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RESEARCH ARTICLE

K-stability of birationally superrigid Fano 3-fold weighted hypersurfaces

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Abstract
We prove that the alpha invariant of a quasi-smooth Fano 3-fold weighted hypersurface of index 1 is greater than or equal to $1/2$. Combining this with the result of Stibitz and Zhuang [SZ19] on a relation between birational superrigidity and K-stability, we prove the K-stability of a birationally superrigid quasi-smooth Fano 3-fold weighted hypersurfaces of index 1.

Contents

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1. Introduction

Throughout the article, the ground field is assumed to be the complex number field $\mathbb{C}$.

1.1. K-stability, birational superrigidity and a conjecture

The notion of K-stability was introduced by Tian [Tia97] as an attempt to characterize the existence of Kähler–Einstein metrics (KE metrics, for short) on Fano manifolds. Later, K-stability was extended and reformulated by Donaldson [Don02] in algebraic terms. The notion of K-stability emerged in the study of KE metrics (see [Don02], [Tia97]), and it gives a characterization of the existence of a KE metric for smooth Fano manifolds (see [CDS15], [Tia15]).

Birational (super)rigidity means the uniqueness of a Mori fiber space in the birational equivalence class (see Definition 2.2), and this notion has its origin in the rationality problem of Fano varieties. Specifically, it grew out of the study of birational self-maps of smooth quartic 3-folds by Iskovskikh and Manin [IM71] (see [Puk13] and [Che05] for surveys).

K-stability and birational superrigidity have completely different origins, and we are unable to find a similarity in their definitions. However, both of them are closely related to some mildness of singularities of pluri-anticanonical divisors (or linear systems). For example, it is proved by Odaka and Sano [OS12] (see also [Tia87]) that a Fano variety X of dimension n is K-stable if $\alpha(X) > n/(n+1)$. Here,

$$\alpha(X) = \sup\{\, c \in \mathbb{Q}_{>0} \mid (X, cD) \text{ is log canonical for any } D \in |-K_X|_{\mathbb{Q}} \,\}$$

is called the *alpha invariant* of X and it measures singularities of pluri-anticanonical divisors. We refer readers to [Fuj19b], [Li17], [FO18] and [BJ20] for criteria for K-stability in terms of beta and delta invariants which are more or less related to singularities of pluri-anticanonical divisors. On the other hand, it is known that a Fano variety of Picard number one is birationally superrigid if and only if the pair $(X, \lambda\mathcal{M})$ is canonical for any $\lambda \in \mathbb{Q}_{>0}$ and any movable linear system $\mathcal{M}$ such that $\lambda\mathcal{M} \sim_{\mathbb{Q}} -K_X$ (see Theorem 2.4). With these relations in mind, one may expect a positive answer to the following.

Conjecture 1.1. *A birationally superrigid Fano variety is K-stable.*

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Actually, we expect stronger conjectures to hold (see Section 7.4 for discussions). The main aim of this article is to verify Conjecture 1.1 for quasi-smooth Fano 3-fold weighted hypersurfaces.

1.2. Evidences for the conjecture

1.2.a. Smooth Fano manifolds

Smooth quartic 3-folds and double covers of $\mathbb{P}^3$ branched along a smooth hypersurface of degree 6 (or equivalently smooth weighted hypersurfaces of degree 6 in $\mathbb{P}(1,1,1,1,3)$) are the only smooth Fano 3-fold that are birationally superrigid (see [IM71], [Isk80], [Che05]). K-stability (and hence the existence of a KE metric) is proved for smooth quartic 3-folds ([Fuj19a, Corollary 1.4]) and for smooth weighted hypersurfaces of degree 6 in $\mathbb{P}(1,1,1,1,3)$ ([CPW14, Corollary 3.4]).

We have evidences in arbitrary dimension $n \geq 3$. After the results established in low-dimensional cases in [IM71], [Puk87] and [dFEM03], it is finally proved by de Fernex [dF13] that any smooth hypersurface of degree $n+1$ in $\mathbb{P}^{n+1}$ is birationally superrigid for $n \geq 3$. On the other hand, it is proved by Fujita [Fuj19a] that any such hypersurface is K-stable (hence admits a KE metric). It is also proved in [Zhu20b] that a smooth Fano complete intersection $X \subset \mathbb{P}^{n+r}$ of Fano index 1, codimension r and dimension $n \geq 10r$ is birationally superrigid and K-stable.

