat{H}\widehat{\varphi}^i(\hat{z})) \leq t$. Since $\text{supp}(\hat{c}_z) \subseteq \bigcup_{i=0}^{\infty} \text{supp}(\widehat{H}\widehat{\varphi}^i(\hat{z}))$ and each $\widehat{H}\widehat{\varphi}^i(\hat{z})$ is a horochain, it follows that there are only finitely many $gx \in \text{supp } \hat{c}$ such that $v(gx) \leq t$, so \hat{c} is a horochain.

Finally, we have

$$\partial\hat{c} = \sum_{i=0}^{\infty} \partial\widehat{H}\widehat{\varphi}^i(\hat{z}) = \sum_{i=0}^{\infty} (\text{id}_{\widehat{F}^{(n)}} - \widehat{\varphi} - \widehat{H}\partial)\widehat{\varphi}^i(\hat{z}) = \sum_{i=0}^{\infty} (\widehat{\varphi}^i - \widehat{\varphi}^{i+1})(\hat{z}) = \hat{z}. \quad \square$$

Proof of (3) \Rightarrow (4). By [Bro94, Lemma I.7.4], there is a chain homotopy $H: F \rightarrow F$ such that $\partial H + H\partial = \text{id}_F - \varphi$. If $\hat{z} \in \widehat{F}^{(n)}$ is a horocycle, then $\partial\hat{c}_z = \hat{z}$ by Lemma 5.4. \square

Proof of (4) \Rightarrow (2). We will prove that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is essentially acyclic in dimension k for all $k < n$ by induction on k . For the base case, we show that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is exact at M , which implies essential acyclicity in dimension -1 . Let $m \in M$. By exactness of $F_{\bullet} \rightarrow M \rightarrow 0$, there is a chain $c \in F_0$ such that $\partial c = m$. By horo-acyclicity in dimension 0, there is some horochain $\hat{c} \in \widehat{F}_1$ such that $\partial\hat{c} = c$. There are $c_- \in F_1$ and $\hat{c}_+ \in \widehat{F}_1$ such that $\hat{c} = c_- + \hat{c}_+$, where $v(c_-) < 0$ and $v(\hat{c}_+) \geq 0$. Then $\partial(c - \partial c_-) = m$ and

$$v(c - \partial c_-) = v(c - \partial(\hat{c} - \hat{c}_+)) = v(\partial\hat{c}_+) \geq v(\hat{c}_+) \geq 0.$$

This shows that $c - \partial c_0 \in F_0^v$, which proves that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is exact at M .

Let $k > -1$ and suppose that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is essentially acyclic in dimensions $< k$. By (2) \Rightarrow (3) applied at $k - 1$, there is a chain map $\varphi: F \rightarrow F$ lifting id_M such that $v(\varphi(c)) > v(c)$ for all $c \in F^{(k)}$. Since id_F and φ both lift id_M and $F_{\bullet} \rightarrow M \rightarrow 0$ is acyclic, there is a chain homotopy $H: F \rightarrow F$ such that $\partial H + H\partial = \text{id}_F - \varphi$ (see [Bro94, Lemma I.7.4]). As in the proof of Lemma 5.4, there are constants $\alpha > 0$ and $\beta < 0$ such that

$$v(\varphi(c)) \geq v(c) + \alpha \quad \text{and} \quad v(H(c)) \geq v(c) + \beta$$

for every $c \in F^{(k)}$.

Let $z \in F_k^v$ be a cycle. Since $F_{\bullet} \rightarrow M \rightarrow 0$ is acyclic, there is some $d \in F_{k+1}$ such that $\partial d = z$. Consider the horocycle $\hat{z} := d - \hat{d}_z$, where $\hat{d}_z = \sum_{i=0}^{\infty} H\varphi^i(z)$ is defined as in Lemma 5.4. Note that

$$v(H\varphi^i(z)) \geq v(z) + i\alpha + \beta \geq \beta$$

for every $i \geq 0$, and therefore that $v(\hat{d}_z) \geq \beta$. By horo-acyclicity in dimension $k + 1$, there is a $(k + 2)$ -horochain \hat{d} such that $\partial\hat{d} = \hat{z}$. As in the base case, there are $d_- \in F_{k+2}$ and $\hat{d}_+ \in \widehat{F}_{k+2}$ such that $\hat{d} = d_- + \hat{d}_+$, where $v(d_-) < 0$ and $v(\hat{d}_+) \geq 0$. Then $\partial(d - \partial d_-) = \partial d = z$, and

$$\begin{aligned} v(d - \partial d_-) &= v(\hat{d}_z + \hat{z} - \partial(\hat{d} - \hat{d}_+)) \\ &= v(\hat{d}_z + \partial\hat{d}_+) \\ &\geq \min\{v(\hat{d}_z), v(\partial\hat{d}_+)\} \\ &\geq \beta \end{aligned}$$

since $v(\hat{d}_z) \geq \beta$ and $v(\partial d_{\infty}) \geq v(d_{\infty}) \geq 0 > \beta$. Letting $D = -\beta$ in the definition of essential acyclicity, we see that $F_{\bullet}^v \rightarrow M \rightarrow 0$ is essentially acyclic in dimension k . \square

Proof of (5) ⇒ (4). Suppose that $\text{Tor}_i^{RG}(\widehat{RG}^X, M) = 0$ for $0 \leq i \leq n$. Consider the chain map

$$\psi: \widehat{RG}^X \otimes_{RG} F \rightarrow \widehat{F}, \quad \alpha \otimes c \mapsto \alpha c$$

of left RG -modules. It is clear that ψ is injective. We claim that ψ induces an isomorphism $\widehat{RG}^X \otimes_{RG} F^{(n)} \rightarrow \widehat{F}^{(n)}$. To see this, simply note that for an arbitrary horochain

$$\hat{c} = \sum_{g \in G, x \in X^{(n)}} r_{g,x} g x$$

in $\widehat{F}^{(n)}$, we have

$$\sum_{x \in X^{(n)}} \left(\sum_{g \in G} r_{g,x} g \right) \otimes x \xrightarrow{\psi} \hat{c}.$$

The horochain condition implies that the sums $\sum_{g \in G} r_{g,x} g$ are elements of \widehat{RG}^X . Thus, ψ is surjective on the n -skeleta and is therefore an isomorphism. Note that this only works because $X^{(n)}$ is finite; in general, we cannot expect ψ to be surjective since the support of a horochain might intersect infinitely many of the modules F_n . Since $\text{Tor}_i^{RG}(\widehat{RG}^X \otimes_{RG} F, M) = 0$ for all $0 \leq i \leq n$, we conclude that $\text{Tor}_i(\widehat{F}, M) = 0$ for $0 \leq i \leq n$ as well. \square

Proof of (4) ⇒ (5). The map ψ defined above is an isomorphism of the n -skeleta, so we immediately have that $\text{Tor}_i^{RG}(\widehat{RG}^X, M) = 0$ for $0 \leq i \leq n - 1$. Since ψ is not necessarily surjective as a map of the $(n + 1)$ -skeleta, we must work harder to show that $\text{Tor}_n^{RG}(\widehat{RG}^X, M) = 0$. Let $z \in \widehat{RG}^X \otimes_{RG} F_n$ be an n -cycle, and let $\hat{z} = \psi(z)$. Since we are assuming that part (4) holds, we may also assume that part (3) holds and use the horochain \hat{c}_z from Lemma 5.4. Since $\hat{c}_z \in \widehat{H}(\widehat{F}_n)$, we have that \hat{c}_z is in the \widehat{RG}^X -submodule of \widehat{F}_{n+1} generated by $\widehat{H}(X_n)$ and, thus, $\hat{c}_z \in \text{im } \psi$ since this is a finite set. Let $c \in \widehat{RG}^X \otimes_{RG} F_{n+1}$ such that $\psi(c) = \hat{c}_z$. Then $\psi \partial(c) = \partial \psi(c) = \partial \hat{c}_z = \hat{z}$. However, ψ is injective, so $\partial c = z$, proving that $\text{Tor}_n^{RG}(\widehat{RG}^X, M) = 0$. \square

We pause again before proving the equivalence of parts (1) and (2) to prove another lemma.