1.2.b. Fano 3-fold weighted hypersurfaces

By a *quasi-smooth Fano* 3*-fold weighted hypersurface*, we mean a Fano 3-fold (with only terminal singularities) embedded as a quasi-smooth hypersurface in a well-formed weighted projective 4-space $\mathbb{P}(a_0, \dots, a_4)$ (see Section 2.2.b for quasi-smoothness and well-formedness). Let $X = X_d \subset \mathbb{P}(a_0, \dots, a_4)$ be a quasi-smooth Fano 3-fold weighted hypersurface of degree d. Then the class group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}$ and is generated by $\mathcal{O}_X(1)$ (see, for example, [Oka19, Remark 4.2]). By adjunction, we have $\mathcal{O}_X(-K_X) \cong \mathcal{O}_X(\iota_X)$, where

$$\iota_X := \sum_{i=0}^{4} a_i - d \in \mathbb{Z}_{>0}.$$

We call ι_X the *Fano index* (or simply *index*) of X.

By [IF00] and [CCC11], quasi-smooth Fano 3-fold weighted hypersurfaces of index 1 are classified and they consist of 95 families. Among them, quartic 3-folds and weighted hypersurfaces of degree 6 in $\mathbb{P}(1,1,1,1,3)$ are smooth and the remaining 93 families consist of singular Fano 3-folds (with terminal quotient singularities). The descriptions of these 93 families are given in Table 7.

Theorem 1.2 [CP17], [CPR00]. *Any quasi-smooth Fano* 3*-fold weighted hypersurface of index* 1 *is birationally rigid.*

Among the 95 families, any quasi-smooth member of each of specific 50 families is birationally superrigid. The 50 families consist of 48 families in Table 7 which do not admit singularity with ‘quadratic involution (QI)’ or ‘elliptic involution (EI)’ in the fourth column plus the two families of smooth Fano weighted hypersurfaces. For each of the remaining 45 families, a general quasi-smooth member is strictly birationally rigid (meaning that it is not birationally superrigid) but some special quasi-smooth members are birationally superrigid (see Section 2.3 for details).

Theorem 1.3 [Che09, Corollary 1.45]. *A general quasi-smooth member of each of the 95 families is K-stable and admits a KE metric.*

The generality assumption is crucial in Theorem 1.3. In particular, it is highly likely that birationally superrigid special members of each of the above mentioned 45 families are not treated in Theorem 1.3. Note that openness of K-stability is known (see [Oda13], [Don15] and [BL22]), and this implies the difficulty in determining which Fano varieties (in a given family) are K-stable. Although Theorems 1.2 and 1.3 give strong evidence for Conjecture 1.1, it is very important to consider special (quasi-smooth) members for Conjecture 1.1.

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1.2.c. Conceptual evidences

Apart from evidences by concrete examples given in Sections 1.2.a and 1.2.b, we have conceptual results supporting Conjecture 1.1.

The notion of slope stability for polarized varieties was introduced by Ross and Thomas (cf. [RT07]). For a Fano variety X, slope stability of $(X, -K_X)$ is a weaker version of K-stability.

Theorem 1.4 [OO13, Theorem 1.1]. *Let X be a birationally superrigid Fano manifold of Fano index* 1. *If $|-K_X|$ is base point free, then $(X, -K_X)$ is slope stable.*

As it is explained in Section 1.1, K-stability of a Fano variety X of dimension n follows from the inequality $\alpha(X) > n/(n+1)$. In practice, the computations of alpha invariants are very difficult and hence it is not easy to prove the inequality $\alpha(X) > n/(n+1)$.

Remark 1.5. In fact, our results show that there exists a birationally superrigid Fano 3-fold X such that $\alpha(X) < 3/4$ (see Example 5.17).

Recently, Stibitz and Zhuang relaxed the assumption on the alpha invariants significantly under the assumption of birational superrigidity and obtained the following.

Theorem 1.6 [SZ19, Theorem 1.2, Corollary 3.1]. *Let X be a birationally superrigid Fano variety. If $\alpha(X) \geq 1/2$, then X is K-stable.*

Note that the assumption on the alpha invariant is $\alpha(X) > 1/2$ in [SZ19, Theorem 1.2], but the equality is allowed by [SZ19, Corollary 3.1]. It is informed by C. Xu and Z. Zhuang that one can even conclude the uniform K-stability of X in Theorem 1.6 under the same assumption. The notion of uniform K-stability is originally introduced in [Der16b] and [BHJ17] (see also [Fuj19b] and [BJ20]) and it is stronger than K-stability[1]. Moreover, it is very important to mention that uniform K-stability implies the existence of a KE metric ([LTW19]). Combining these results, we have the following.

Theorem 1.7 [Xu21, Theorem 9.6], [SZ19], [LTW19]. *Let X be a birationally superrigid Fano variety, and assume that $\alpha(X) \geq 1/2$. Then X is uniformly K-stable. In particular, X is K-stable and it admits a KE metric.*

1.3. Main results

We state main theorem of this article.

Theorem 1.8 (Main theorem). *Let X be a quasi-smooth Fano* 3*-fold weighted hypersurface of index* 1. *Then $\alpha(X) \geq 1/2$.*

The following is a direct consequence of Theorems 1.8 and 1.7.

Corollary 1.9. *Any birationally superrigid quasi-smooth Fano* 3*-fold weighted hypersurface of index* 1 *is K-stable and admits a KE metric.*

By [ACP20, Corollary 1.3], a birationally superrigid quasi-smooth Fano 3-fold weighted hypersurface necessarily has Fano index 1. Thus, we obtain the following.

Corollary 1.10. *Conjecture 1.1 is true for quasi-smooth Fano 3-fold weighted hypersurfaces.*

It is natural to consider a generalization of Conjecture 1.1 by relaxing the assumption of birational superrigidity to birational rigidity (see Section 7.4) or to expect that the conclusion of Corollary 1.9 holds without the assumption of birational superrigidity. We are unable to relax the assumption of birational superrigidity to birational rigidity in Theorem 1.6 or 1.7, and thus we cannot conclude K-stability for strictly birationally rigid members as a direct consequence of Theorem 1.8. By the arguments delivered in this article, we are able to prove $\alpha(X) > 3/4$ for any quasi-smooth member X of suitable families. As a consequence, we have the following (see Section 7.3 for details).

[1] Soon after this paper was completed, it wad proved in [LXZ22] that uniform K-stability is equivalent to K-stability.

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Theorem 1.11 (= Theorem 7.7). *Let X be any quasi-smooth member of a family which is given 'KE' in the right-most column of Table 7. Then X is K-stable and admits a KE metric.*

We can also prove K-stability for any quasi-smooth member (which is not necessarily birationally superrigid) of suitable families.

Theorem 1.12 (= Corollary 7.13). *Let X be any quasi-smooth member of a family which is given 'K' or 'KE' in the right-most column of Table 7. Then X is K-stable.*

We explain the organization of this article. In Section 2, we recall definitions and basic properties of relevant notions such as birational (super)rigidity, log canonical thresholds, alpha invariants and weighted projective varieties. In Section 3, we explain methods of computing log canonical thresholds and alpha invariants. By applying these methods, we compute local alpha invariants $\alpha_{\mathsf{p}}(X)$ for any point p on a quasi-smooth Fano 3-fold weighted hypersurface X of index 1. In Sections 4 and 5, we compute local alpha invariants at smooth and singular points, respectively. At this stage, Theorem 1.8 is proved except for seven specific families. These exceptional families are families No. 2, 4, 5, 6, 8, 10 and 14, and we need extra arguments to prove $\alpha(X) \geq 1/2$, which will be done in Section 6. In Section 7, we will consider and prove further results such as Theorems 1.11 and 1.12. We will also discuss related problems that arise naturally through the experience of huge amount of computations. Finally, in Section 8, various information on the families of quasi-smooth Fano 3-fold weighted hypersurfaces of index 1 are summarized, and we also make it clear in Remark 8.1 what is left about K-stability for quasi-smooth Fano 3-fold weighted hypersurfaces of index 1.

1.4. Relevant results in K-stability

Recently both theoretical and explicit studies of K-stability of Fano varieties have been developed drastically. We refer readers to [Xu21] for up-to-date surveys. Following the suggestion from the referee, we add Section 1.4 to explain some of them that are developed during the preparation or after the completion of this article.