LEMMA 5.5. *Free RG -modules are flat over RG_χ .*

Proof. It suffices to prove that RG is flat as an RG_χ -module, since the direct sum of flat modules is flat. To this end, let $\iota: M \hookrightarrow N$ be an injection of right RG_χ -modules; our goal is to show that $\iota \otimes \text{id}: M \otimes_{RG_\chi} RG \rightarrow N \otimes_{RG_\chi} RG$ is injective. Let $g \in G$ be such that $\chi(g) < 0$ and consider the left RG_χ -module $RG_\chi g^k = \{\alpha g^k : \alpha \in RG_\chi, k \in \mathbb{Z}\}$. The modules $RG_\chi g^k$ form a directed system with respect to the inclusion maps $RG_\chi g^k \hookrightarrow RG_\chi g^l$ for $k \leq l$ and the direct limit is $\varinjlim RG_\chi g^k \cong RG$.

There are left RG_χ -module isomorphisms $RG_\chi g^k \rightarrow RG_\chi$ given by right multiplication by g^{-k} . Then $RG_\chi g^k$ is flat over RG_χ , so $M \otimes_{RG_\chi} RG_\chi g^k \rightarrow N \otimes_{RG_\chi} RG_\chi g^k$ is injective for all $k \in \mathbb{Z}$. By exactness of the direct limit,

$$\varinjlim (M \otimes_{RG_\chi} RG_\chi g^k) \rightarrow \varinjlim (N \otimes_{RG_\chi} RG_\chi g^k)$$

is injective. Since the direct limit commutes with the tensor product, the previous line implies $\iota \otimes \text{id}_M$ is injective. \square

We now return to the proof of Theorem 5.3.

Proof of (1) ⇔ (2). Let $g \in G$ be such that $\chi(g) < 0$ and let E_k be the left RG_χ -module $g^k F^v$. We denote the chain complexes $F_\bullet^v \rightarrow M \rightarrow 0$ and $(E_k)_\bullet \rightarrow M \rightarrow 0$ by \tilde{F}^v and \tilde{E}_k , respectively.

Essential acyclicity in dimension j is equivalent to the existence of an integer $D \geq 0$ such that the inclusion-induced homomorphism $H_j(\tilde{E}_k) \rightarrow H_j(\tilde{E}_{k+D})$ is the zero map for all $k \in \mathbb{N}$. This in turn is equivalent to $\varinjlim \prod_I H_j(\tilde{E}_k) = 0$ for any index set I . Here, for fixed I and j , the powers $\prod_I H_j(\tilde{E}_k)$ form a directed system with respect to the inclusion-induced maps $\prod_I H_j(\tilde{E}_k) \rightarrow \prod_I H_j(\tilde{E}_l)$ for $k \leq l$. Indeed, if $D \geq 0$ is such that $H_j(\tilde{E}_k) \rightarrow H_j(\tilde{E}_{k+D})$ is the zero map, it is clear that the direct limit will be zero. Conversely, let $I = Z_j(\tilde{E}_0) = Z_j(\tilde{F}^v)$ be the set of j -cycles of \tilde{F}^v and consider the element $([x])_{x \in I} \in \prod_I H_j(\tilde{E}_0)$. Since the direct limit is zero, there is some $D \geq 0$ such that $([x])_{x \in I} = 0$ in $\prod_I H_j(\tilde{E}_D)$, which means that \tilde{F}^v is essentially acyclic in dimension j .

There is a short exact sequence of chain complexes $0 \rightarrow M \rightarrow \tilde{E}_k \rightarrow E_k \rightarrow 0$, where, by abuse of notation, M is a chain complex concentrated in dimension -1 and E_k is the chain complex $(E_k)_\bullet \rightarrow 0$ with $(E_k)_0$ in dimension 0 . The long exact sequence in homology associated to the short exact sequence gives $H_j(\tilde{E}_k) \cong H_j(E_k)$ for $j \geq 1$. The interesting part of the long exact sequence is

$$0 \rightarrow H_0(\tilde{E}_k) \rightarrow H_0(E_k) \xrightarrow{\delta} M \rightarrow H_{-1}(\tilde{E}_k) \rightarrow 0,$$

where δ is the connecting homomorphism. By exactness of the direct power and direct limit functors, the sequence

$$0 \rightarrow \varinjlim \prod_I H_0(\tilde{E}_k) \rightarrow \varinjlim \prod_I H_0(E_k) \xrightarrow{\prod_I \delta} \prod_I M \rightarrow \varinjlim \prod_I H_{-1}(\tilde{E}_k) \rightarrow 0$$

is exact. Then \tilde{F}^v is essentially acyclic in dimension 0 if and only if δ induces an injection $\varinjlim \prod_I H_0(E_k) \rightarrow \prod_I M$ for every I . Moreover, \tilde{F}^v is essentially acyclic in dimension -1 if and only if δ induces a surjection $\varinjlim \prod_I H_0(E_k) \rightarrow \prod_I M$ for every I .

By Lemma 5.5, $F_\bullet \rightarrow M \rightarrow 0$ is a flat resolution of M by left RG_χ -modules, so

$$\text{Tor}_j^{RG_\chi} \left(\prod_I RG_\chi, M \right) = H_j \left(\left(\prod_I RG_\chi \right) \otimes_{RG_\chi} F \right)$$

and, therefore,

$$\text{Tor}_j^{RG_\chi} \left(\prod_I RG_\chi, M \right) = \varinjlim H_j \left(\left(\prod_I RG_\chi \right) \otimes_{RG_\chi} E_k \right),$$

as $F = \varinjlim E_k$ and direct limits commute with tensor products and homology. Since $(E_k)_j$ is a finitely generated free RG_χ -module for $j \leq n$, we have $(\prod_I RG_\chi) \otimes_{RG_\chi} (E_k)_j \cong \prod_I (E_k)_j$. Hence, $\text{Tor}_j^{RG_\chi}(\prod_I RG_\chi, M) = \varinjlim H_j(\prod_I E_k)$ for $j < n$.

To summarise the work done above, we have \tilde{F}^v is essentially acyclic in dimensions $-1 \leq j < n$ if and only if:

- (a) $(\prod_I RG_\chi) \otimes_{RG_\chi} M \rightarrow \prod_I M$ is surjective if $n = 0$; and
- (b) $(\prod_I RG_\chi) \otimes_{RG_\chi} M \rightarrow \prod_I M$ is an isomorphism and

$$\text{Tor}_j^{RG_\chi} \left(\prod_I RG_\chi, M \right)$$

vanishes for $1 \leq j < n$ otherwise.

Here we have used the general fact that $\text{Tor}_0^R(A, B) \cong A \otimes_R B$. Together with Lemma 1.1 and Proposition 1.2 of [BE74], parts (a) and (b) are equivalent to M being of type $\text{FP}_n(RG_\chi)$.

Thus, we conclude that $[\chi] \in \Sigma_R^m(G; M)$ if and only if \tilde{F}^v is essentially acyclic in dimensions $j = -1, 0, 1, \dots, n - 1$. □

6. Agrarian homology and main result

DEFINITION 6.1 (Agrarian groups and \mathcal{D} -homology). Let R be a ring. A group G is *agrarian over R* if there is a skew-field \mathcal{D} and an injective ring homomorphism $RG \hookrightarrow \mathcal{D}$. In this case, we will say that G is \mathcal{D} -agrarian over R if we wish to specify the skew-field.