One of the most striking one is the equivalence of the notions of K-stability and uniform K-stability for klt Fano varieties that is proved in [LXZ22]. It in particular follows that the K-stability implies the existence of KE metric for klt Fano varieties. As a consequence, we are now able to conclude in Theorem 1.12 not only the K-stability of X but also the existence of a KE metric on X.

It should be mentioned that currently there are various methods in hand to check K-stability: The most powerful methods at present are the induction method of Abban and Zhuang [AZ22] computing (local) delta invariants or the moduli method of Liu et al. (see, e.g., [Liu22], [LX19]). These methods are developed parallel to the preparation of this article, and we do not use them. As it is explained in Section 1.3, the proofs of the main results of this article rely on the computation of (local) alpha invariants.

This article aims the systematic study of singular Fano 3-folds. There are on-going work by Cheltsov and collaborators on smooth Fano 3-folds. In the book [Ara+23], it is completely determined whether the general member of each of the 105 irreducible families of smooth Fano 3-folds admits a KE metric or not. Very recently, there have been a lot of works aiming to drop the generality assumption in the above result and to classify K-(poly)stable smooth Fano 3-folds in each family completely (see [Liu23], [CP22], [CFKO22], [CFKP23], [BL22], [Den22], [CDF22], [Mal23]).

2. Preliminaries

2.1. Basic definitions and properties

We refer readers' to [KM98] for standard notions of birational geometry which are not explained in this article.

Definition 2.1. By a *Fano variety*, we mean a normal projective $\mathbb{Q}$-factorial variety with at most terminal singularities whose anticanonical divisor is ample.

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For a variety X, we denote by $\mathrm{Sm}(X)$ the smooth locus of X and $\mathrm{Sing}(X) = X \setminus \mathrm{Sm}(X)$ the singular locus of X. For a subset $\Gamma \subset X$, we define $\mathrm{Sing}_\Gamma(X) := \mathrm{Sing}(X) \cap \Gamma$. Let X be a normal variety and D a Weil divisor (class) on X. We denote by $|D|_{\mathbb{Q}}$ the set of effective $\mathbb{Q}$-divisors on X which are $\mathbb{Q}$-linearly equivalent to D. For a smooth point $\mathsf{p} \in X$, we define $|\mathcal{I}_{\mathsf{p}}(D)|$ to be the linear subspace of $|D|$ consisting of members of $|D|$ passing through p.

2.1.a. Birational (super)rigidity of Fano varieties

Let X be a normal $\mathbb{Q}$-factorial variety, D a $\mathbb{Q}$-divisor on X and $\mathcal{M}$ a movable linear system on X. For a prime divisor E over X, we define $\mathrm{ord}_E(D)$ to be the coefficient of E in $\varphi^* D$, where $\varphi\colon Y \to X$ is a birational morphism such that $E \subset Y$, and we set $m_E(\mathcal{M}) := \mathrm{ord}_E(M)$, where M is a general member of $\mathcal{M}$. For a positive rational number λ, we say that a pair $(X, \lambda\mathcal{M})$ is *canonical* if

$$a_E(K_X) \geq \lambda m_E(\mathcal{M})$$

for any exceptional prime divisor E over X.

Let X be a Fano variety of Picard number one. Note that we can view X (or more precisely the structure morphism $X \to \mathrm{Spec}\,\mathbb{C}$) as a Mori fiber space.

Definition 2.2. We say that X is *birationally rigid* if the existence of a Mori fiber space $Y \to T$ such that Y is birational to X implies that Y is isomorphic to X (and $T = \mathrm{Spec}\,\mathbb{C}$). We say that X is *birationally superrigid* if X is birationally rigid and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

Definition 2.3. A closed subvariety $\Gamma \subset X$ is called a *maximal center* if there exists a movable linear system $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$ and an exceptional prime divisor E over X such that $m_E(\mathcal{M}) > n a_E(K_X)$.

Theorem 2.4 [CS08, Theorem 1.26]. *A Fano variety X of Picard number 1 is birationally superrigid if and only if the pair $(X, \frac{1}{n}\mathcal{M})$ is canonical for any movable linear system $\mathcal{M}$ on X, where $n \in \mathbb{Q}_{>0}$ is such that $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$, or equivalently if and only if there is no maximal center on X.*

2.1.b. Log canonical thresholds and alpha invariants

Definition 2.5. Let (X, Δ) be a pair, D an effective $\mathbb{Q}$-divisor on X, and let $\mathsf{p} \in X$ be a point. Assume that (X, Δ) has at most log canonical singularities. We define the *log canonical threshold* (abbreviated as LCT) of $(X, \Delta; D)$ *at* p and the *log canonical threshold* of $(X, \Delta; D)$ to be the numbers

$$\mathrm{lct}_{\mathsf{p}}(X, \Delta; D) = \sup\{\, c \in \mathbb{Q}_{\geq 0} \mid (X, \Delta + cD) \text{ is log canonical at } \mathsf{p} \,\},$$
$$\mathrm{lct}(X, \Delta; D) = \sup\{\, c \in \mathbb{Q}_{\geq 0} \mid (X, \Delta + cD) \text{ is log canonical} \,\},$$

respectively. We set $\mathrm{lct}_{\mathsf{p}}(X; D) = \mathrm{lct}_{\mathsf{p}}(X, 0; D)$ and $\mathrm{lct}_{\mathsf{p}}(X; D) = \mathrm{lct}(X, \Delta; D)$ when $\Delta = 0$. Assume that $|-K_X|_{\mathbb{Q}} \neq \emptyset$. Then we define the *alpha invariant of X at* p and the *alpha invariant of X* to be the numbers

$$\alpha_{\mathsf{p}}(X) = \inf\{\, \mathrm{lct}_{\mathsf{p}}(X, D) \mid D \in |-K_X|_{\mathbb{Q}} \,\},$$
$$\alpha(X) = \inf\{\, \alpha_{\mathsf{p}}(X) \mid \mathsf{p} \in X \,\},$$

respectively.

The following fact is frequently used.

Remark 2.6. Let p be a point on X, and let D_1, D_2 be effective $\mathbb{Q}$-divisors on X. If both (X, D_1) and (X, D_2) are log canonical at p, then the pair

$$(X, \lambda D_1 + (1 - \lambda) D_2)$$

is log canonical at p for any $\lambda \in \mathbb{Q}$ such that $0 \leq \lambda \leq 1$. In particular, if $\alpha_{\mathsf{p}}(X) < c$ for some number $c > 0$, then there exists an irreducible $\mathbb{Q}$-divisor $D \in |-K_X|_{\mathbb{Q}}$ such that (X, cD) is not log canonical at p. Here, a $\mathbb{Q}$-divisor is *irreducible* if its support $\mathrm{Supp}(D)$ is irreducible.

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2.1.c. Cyclic quotient singularities and orbifold multiplicities

Definition 2.7. Let $r > 0$ and $a_1, \dots, a_n$ be integers. Suppose that the cyclic group $\boldsymbol{\mu}_r$ of rth roots of unity in $\mathbb{C}$ acts on the affine n-space $\mathbb{A}^n$ with affine coordinates $x_1, \dots, x_n$ via

$$(x_1, \dots, x_n) \mapsto (\zeta^{a_1} x_1, \dots, \zeta^{a_n} x_n),$$

where $\zeta \in \boldsymbol{\mu}_r$ is a fixed primitive rth root of unity. We denote by $\bar{o} \in \mathbb{A}^n/\boldsymbol{\mu}_r$ the image of the origin $o \in \mathbb{A}^n$ under the quotient morphism $\mathbb{A}^n \to \mathbb{A}^n/\boldsymbol{\mu}_r$. A singularity $\mathsf{p} \in X$ is a *cyclic quotient singularity* of type $\frac{1}{r}(a_1, \dots, a_n)$ if $\mathsf{p} \in X$ is analytically isomorphic to an analytic germ $\bar{o} \in \mathbb{A}^n/\boldsymbol{\mu}_r$. In this case, r is called the *index* of the cyclic quotient singularity $\mathsf{p} \in X$.