If G is \mathcal{D} -agrarian over R , we define its p -dimensional \mathcal{D} -homology to be

$$H_p^{\mathcal{D}}(G) := \text{Tor}_p^{RG}(\mathcal{D}, R),$$

where R is the trivial RG -module and \mathcal{D} is viewed as a \mathcal{D} - RG -bimodule via the embedding $RG \hookrightarrow \mathcal{D}$. The p th \mathcal{D} -Betti number of G is then

$$b_p^{\mathcal{D}}(G) := \dim_{\mathcal{D}} H_p^{\mathcal{D}}(G).$$

Note that $b_p^{\mathcal{D}}(G)$ is well-defined and integral or infinite, since a module over a skew-field has a well-defined dimension.

Remark 6.2. The term ‘agrarian’ was introduced by Kielak in [Kie20a] in the case $R = \mathbb{Z}$. Using strong Hughes-freeness, i.e. condition (2') after Definition 2.3, it follows that if G is a locally indicable group and \mathbb{F} is a skew-field such that $\mathcal{D}_{\mathbb{F}G}$ exists, then G is $\mathcal{D}_{\mathbb{F}G}$ -agrarian over \mathbb{F} . If R is a skew-field or an integral domain, then there are no known examples of torsion-free groups that are not agrarian over R . For the remainder of the section, we will be interested in the $\mathcal{D}_{\mathbb{F}G}$ -homology of G .

In what follows, we will need that $\mathcal{D}_{\mathbb{F}G}$ -Betti numbers have good scaling properties when passing to finite index subgroups. This is analogous to the fact that ℓ^2 -Betti numbers also scale under passage to finite index subgroups.

LEMMA 6.3. *Let H be a finite index subgroup of G and let \mathbb{F} be a skew-field such that $\mathcal{D}_{\mathbb{F}G}$ exists. Then*

$$b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = \frac{b_p^{\mathcal{D}_{\mathbb{F}H}}(H)}{[G : H]}.$$

Proof. It suffices to prove the claim when H is normal in G . To see this, if $H \leq G$ is any subgroup of finite index, then there is normal subgroup $N \triangleleft G$ of finite index such that $N \leq H \leq G$ (we can take N to be the *normal core* of H , that is, the intersection of all the conjugates of H). Then

$$b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = \frac{b_p^{\mathcal{D}_{\mathbb{F}N}}(N)}{[G : N]} = \frac{[H : N] b_p^{\mathcal{D}_{\mathbb{F}H}}(H)}{[G : N]} = \frac{b_p^{\mathcal{D}_{\mathbb{F}H}}(H)}{[G : H]},$$

by the claim for normal subgroups.

Assume that H is a finite index normal subgroup of G and let $\{t_1, \dots, t_n\}$ be a transversal for H in G . By Proposition 2.5(2), we have a $\mathcal{D}_{\mathbb{F}H}$ - $\mathbb{F}G$ -bimodule isomorphism $\mathcal{D}_{\mathbb{F}G} \cong \mathcal{D}_{\mathbb{F}H} \otimes_{\mathbb{F}H} \mathbb{F}G$. Take a free resolution $F_{\bullet} \rightarrow \mathbb{F} \rightarrow 0$ of the trivial left $\mathbb{F}G$ -module \mathbb{F} ; there are chain isomorphisms

$$\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_{\bullet} \cong (\mathcal{D}_{\mathbb{F}H} \otimes_{\mathbb{F}H} \mathbb{F}G) \otimes_{\mathbb{F}G} F_{\bullet} \cong \mathcal{D}_{\mathbb{F}H} \otimes_{\mathbb{F}H} (\mathbb{F}G \otimes_{\mathbb{F}G} F_{\bullet}) \cong \mathcal{D}_{\mathbb{F}H} \otimes_{\mathbb{F}H} F_{\bullet}$$

of left $\mathcal{D}_{\mathbb{F}H}$ -modules, so $H_p^{\mathcal{D}_{\mathbb{F}G}}(G) \cong H_p^{\mathcal{D}_{\mathbb{F}H}}(H)$ as $\mathcal{D}_{\mathbb{F}H}$ -modules. Therefore,

$$\begin{aligned} b_p^{\mathcal{D}_{\mathbb{F}H}}(H) &= \dim_{\mathcal{D}_{\mathbb{F}H}} H_p^{\mathcal{D}_{\mathbb{F}H}}(H) \\ &= \dim_{\mathcal{D}_{\mathbb{F}H}} H_p^{\mathcal{D}_{\mathbb{F}G}}(G) \\ &= [G : H] \cdot \dim_{\mathcal{D}_{\mathbb{F}G}} H_p^{\mathcal{D}_{\mathbb{F}G}}(G) \\ &= [G : H] \cdot b_p^{\mathcal{D}_{\mathbb{F}G}}(G). \end{aligned} \quad \square$$

In view of Lemma 6.3, if G is a group with a finite index subgroup H such that $\mathcal{D}_{\mathbb{F}H}$ exists, we can define $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = b_p^{\mathcal{D}_{\mathbb{F}H}}(H)/[G : H]$. This is an abuse of notation since $\mathcal{D}_{\mathbb{F}G}$ might not exist.

The following theorem is an analogue of a theorem of Lück which holds for ℓ^2 -Betti numbers [Lüc02, Theorem 7.2].

THEOREM 6.4. *Let $1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$ be a short exact sequence of groups and suppose that $\mathcal{D}_{\mathbb{F}G}$ exists for some skew-field \mathbb{F} . If $b_p^{\mathcal{D}_{\mathbb{F}K}}(K) < \infty$ for some $p \geq 0$, then $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$.*

Proof. Let $F_\bullet \rightarrow \mathbb{F} \rightarrow 0$ be a free resolution of \mathbb{F} by left $\mathbb{F}G$ -modules. Note that the modules F_j are also free left $\mathbb{F}K$ -modules and there are chain maps $\iota_j : \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_j \rightarrow \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_j$, induced by the inclusion $\mathcal{D}_{\mathbb{F}K} \hookrightarrow \mathcal{D}_{\mathbb{F}G}$. We claim that the maps ι_j are injective. To see this, it is enough to consider the case where $F_j = \mathbb{F}G$. Choose $t \in G$ such that $\{t^n : n \in \mathbb{Z}\}$ is a transversal for $K \leq G$. By Proposition 2.5, there is an embedding $\bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\mathbb{F}K} \cdot \varphi(t^n) \hookrightarrow \mathcal{D}_{\mathbb{F}G}$. Since $\mathbb{F}G$ is free over $\mathbb{F}K$, there is also an isomorphism $\mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} \mathbb{F}G \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\mathbb{F}K} \cdot \varphi(t^n)$ determined by $\alpha \otimes t^n \mapsto \alpha \varphi(t^n)$. Then the diagram

$$\begin{array}{ccc} \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} \mathbb{F}G & \xrightarrow{\cong} & \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\mathbb{F}K} \cdot \varphi(t^n) \\ \iota_j \downarrow & & \downarrow \\ \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} \mathbb{F}G & \xrightarrow{\cong} & \mathcal{D}_{\mathbb{F}G} \end{array}$$

of left $\mathcal{D}_{\mathbb{F}K}$ -modules commutes, proving that ι_j is an injection. From now on, we will treat the maps ι_j as inclusions.

Consider the following portions of the chain complexes computing $b_p^{\mathcal{D}_{\mathbb{F}G}}(G)$ and $b_p^{\mathcal{D}_{\mathbb{F}K}}(K)$.

$$\begin{array}{ccccccc} \cdots & \rightarrow & \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_{p+1} & \rightarrow & \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_p & \rightarrow & \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_{p-1} \rightarrow \cdots \\ & & \downarrow \iota_{p+1} & & \downarrow \iota_p & & \downarrow \iota_{p-1} \\ \cdots & \rightarrow & \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_{p+1} & \rightarrow & \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_p & \rightarrow & \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_{p-1} \rightarrow \cdots \end{array}$$

Let x be a cycle in $\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_p$. By [JZ21, Proposition 2.2(2)], $\mathcal{D}_{\mathbb{F}G} \cong \text{Ore}(\mathcal{D}_{\mathbb{F}K} * \mathbb{Z})$, where we are making the identifications $\mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} \mathbb{F}G \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_{\mathbb{F}K} \cdot \varphi(t^n) \cong \mathcal{D}_{\mathbb{F}K} * \mathbb{Z}$. Hence, there is some nonzero $a \in \mathcal{D}_{\mathbb{F}K} * \mathbb{Z}$ such that $ax \in \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_p$. Since $(\mathcal{D}_{\mathbb{F}K} * \mathbb{Z}) \cdot ax \subseteq Z_p(\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_\bullet)$ and ι_{p-1} is injective, $(\mathcal{D}_{\mathbb{F}K} * \mathbb{Z}) \cdot ax$ is an infinite-dimensional $\mathcal{D}_{\mathbb{F}K}$ -subspace of $Z_p(\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_\bullet)$. Since $b_p^{\mathcal{D}_{\mathbb{F}K}}(K) < \infty$, there is a nonzero $b \in \mathcal{D}_{\mathbb{F}K} * \mathbb{Z}$ such that $ba x = \partial y$ for some $y \in \mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_{p+1}$. However, then $x = \partial((ba)^{-1}y)$, so we conclude that $H_p(\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_\bullet) = 0$. \square

For the proof of the main theorem, we will need the following version of a theorem of Bieri and Renz. The details of the proof are given in [BR88, Theorem 5.1] in the case $R = \mathbb{Z}$, though the proof goes through in exactly the same way after replacing the ring \mathbb{Z} by an arbitrary ring R .

THEOREM 6.5 (Bieri–Renz). *Let G be a finitely generated group and let $N \triangleleft G$ be a normal subgroup containing the commutator subgroup $[G, G]$. Let R be a unital ring and let M be an*

RG-module of type $\text{FP}_n(RG)$. Then $M \in \text{FP}_n(RN)$ if and only if $\Sigma_R^n(G; M) \supseteq S(G, N) := \{[\chi] \in S(G) : \chi(N) = 0\}$.

We will also need the following result due to Kielak and Jaikin-Zapirain. Kielak first proved the result in [Kie20c, Theorem 5.2] by giving an explicit construction of the Linnell skew-field $\mathcal{D}(G)$ when G is RFRS. In the appendix to [JZ21], Jaikin-Zapirain showed that when G is RFRS and \mathbb{F} is any skew-field, then $\mathcal{D}_{\mathbb{F}G}$ exists and admits a completely analogous construction to $\mathcal{D}(G)$ (in fact, $\mathcal{D}(G) = \mathcal{D}_{\mathbb{Q}G}$ for a RFRS group G). As a consequence, Kielak’s proof of Theorem 6.6 still holds after making the replacements $\mathbb{Q} \rightsquigarrow \mathbb{F}$ and $\mathcal{D}(G) \rightsquigarrow \mathcal{D}_{\mathbb{F}G}$.

THEOREM 6.6 (Kielak and Jaikin-Zapirain). *Let \mathbb{F} be a skew-field, G a finitely generated RFRS group, and $n \in \mathbb{N}$. Let F_\bullet be a chain complex of free $\mathbb{F}G$ -modules with F_p finitely generated and $H_p(\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_\bullet) = 0$ for all $p \leq N$. Then, there exist a finite index subgroup $H \leq G$ and an open subset $U \subseteq S(H)$ such that:*

- (1) *the closure of U contains $S(G)$;*
- (2) *U is invariant under the antipodal map;*
- (3) *$H_p(\widehat{\mathbb{F}H}^X \otimes_{\mathbb{F}H} F_\bullet) = 0$ for every $p \leq n$ and every $[\chi] \in U$.*

We are now ready to prove the main theorem.

THEOREM 6.7. *Let \mathbb{F} be a skew-field and let G be a virtually RFRS group of type $\text{FP}_n(\mathbb{F})$. Then there is a finite index subgroup $H \leq G$ admitting a homomorphism onto \mathbb{Z} with kernel of type $\text{FP}_n(\mathbb{F})$ if and only if $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$ for $p = 0, \dots, n$.*

Proof. (\Rightarrow) Let $\varphi: H \rightarrow \mathbb{Z}$ be an epimorphism with kernel $K \in \text{FP}_n(\mathbb{F})$. Then there is a free resolution $F_\bullet \rightarrow \mathbb{F} \rightarrow 0$ of the trivial $\mathbb{F}K$ -module \mathbb{F} with finitely generated n -skeleton. Therefore,

$$b_p^{\mathcal{D}_{\mathbb{F}K}}(K) = \dim_{\mathcal{D}_{\mathbb{F}K}} H_p(\mathcal{D}_{\mathbb{F}K} \otimes_{\mathbb{F}K} F_\bullet) < \infty$$

for $p \leq n$ and we have a short exact sequence $1 \rightarrow K \rightarrow H \rightarrow \mathbb{Z} \rightarrow 1$, so $b_p^{\mathcal{D}_{\mathbb{F}H}}(H) = 0$ for $p \leq n$ by Theorem 6.4. Then $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$ for $p \leq n$ by Lemma 6.3.

(\Leftarrow) The properties of being of type $\text{FP}_n(\mathbb{F})$ and of having vanishing p th $\mathcal{D}_{\mathbb{F}G}$ -Betti number pass to finite index subgroups. Moreover, the property of being virtually fibred passes to finite index overgroups (the same is, of course, true of any virtual property). Hence, we may assume that G is RFRS and of type $\text{FP}_n(\mathbb{F})$. Then $H^1(G; \mathbb{R}) \neq 0$ by Proposition 2.7.

Since $G \in \text{FP}_n(\mathbb{F})$, there is a free resolution $F_\bullet \rightarrow \mathbb{F} \rightarrow 0$ of the trivial $\mathbb{F}G$ -module \mathbb{F} with F_p finitely generated for $p \leq n$. By assumption, $H_p(\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} F_\bullet) = 0$ for $p \leq n$. By Theorem 6.6, there is a finite index subgroup $H \leq G$ and an open subset $U \subseteq H^1(H; \mathbb{R})$ such that the closure of U contains $H^1(G; \mathbb{R})$, is invariant under nonzero scalar multiplication, and $H_p(\widehat{\mathbb{F}H}^\phi \otimes_{\mathbb{F}H} F_\bullet) = 0$ for $p \leq n$ and all $\phi \in U$. Since U is nonempty, we can find a surjective character $\varphi: H \rightarrow \mathbb{Z}$ in U . To see this, let $\varphi \in H^1(G; \mathbb{R})$ be a nontrivial character. Since U is open, φ can be perturbed so that its image is in \mathbb{Q} . Finally, since H is finitely generated, we can rescale the character so that it maps H onto \mathbb{Z} . Then $[\pm\varphi] \in \Sigma_{\mathbb{F}}^n(H; \mathbb{F})$ by Theorem 5.3. It is not hard to show that $[\chi] = [\pm\varphi]$ for any character $\chi: H \rightarrow \mathbb{R}$ with $\ker \varphi \subseteq \ker \chi$. Thus, $\ker \varphi \in \text{FP}_n(\mathbb{F})$ by Theorem 6.5. \square

COROLLARY 6.8. *Let G be virtually RFRS and of type $\text{FP}(\mathbb{F})$. Then G virtually algebraically fibres with kernel of type $\text{FP}(\mathbb{F})$ if and only if G is $\mathcal{D}_{\mathbb{F}G}$ -acyclic.*

Proof. One direction is clear. If G is $\mathcal{D}_{\mathbb{F}G}$ -acyclic, then G virtually algebraically fibres with kernel K of type $\text{FP}_n(\mathbb{F})$ for $n > \text{cd}_{\mathbb{F}}(G)$. However, then $n > \text{cd}_{\mathbb{F}}(K)$, so K is of type $\text{FP}(\mathbb{F})$. \square

We now apply Theorem 6.7 to the case $\mathbb{F} = \mathbb{Q}$.