Remark 2.8. Let $\mathsf{p} \in X$ be an n-dimensional cyclic quotient singular point. Then we have a suitable action of $\boldsymbol{\mu}_r$ on $\mathbb{A}^n$ such that there is an analytic isomorphism $\bar{o} \in \mathbb{A}^n/\boldsymbol{\mu}_r \cong \mathsf{p} \in X$ of (analytic) germs. In the following, the germ $o \in \mathbb{A}^n$ is often denoted by $\check{\mathsf{p}} \in \check{X}$. By identifying $\mathsf{p} \in X \cong \bar{o} \in \mathbb{A}^n/\boldsymbol{\mu}_r$, the quotient morphism $o \in \mathbb{A}^n \to \bar{o} \in \mathbb{A}^n/\boldsymbol{\mu}_r$ is denoted by $q_{\mathsf{p}} \colon \check{X} \to X$ and is called the *quotient morphism* of $\mathsf{p} \in X$.

Note that, by convention, the case $r = 1$ is allowed in the definition of cyclic quotient singularity. A cyclic quotient singularity $\mathsf{p} \in X$ of index 1 is nothing but a smooth point $\mathsf{p} \in X$ and in that case the quotient morphism $q_{\mathsf{p}} \colon \check{X} \to X$ is simply an isomorphism.

Definition 2.9. Let $\mathsf{p} \in X$ be a cyclic quotient singularity, and let $q_{\mathsf{p}} \colon \check{X} \to X$ be its quotient morphism with $\check{\mathsf{p}} \in \check{X}$ the preimage of p. For an effective $\mathbb{Q}$-divisor D on X, we define

$$\operatorname{omult}_{\mathsf{p}}(D) := \operatorname{mult}_{\check{\mathsf{p}}}(q_{\mathsf{p}}^* D)$$

and call it the *orbifold multiplicity* of D at p. By convention, we set $\operatorname{omult}_{\mathsf{p}}(D) = \operatorname{mult}_{\mathsf{p}}(D)$ when $\mathsf{p} \in X$ is a smooth point.

2.1.d. Kawamata blowup

Let $\mathsf{p} \in V$ be a three-dimensional terminal quotient singularity. Then it is of type $\frac{1}{r}(1, a, r-a)$, where r and a are coprime positive integers with $r > a$ (see [MS84]). Let $\varphi \colon W \to V$ be the weighted blowup of V at p with weight $\frac{1}{r}(1, a, r-a)$. By [Kaw96], φ is the unique divisorial contraction centered at p and we call φ the *Kawamata blowup* of V at p. If we denote by E the φ-exceptional divisor, then $E \cong \mathbb{P}(1, a, r-a)$ and we have

$$K_W = \varphi^* K_V + \frac{1}{r} E,$$

and

$$(E^3) = \frac{r^2}{a(r-a)}.$$

2.2. *Weighted projective varieties*

We recall basic definitions of various notions concerning weighted projective spaces and their subvarieties. We refer readers to [IF00] for details.

2.2.a. Weighted projective space

Let N be a positive integer. For positive integers $a_0, \dots, a_N$, let

$$R(a_0, \dots, a_N) := \mathbb{C}[x_0, \dots, x_N]$$

be the graded ring whose grading is given by $\deg x_i = a_i$. We define

$$\mathbb{P}(a_0, \dots, a_N) := \operatorname{Proj} R(a_0, \dots, a_N)$$

and call it the *weighted projective space* with homogeneous coordinates $x_0, \dots, x_N$ (of degree $\deg x_i = a_i$). We sometimes denote

$$\mathbb{P}(a_0, \dots, a_N)_{x_0, \dots, x_N}$$

in order to make it clear the homogeneous coordinates $x_0, \dots, x_N$. For $i = 0, \dots, N$, we denote by

$$\mathsf{p}_{x_i} = (0 : \cdots : 1 : \cdots : 0) \in \mathbb{P}(a_0, \dots, a_N) \tag{2.1}$$

the coordinate point at which only the coordinate x_i does not vanish. Let $f \in R := R(a_0, \dots, a_N) = \mathbb{C}[x_0, \dots, x_N]$ be a polynomial. We say that f is *quasi-homogeneous* (resp. *homogeneous*) if it is homogeneous with respect to the grading $\deg x_i = a_i$ (resp. $\deg x_i = 1$) for $i = 0, 1, \dots, N$. For a polynomial $f \in \mathbb{C}[x_0, \dots, x_N]$ and a monomial $M = x_0^{m_0} \cdots x_N^{m_N}$, we denote by