DEFINITION 6.9 (ℓ^2 -Betti numbers). Let G be a torsion-free group satisfying the Atiyah conjecture and let $\mathcal{D}(G)$ be the Linnell skew-field of G (see the following remark). Define

$$b_p^{(2)}(G) = \dim_{\mathcal{D}(G)} \operatorname{Tor}_p^{\mathbb{Q}G}(\mathcal{D}(G), \mathbb{Q})$$

to be the p th ℓ^2 -Betti number of G .

If a group G has a torsion-free subgroup H of finite index satisfying the Atiyah conjecture, we extend the definition of ℓ^2 -Betti numbers by declaring that

$$b_n^{(2)}(G) = \frac{b_n^{(2)}(H)}{[G : H]}.$$

REMARK 6.10. This definition of ℓ^2 -Betti numbers for torsion-free groups satisfying the Atiyah conjecture agrees with the usual definition by [Lüc02, Lemma 10.28(3)]. Moreover, ℓ^2 -Betti numbers for virtually torsion-free groups are well-defined and coincide with the usual definition by [Lüc02, Theorem 6.54(6)]. We will not give the definition of the Linnell ring $\mathcal{D}(G)$ since that would take us too far afield. We take the Atiyah conjecture to be the statement that $\mathcal{D}(G)$ is a skew-field, which allows us to make Definition 6.9. Linnell showed that this formulation implies the strong Atiyah conjecture over \mathbb{Q} for torsion-free groups [Lin93]. The details of the reverse implication can be found in Lück’s book [Lüc02, § 10].

In the case where G is finitely generated and RFRS, $\mathcal{D}_{\mathbb{Q}G}$ exists and is isomorphic to the Linnell skew-field $\mathcal{D}(G)$ by Jaikin-Zapirain’s appendix to [JZ21]. Then, the ℓ^2 -Betti numbers of G are defined and $b_p^{(2)}(G) = b_p^{\mathcal{D}_{\mathbb{Q}G}}(G)$. Hence, applying Theorem 6.7 to the case $\mathbb{F} = \mathbb{Q}$ yields the following result stated in the introduction.

THEOREM 6.11. Let G be a virtually RFRS group of type $\operatorname{FP}_n(\mathbb{Q})$. Then there is a finite index subgroup $H \leq G$ admitting a homomorphism onto \mathbb{Z} with kernel of type $\operatorname{FP}_n(\mathbb{Q})$ if and only if $b_p^{(2)}(G) = 0$ for $p = 0, \dots, n$.

Thanks to Jaikin-Zapirain’s work on rank functions in [JZ21], we can give a third characterisation of algebraic fibring. First, we set up some notation and terminology. If R is a ring and $\varphi: R \rightarrow \mathcal{D}$ is a division R -ring, then there is a natural rank function on matrices over R , which we denote $\operatorname{rk}_{\mathcal{D}, \varphi}$ or simply $\operatorname{rk}_{\mathcal{D}}$ when the map φ is understood. If φ is epic, meaning that $\varphi(R)$ generates \mathcal{D} as a division ring, then we call \mathcal{D} universal if the $\operatorname{rk}_{\mathcal{D}} \geq \operatorname{rk}_{\mathcal{E}}$ for every division R -ring $\psi: R \rightarrow \mathcal{E}$. Note that if a universal division R -ring exists, then it is unique up to R -isomorphism by a result of Cohn [Coh95, Theorem 4.4.1]. If, additionally, φ is an injection, then we call \mathcal{D} the universal division ring of fractions for R .

THEOREM 6.12 (Jaikin-Zapirain [JZ21, Corollary 1.3]). If G is a residually (locally indicable and amenable) group and \mathbb{F} is a skew-field, then the Hughes-free division ring $\mathcal{D}_{\mathbb{F}G}$ exists and is the universal division ring of fractions for $\mathbb{F}G$.

Following Jaikin-Zapirain, whenever G has a Hughes-free division ring $\mathcal{D}_{\mathbb{F}G}$, we denote the rank function $\operatorname{rk}_{\mathcal{D}_{\mathbb{F}G}}$ by $\operatorname{rk}_{\mathbb{F}G}$. Note, in particular, that Theorem 6.12 holds for RFRS groups since they are residually poly- \mathbb{Z} (see, e.g., [JZ21, Proposition 4.4]). Therefore, we have the following corollary, which will be useful to us later.

COROLLARY 6.13. Let G be a RFRS group and let $\varphi: G \rightarrow \mathbb{Z}$ be a homomorphism. Let A be a matrix over $\mathbb{F}G$ and let $A^{\mathbb{Z}}$ be the matrix over $\mathbb{F}[\mathbb{Z}]$ obtained by applying φ to A . Then $\operatorname{rk}_{\mathbb{F}G} A \geq \operatorname{rk}_{\mathbb{F}[\mathbb{Z}]} A^{\mathbb{Z}}$.

Proof. There is a map $\bar{\varphi}: \mathbb{F}G \rightarrow \mathbb{F}[\mathbb{Z}] \hookrightarrow \mathcal{D}_{\mathbb{F}[\mathbb{Z}]}$ induced by φ and, therefore, $\operatorname{rk}_{\mathbb{F}G} A \geq \operatorname{rk}_{\mathcal{D}_{\mathbb{F}[\mathbb{Z}]}, \bar{\varphi}} A = \operatorname{rk}_{\mathbb{F}[\mathbb{Z}]} A^{\mathbb{Z}}$ by universality of $\mathcal{D}_{\mathbb{F}G}$. \square

The following theorem generalises Corollary 1.5 of [JZ21], where the result is proven for $n = 1$. In the proof, if M is a finitely generated $\mathbb{F}[\mathbb{Z}]$ -module, then we define $\dim M := \dim_{\mathcal{D}_{\mathbb{F}[\mathbb{Z}]}}(M \otimes_{\mathbb{F}[\mathbb{Z}]} \mathcal{D}_{\mathbb{F}[\mathbb{Z}]})$. We will also write rk_G instead of $\text{rk}_{\mathbb{F}G}$ to lighten the notation.

THEOREM 6.14. *Let \mathbb{F} be a skew-field and let G be a virtually RFRS group of type $\text{FP}_n(\mathbb{F})$. The following are equivalent:*

- (1) *there is a finite index subgroup $H_0 \leq G$ and a surjection $\varphi_0: H_0 \rightarrow \mathbb{Z}$ with $\ker \varphi_0$ of type $\text{FP}_n(\mathbb{F})$;*
- (2) *there is a finite index subgroup $H_1 \leq G$ and a surjection $\varphi_1: H_1 \rightarrow \mathbb{Z}$ with $b_p(\ker \varphi_1; \mathbb{F}) < \infty$ for $p = 0, \dots, n$.*

Proof. If G algebraically fibres with kernel K of type $\text{FP}_n(\mathbb{F})$, then there is a free resolution $F_\bullet \rightarrow \mathbb{F} \rightarrow 0$ of the trivial $\mathbb{F}K$ module \mathbb{F} such that F_p is finitely generated for all $p \leq n$. This resolution can be used to compute the homology of K and, therefore, $b_p(K; \mathbb{F}) < \infty$ for all $p \leq n$.

In view of Theorem 6.7, to prove the converse it suffices to show that $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$ for all $p \leq n$. By multiplicativity of $\mathcal{D}_{\mathbb{F}G}$ -Betti numbers (Lemma 6.3), we may assume that $H_1 = G$. Let $K = \ker \varphi_1$ and write $\mathbb{F}[\mathbb{Z}]$ for the group algebra $\mathbb{F}[G/K]$. Moreover, note that $H_p(K; \mathbb{F}) \cong H_p(G; \mathbb{F}[\mathbb{Z}])$ for all p . Let

$$\dots \rightarrow \mathbb{F}G^{d_p} \rightarrow \dots \rightarrow \mathbb{F}G^{d_0} \rightarrow \mathbb{F} \rightarrow 0$$

be a free resolution of the trivial $\mathbb{F}G$ -module \mathbb{F} , where d_p is some cardinal for each p and d_p is finite for each $p \leq n$, and we use the (non-standard) notation $\mathbb{F}G^{d_p}$ to denote the d_p -fold direct sum of $\mathbb{F}G$, as opposed to the d_p -fold direct product. The quotient map $G \rightarrow \mathbb{Z}$ induces a chain map

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_{n+2}} & \mathbb{F}G^{d_{n+1}} & \xrightarrow{\partial_{n+1}} & \mathbb{F}G^{d_n} & \xrightarrow{\partial_n} & \dots & \xrightarrow{\partial_1} & \mathbb{F}G^{d_0} & \xrightarrow{\partial_0} & 0 \\ & & \downarrow & & \downarrow & & & & \downarrow & & \\ \dots & \xrightarrow{\partial_{n+2}^{\mathbb{Z}}} & \mathbb{F}[\mathbb{Z}]^{d_{n+1}} & \xrightarrow{\partial_{n+1}^{\mathbb{Z}}} & \mathbb{F}[\mathbb{Z}]^{d_n} & \xrightarrow{\partial_n^{\mathbb{Z}}} & \dots & \xrightarrow{\partial_1^{\mathbb{Z}}} & \mathbb{F}[\mathbb{Z}]^{d_0} & \xrightarrow{\partial_0^{\mathbb{Z}}} & 0, \end{array}$$

where the boundary maps are viewed as matrices and $\partial_p^{\mathbb{Z}}$ is obtained by applying the map $G \rightarrow \mathbb{Z}$ to each entry of the matrix ∂_p . Note that the homology of the bottom chain complex is $H_\bullet(G; \mathbb{Q}[\mathbb{Z}])$.

To apply Jaikin-Zapirain’s results on rank functions, we will need the boundary maps to be between finitely generated free modules. However, d_{n+1} is not finite in general, so we must modify the chain complexes as follows. Since $\mathbb{F}[\mathbb{Z}]$ is Noetherian and $\mathbb{F}[\mathbb{Z}]^{d_n}$ is finitely generated, $\text{im } \partial_{n+1}^{\mathbb{Z}}$ is a finitely generated submodule of $\mathbb{F}[\mathbb{Z}]^{d_n}$. The preimage of a finite generating set of $\text{im } \partial_{n+1}^{\mathbb{Z}}$ is contained in a finitely generated free summand F of $\mathbb{F}[\mathbb{Z}]^{d_{n+1}}$. Note that the homology of

$$F \rightarrow \mathbb{F}[\mathbb{Z}]^{d_n} \rightarrow \dots \rightarrow \mathbb{F}[\mathbb{Z}]^{d_0} \rightarrow 0$$

is still $H_p(G; \mathbb{F}[\mathbb{Z}])$ for $p \leq n$. The preimage of F in $\mathbb{F}G^{d_{n+1}}$ is again a finitely generated free summand \widehat{F} of $\mathbb{F}G^{d_{n+1}}$. Note that it suffices to show that the homology of

$$\mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{Q}G} \widehat{F} \rightarrow \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} \mathbb{F}G^{d_n} \rightarrow \dots \rightarrow \mathcal{D}_{\mathbb{F}G} \otimes_{\mathbb{F}G} \mathbb{F}G^{d_0} \rightarrow 0$$

vanishes in degrees $\leq n$ to show that $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$ for all $p \leq n$.

We assume that d_{n+1} is finite and that $F = \mathbb{F}[\mathbb{Z}]^{d_{n+1}}$ and $\widehat{F} = \mathbb{F}G^{d_{n+1}}$. Since, for every $p \leq n$, the homology $H_p(G; \mathbb{F}[\mathbb{Z}])$ is finite-dimensional as an \mathbb{F} -vector space, it must be torsion as an $\mathbb{F}[\mathbb{Z}]$ -module. Therefore, $\text{rk}_{\mathbb{Z}} \partial_{p+1}^{\mathbb{Z}} = \dim \ker \partial_p^{\mathbb{Z}}$ for every $p \leq n$. Now, for each $p \leq n$, we have

short exact sequences

$$0 \rightarrow \ker \partial_p^{\mathbb{Z}} \rightarrow \mathbb{F}G^{d_p} \rightarrow \text{im } \partial_p^{\mathbb{Z}} \rightarrow 0,$$

which implies that $d_p = \dim \ker \partial_p^{\mathbb{Z}} + \text{rk}_{\mathbb{Z}} \partial_p^{\mathbb{Z}} = \text{rk}_{\mathbb{Z}} \partial_{p+1}^{\mathbb{Z}} + \text{rk}_{\mathbb{Z}} \partial_p^{\mathbb{Z}}$. Hence,

$$\begin{aligned} d_p - \text{rk}_G \partial_p &= \dim_{\mathcal{D}_{\mathbb{F}G}} \ker(\mathcal{D}_{\mathbb{F}G}^{d_p} \xrightarrow{\partial_p} \mathcal{D}_{\mathbb{F}G}^{d_{p-1}}) \\ &\geq \text{rk}_G \partial_{p+1} \\ &\geq \text{rk}_{\mathbb{Z}} \partial_{p+1}^{\mathbb{Z}} \\ &= d_p - \text{rk}_{\mathbb{Z}} \partial_p^{\mathbb{Z}} \\ &\geq d_p - \text{rk}_G \partial_p, \end{aligned}$$

where we have used Corollary 6.13. Thus,

$$\text{rk}_G \partial_{p+1} = \dim_{\mathcal{D}_{\mathbb{F}G}} \ker(\mathcal{D}_{\mathbb{F}G}^{d_p} \xrightarrow{\partial_p} \mathcal{D}_{\mathbb{F}G}^{d_{p-1}}),$$

and therefore $b_p^{\mathcal{D}_{\mathbb{F}G}}(G) = 0$ for all $p \leq n$. □

COROLLARY 6.15. *Let G be a virtually RFRS group and let $n \in \mathbb{N}$.*

- (1) *If \mathbb{F} and \mathbb{F}' are skew-fields of the same characteristic, then G virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F})$ if and only if it virtually algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F}')$.*
- (2) *If p is a prime such that G algebraically fibres with kernel of type $\text{FP}_n(\mathbb{F}_p)$, then it fibres with kernel of type $\text{FP}_n(\mathbb{Q})$.*

Proof. Item (1) follows from the fact that the Betti numbers of a group with trivial skew-field coefficients depend only on the characteristic of the skew-field. Item (2) follows from the fact that $b_k(G; \mathbb{F}_p) \geq b_k(G; \mathbb{Q})$ for any group G and any prime p (this is a consequence of the universal coefficient theorem). □

7. Applications

7.1 Amenable RFRS groups

DEFINITION 7.1. A group G is *amenable* if for every continuous G -action on a compact, Hausdorff space X , there is a G -invariant probability measure on X .