$$\operatorname{coeff}_f(M) \in \mathbb{C}$$

the coefficient of M in f, and, by a slight abuse of notation, we write $M \in f$ if $\operatorname{coeff}_f(M) \neq 0$. For quasi-homogeneous polynomials $f_1, \dots, f_k \in R$, we denote by

$$(f_1 = \cdots = f_k = 0) \subset \mathbb{P}(a_0, \dots, a_N)$$

the closed subscheme defined by the quasi-homogeneous ideal $(f_1, \dots, f_k) \subset R$. Moreover, for a closed subscheme $X \subset \mathbb{P}(a_0, \dots, a_N)$ and quasi-homogeneous polynomials $g_1, \dots, g_l \in R$, we define

$$(g_1 = \cdots = g_l = 0)_X := (g_1 = \cdots = g_l = 0) \cap X,$$

which is a closed subscheme of X. For $i = 0, \dots, N$, we define

$$\begin{aligned} \mathcal{H}_{x_i} &:= (x_i = 0) \subset \mathbb{P}(a_0, \dots, a_N), \\ \mathcal{U}_{x_i} &:= \mathbb{P}(a_0, \dots, a_N) \setminus \mathcal{H}_{x_i}. \end{aligned} \tag{2.2}$$

Remark 2.10. The weighted projective space $\mathbb{P}(a, b, c, d, e)$ with homogeneous coordinates x, y, z, t, w of degrees a, b, c, d, e, respectively, is sometimes denoted by

$$\mathbb{P}(a, b, c, d, e)_{x,y,z,t,w}$$

in order to emphasize the homogeneous coordinates. For a coordinate $v \in \{x, \dots, w\}$, the point $\mathsf{p}_v \in \mathbb{P}(a, b, c, d, e)$, the quasi-hyperplane $\mathcal{H}_v = (v = 0) \subset \mathbb{P}(a, b, c, d, e)$ and the open set $\mathcal{U}_v = \mathbb{P}(a, b, c, d, e) \setminus \mathcal{H}_v$ are similarly defined as in equations (2.1) and (2.2).

2.2.b. Well-formedness and quasi-smoothness

Definition 2.11. We say that a weighted projective space $\mathbb{P}(a_0, \dots, a_N)$ is *well-formed* if

$$\gcd\{a_0, \dots, \hat{a}_i, \dots, a_N\} = 1$$

for any $i = 0, 1, \dots, N$.

Definition 2.12. Let $\mathbb{P}(a_0, \dots, a_N)$ be a weighted projective space such that $\gcd\{a_0, \dots, a_N\} = 1$. For $j = 0, 1, \dots, N$, we set

$$\begin{aligned} l_j &:= \gcd\{a_0, a_1, \dots, \hat{a}_j, \dots, a_N\}, \\ m_j &:= l_0 l_1 \cdots \hat{l}_j \cdots l_N, \\ b_j &:= \frac{a_j}{m_j}. \end{aligned}$$

We then define

$$\mathbb{P}(a_0, \dots, a_N)^{\mathrm{wf}} := \mathbb{P}(b_0, \dots, b_N)$$

and call it the *well-formed model* of $\mathbb{P}(a_0, \dots, a_N)$.

Remark 2.13. Any weighted projective space is isomorphic to a well-formed one (see, e.g., [IF00, Lemma 5.7]). More precisely, for a weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \dots, a_N)$ with $\gcd\{a_0, \dots, a_N\} = 1$, there exists an isomorphism

$$\phi \colon \mathbb{P}(a_0, \dots, a_N)_{x_0, \dots, x_N} \to \mathbb{P}^{\mathrm{wf}} = \mathbb{P}(b_0, \dots, b_N)_{y_0, \dots, y_N}$$

such that $\phi^* \mathcal{H}_{y_i} = m_i \mathcal{H}_{x_i}$ for $i = 0, 1, \dots, N$, where $\mathcal{H}_{x_i} = (x_i = 0) \subset \mathbb{P}$ and $\mathcal{H}_{y_i} = (y_i = 0) \subset \mathbb{P}^{\mathrm{wf}}$.