DEFINITION 7.2 (Elementary amenable groups). The class \mathcal{E} of *elementary amenable* groups is the smallest class such that:

- \mathcal{E} contains all finite groups and all Abelian groups;
- if $G \in \mathcal{E}$ then the entire isomorphism class of G is contained in \mathcal{E} ;
- \mathcal{E} is closed under taking subgroups, quotients, extensions, and directed unions.

All elementary amenable groups are amenable, however there are many examples of amenable groups that are not elementary amenable, the earliest being Grigorchuk’s group of intermediate growth [Gri80]. For a discussion of more examples, we refer the reader to the introduction of Juschenko’s paper [Jus18]. The known examples of non-elementary amenable groups all have infinite cohomological dimension over any field. Moreover, elementary amenable groups of finite cohomological dimension over \mathbb{Z} are virtually solvable by [Hil91, Lemma 2] and [HL92, Corollary 1]. We are led to the following question, which was stated in the introduction.

QUESTION 7.3. Are amenable groups of finite cohomological dimension over \mathbb{Z} virtually solvable?

We obtain a partial answer in the positive direction as an application of Theorem 6.11, and extends the well-known fact that nilpotent RFRS groups are Abelian. The author thanks Sami Douba for pointing out the fact (and the proof) that polycyclic RFRS groups are virtually Abelian, which allows us to strengthen the following theorem.

THEOREM 7.4. *If G is an amenable RFRS group of type $\text{FP}(\mathbb{Q})$, then G is virtually Abelian.*

Proof. Since G is amenable, $b_p^{\mathcal{D}_{\mathbb{Q}G}}(G) = 0$ for all p by [Lüc02, Theorem 7.2(1)]. By Theorem 6.11 we obtain a finite index subgroup $H \leq G$ and an epimorphism $\varphi: H \rightarrow \mathbb{Z}$ such that $N = \ker \varphi \in \text{FP}_n(\mathbb{Q})$. It will be necessary to require that H be normal in G , which is not an issue since we can replace H with its normal core. Since $\text{cd}_{\mathbb{Q}} N \leq \text{cd}_{\mathbb{Q}} G \leq n$, we also have $N \in \text{FP}(\mathbb{Q})$ (see [Bro94, Proposition VIII.6.1]). Because we have a short exact sequence

$$1 \rightarrow N \rightarrow H \rightarrow \mathbb{Z} \rightarrow 1$$

with $N \in \text{FP}(\mathbb{Q})$, a theorem of Fel'dman [Fel71, Theorem 2.4] (see also [Bie76, Proposition 2.5]) gives $\text{cd}_{\mathbb{Q}} N = \text{cd}_{\mathbb{Q}} H - \text{cd}_{\mathbb{Q}} \mathbb{Z} = n - 1$.

Since subgroups of amenable RFRS groups are amenable and RFRS, we can repeat the argument above with N instead of G . Iterating this process, we obtain a subnormal series

$$G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

of G such that $\text{cd}_{\mathbb{Q}} G_j = j$ for each j (note that $N = G_{n-1}$ here). However, the only torsion-free group of cohomological dimension 0 over \mathbb{Q} is the trivial group, so we conclude that G is polycyclic-by-finite.

Let $G' \leq G$ be a polycyclic group of finite index. We will show, by induction on the Hirsch length of G' , that G' must be virtually Abelian. Indeed, by induction we may assume that $G' \cong \mathbb{Z}^m \rtimes_{\psi} \mathbb{Z}$. If ψ is of finite order, then G' is virtually Abelian. If ψ is of infinite order, then the argument of [BH99, Proposition 4.17] shows that \mathbb{Z}^m cannot be quasi-isometrically embedded in G' . However, Proposition 2.7(2) implies that all Abelian subgroups of a finitely generated RFRS group are undistorted, a contradiction. \square

Remark 7.5. The assumption that G be RFRS is necessary. For example, the Baumslag–Solitar group $\text{BS}(1, n)$, with $n > 1$, is locally indicable, solvable, and of finite type, but it is not virtually Abelian (nor is it polycyclic-by-finite). The assumption that G be (virtually) of type $\text{FP}(\mathbb{Q})$ is also necessary; for example, $\mathbb{Z} \wr \mathbb{Z}$ is RFRS and amenable, yet it is not virtually Abelian.

Baer's conjecture states that if G is a group with a Noetherian group ring $\mathbb{Z}G$, then G is a polycyclic-by-finite group. The author is grateful to Sam Hughes for pointing out the following consequence of Theorem 7.4, which resolves a special case of Baer's conjecture.

COROLLARY 7.6. *Let G be a virtually RFRS group of type $\text{FP}(\mathbb{Q})$. If $\mathbb{Z}G$ is Noetherian, then G is virtually Abelian.*

Proof. It is enough to prove the claim in the case that G is RFRS. In this case, $\mathbb{Z}G$ embeds into the Linnell skew-field $\mathcal{D}(G)$, and therefore $\mathbb{Z}G$ is a domain. Since Noetherian domains are Ore domains [MR01, p. 47], we have that $\mathbb{Z}G$ is an Ore domain. It then follows that $\mathbb{Q}G$ is an Ore domain, which implies that G is amenable by Kielak's appendix to [Bar19]. Then G is polycyclic-by-finite by Theorem 7.4. \square

7.2 Arithmetic lattices and subgroups of hyperbolic groups

Recall that hyperbolic groups are always of type F_{∞} (and of type F if they are torsion-free). However, their subgroups can exhibit many interesting finiteness properties. In [Rip82], Rips gave the first example of an incoherent hyperbolic group; phrased in terms of finiteness properties,

this gives an example of a hyperbolic group with a subgroup that is of type F_1 but not F_2 . In [Bra99], Brady constructed a type F_2 subgroup of a hyperbolic group that is not of type F_3 ; this provided the first example of a finitely presented non-hyperbolic subgroup of a hyperbolic group. In the same paper, Brady asked whether there are subgroups of hyperbolic groups that are of type F_n but not F_{n+1} for all n . More examples of F_2 -not- F_3 subgroups of hyperbolic groups were provided by Kropholler [Kro21] and Lodha [Lod18], and in [LMP21], Llosa Isenrich, Martelli and Py constructed the first examples of F_3 -not- F_4 subgroups of hyperbolic groups. This result was extended in a subsequent paper [LP24], where Llosa Isenrich and Py completely answer Brady's question by exhibiting cocompact hyperbolic arithmetic lattices in $\mathrm{PU}(n, 1)$ with F_{n-1} -not- F_n subgroups for all n . We also mention the related work of Italiano, Martelli and Migliorini, who constructed the first example of a non-hyperbolic type F subgroup of a hyperbolic group [IMM23].

In [LMP21], Llosa Isenrich, Martelli and Py remark that one can use Theorem 6.11 to show that there are subgroups of hyperbolic lattices in $\mathrm{PO}(2n, 1)$ that are of type $\mathrm{FP}_{n-1}(\mathbb{Q})$ but not $\mathrm{FP}_n(\mathbb{Q})$. We record their argument here, and also mention that the same line of reasoning shows that there are many hyperbolic lattices in $\mathrm{PO}(2n + 1, 1)$ that virtually fibre with kernel of type $\mathrm{FP}(\mathbb{Q})$.

PROPOSITION 7.7 [LMP21, Proposition 19]. *Let $\Gamma < \mathrm{PO}(n, 1)$ be a cocompact cubulable lattice. If $n = 2k$ is even, then Γ virtually fibres with kernel of type $\mathrm{FP}_{k-1}(\mathbb{Q})$ but not $\mathrm{FP}_k(\mathbb{Q})$. If n is odd, then Γ virtually fibres with kernel of type $\mathrm{FP}(\mathbb{Q})$.*

Remark 7.8. In [BHW11], Bergeron, Haglund and Wise show that any standard arithmetic subgroup of $\mathrm{PO}(n, 1)$ is cubulated, so Proposition 7.7 applies to a nonempty class of groups. We refer the reader to their paper for the definition of standard.

Proof. By Agol's theorem [Ago13, Theorem 1.1], Γ is virtually special and in particular virtually RFRS. By, for instance, [Kam14, Theorem 3.3], the ℓ^2 -Betti numbers of lattices in semisimple Lie groups vanish except in the middle dimension, where they are nonzero. If $n = 2k$, then Γ virtually fibres with kernel of type $\mathrm{FP}_{k-1}(\mathbb{Q})$, but $b_k^{(2)}(\Gamma) \neq 0$ so the kernel cannot be of type $\mathrm{FP}_n(\mathbb{Q})$ by Theorem 6.7. If n is odd, then Γ is ℓ^2 -acyclic and, therefore, virtually fibres with kernel of type $\mathrm{FP}(\mathbb{Q})$ in this case, where we have also used Corollary 6.8. \square

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CONFLICTS OF INTEREST

None.

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REFERENCES

- Ago08 I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), 269–284.
- Ago13 I. Agol, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087, with an appendix by I. Agol, Daniel Groves, and Jason Manning.
- Bar19 L. Bartholdi, *Amenability of groups is characterized by Myhill’s theorem*, J. Eur. Math. Soc. (JEMS) **21** (2019), 3191–3197, with an appendix by Dawid Kielak.
- BHW11 N. Bergeron, F. Haglund and D. T. Wise, *Hyperplane sections in arithmetic hyperbolic manifolds*, J. Lond. Math. Soc. (2) **83** (2011), 431–448.
- Bie76 R. Bieri, *Normal subgroups in duality groups and in groups of cohomological dimension 2*, J. Pure Appl. Algebra **7** (1976), 35–51.
- Bie07 R. Bieri, *Deficiency and the geometric invariants of a group*, J. Pure Appl. Algebra **208** (2007), 951–959, with an appendix by Pascal Schweitzer.
- BE74 R. Bieri and B. Eckmann, *Finiteness properties of duality groups*, Comment. Math. Helv. **49** (1974), 74–83.
- BNS87 R. Bieri, W. D. Neumann and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. **90** (1987), 451–477.
- BR88 R. Bieri and B. Renz, *Valuations on free resolutions and higher geometric invariants of groups*, Comment. Math. Helv. **63** (1988), 464–497.
- Bra99 N. Brady, *Branched coverings of cubical complexes and subgroups of hyperbolic groups*, J. Lond. Math. Soc. (2) **60** (1999), 461–480.
- BH99 M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319 (Springer, Berlin, 1999).
- Bro94 K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87 (Springer, New York, 1994). Corrected reprint of the 1982 original.
- Coh95 P. M. Cohn, *Skew fields: Theory of general division rings*, Encyclopedia of Mathematics and its Applications, vol. 57 (Cambridge University Press, Cambridge, 1995).
- Deg16 D. Degrijse, *Amenable groups of finite cohomological dimension and the zero divisor conjecture*, Preprint (2016), [arXiv:1609.07635](https://arxiv.org/abs/1609.07635).
- Fel71 G. L. Fel’dman, *The homological dimension of group algebras of solvable groups*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1225–1236.
- Grä20 J. Gräter, *Free division rings of fractions of crossed products of groups with Conradian left-orders*, Forum Math. **32** (2020), 739–772.
- Gri80 R. I. Grigorčuk, *On Burnside’s problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), 53–54.
- Hal59 P. Hall, *On the finiteness of certain soluble groups*, Proc. Lond. Math. Soc. (3) **9** (1959), 595–622.
- Hil91 J. A. Hillman, *Elementary amenable groups and 4-manifolds with Euler characteristic 0*, J. Aust. Math. Soc. Ser. A **50** (1991), 160–170.
- HL92 J. A. Hillman and P. A. Linnell, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable*, J. Aust. Math. Soc. Ser. A **52** (1992), 237–241.
- Hug70 I. Hughes, *Division rings of fractions for group rings*, Comm. Pure Appl. Math. **23** (1970), 181–188.

- IMM23 G. Italiano, B. Martelli and M. Migliorini, *Hyperbolic 5-manifolds that fiber over S^1* , *Invent. Math.* **231** (2023), 1–38.
- JZ21 A. Jaikin-Zapirain, *The universality of Hughes-free division rings*, *Selecta Math. (N.S.)* **27** (2021), article 74.
- Jus18 K. Juschenko, *Non-elementary amenable subgroups of automata groups*, *J. Topol. Anal.* **10** (2018), 35–45.
- Kam14 H. Kammeyer, *L^2 -invariants of nonuniform lattices in semisimple Lie groups*, *Algebr. Geom. Topol.* **14** (2014), 2475–2509.
- Kie20a D. Kielak, *The Bieri-Neumann-Strebel invariants via Newton polytopes*, *Invent. Math.* **219** (2020), 1009–1068.
- Kie20b D. Kielak, *Fibring over the circle via group homology*, in *Manifolds and groups*, *Oberwolfach Reports*, vol. 17 (European Mathematical Society, 2020), 428–431.
- Kie20c D. Kielak, *Residually finite rationally solvable groups and virtual fibring*, *J. Amer. Math. Soc.* **33** (2020), 451–486.
- KS20 T. Koberda and A. I. Suciuc, *Residually finite rationally p groups*, *Commun. Contemp. Math.* **22** (2020), 1950016.
- KL19 P. Kropholler and K. Lorensen, *Group-graded rings satisfying the strong rank condition*, *J. Algebra* **539** (2019), 326–338.
- Kro21 R. Kropholler, *Hyperbolic groups with finitely presented subgroups not of type F_3* , *Geom. Dedicata* **213** (2021), 589–619.
- Lin93 P. A. Linnell, *Division rings and group von Neumann algebras*, *Forum Math.* **5** (1993), 561–576.
- LMP21 C. Llosa Isenrich, B. Martelli and P. Py, *Hyperbolic groups containing subgroups of type \mathcal{F}_3 not \mathcal{F}_4* , *J. Differential Geom.* **127** (2024), 1121–1147.
- LP24 C. Llosa Isenrich and P. Py, *Subgroups of hyperbolic groups, finiteness properties and complex hyperbolic lattices*, *Invent. Math.* **235** (2024), 233–254.
- Lod18 Y. Lodha, *A hyperbolic group with a finitely presented subgroup that is not of type FP_3* , in *Geometric and cohomological group theory*, *London Mathematical Society Lecture Note series*, vol. 444 (Cambridge University Press, Cambridge, 2018), 67–81.
- Lüc02 W. Lück, *L^2 -invariants: theory and applications to geometry and K -theory* (Springer, Berlin, 2002).
- MR01 J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, revised edition, *Graduate Studies in Mathematics*, vol. 30 (American Mathematical Society, Providence, RI, 2001), with the cooperation of L. W. Small.
- Min21 A. Minasyan, *Virtual retraction properties in groups*, *Int. Math. Res. Not. IMRN* **2021** (2021), 13434–13477.
- Pas77 D. S. Passman, *The algebraic structure of group rings*, *Pure and Applied Mathematics* (Wiley, 1977).
- Rip82 E. Rips, *Subgroups of small cancellation groups*, *Bull. Lond. Math. Soc.* **14** (1982), 45–47.
- Sik87 J.-C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie de degré un*, *Thèse d'État*, Université Paris-Sud (Orsay) (1987).
- Sta62 J. Stallings, *On fibering certain 3-manifolds*, in *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)* (Prentice-Hall, Englewood Cliffs, NJ, 1962), 95–100.

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