In the following, we set $\mathbb{P} := \mathbb{P}(a_0, \dots, a_N)$ and we denote by

$$\Pi \colon \mathbb{A}^{N+1} \setminus \{o\} \to \mathbb{P}, \quad (\alpha_0, \dots, \alpha_N) \mapsto (\alpha_0 : \cdots : \alpha_N),$$

the canonical projection. Let $X \subset \mathbb{P}$ be a closed subscheme. We set $C_X^* := \Pi^{-1}(X)$ and call it the *punctured affine quasi-cone* over X. The *affine quasi-cone* C_X over X is the closure of C_X^* in $\mathbb{A}^{N+1}$. We set $\pi := \Pi|_{C_X^*} : C_X^* \to X$.

Definition 2.14. We say that a closed subscheme $X \subset \mathbb{P}$ is *well-formed* if $\mathbb{P}$ is well-formed and $\operatorname{codim}_X(X \cap \operatorname{Sing}(\mathbb{P})) \geq 2$.

Definition 2.15. Let $X \subset \mathbb{P}$ be a closed subscheme as above. We define the *quasi-smooth locus* of X as

$$\operatorname{QSm}(X) := \pi(\operatorname{Sm}(C_X^*)) \subset X.$$

Let S be a subset of X. We say that X is *quasi-smooth along* S if $S \subset \operatorname{QSm}(X)$. We simply say that X is *quasi-smooth* when $X = \operatorname{QSm}(X)$.

2.2.c. Orbifold charts

Let $\mathcal{U}_{x_i}$ be the open subset of $\mathbb{P} = \mathbb{P}(a_0, \dots, a_N)_{x_0, \dots, x_N}$ as in equation (2.2), where $i \in \{0, 1, \dots, N\}$. We call $\mathcal{U}_{x_i}$ the *standard affine open subset* of $\mathbb{P}$ containing p_{x_i}. We denote by $\breve{\mathcal{U}}_{x_i}$ the affine N-space $\mathbb{A}^N$ with affine coordinates $\breve{x}_0, \dots, \widehat{\breve{x}_i}, \dots, \breve{x}_N$. Consider the $\boldsymbol{\mu}_{a_i}$-action on $\breve{\mathcal{U}}_i$ defined by

$$\breve{x}_j \mapsto \zeta^{a_j} \breve{x}_j, \quad \text{for } j = 0, \dots, \hat{\imath}, \dots, N,$$

where $\zeta \in \boldsymbol{\mu}_{a_i}$ is a primitive a_ith root of unity. Then the open set $\mathcal{U}_i$ can be naturally identified with the quotient $\breve{\mathcal{U}}_{x_i} / \boldsymbol{\mu}_{a_i}$. In fact, this can be seen by the identification

$$\breve{x}_j = \frac{x_j}{x_i^{a_j/a_i}}, \quad \text{for } j = 0, \dots, \hat{\imath}, \dots, N.$$

The quotient morphism $\breve{\mathcal{U}}_{x_i} \to \breve{\mathcal{U}}_{x_i} / \boldsymbol{\mu}_{a_i} = \mathcal{U}_{x_i}$ is denoted by

$$\rho_{x_i} : \breve{\mathcal{U}}_{x_i} \to \mathcal{U}_{x_i}$$

and is called the *orbifold chart* of $\mathbb{P}$ containing p_{x_i}.

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Let $X \subset \mathbb{P}$ be a subscheme. Usually, we denote by $U_{x_i} \subset X$ the open set $\mathcal{U}_{x_i} \cap X$, and we call U_{x_i} the *standard affine open subset* of X containing p_{x_i}. In this case, we set $\breve{U}_{x_i} = \rho_{x_i}^{-1}(U_{x_i}) \subset \breve{\mathcal{U}}_{x_i}$. By a slight abuse of notation, the morphism $\rho_{x_i}|_{U_i} \colon \breve{U}_{x_i} \to U_{x_i}$ is also denote by

$$\rho_{x_i} \colon \breve{U}_{x_i} \to U_{x_i}$$

and is called the *orbifold chart* of X containing p_{x_i}. When we are using the notation $\mathsf{p} = \mathsf{p}_{x_i}$, the morphism ρ_{x_i} is sometimes denoted by ρ_{p}. Note that $\breve{U}_{x_i}$ is not necessary smooth in general.

Suppose that $X \subset \mathbb{P}$ is a closed subvariety containing the point $\mathsf{p} = \mathsf{p}_{x_i}$. The preimage $\breve{\mathsf{p}}$ of p is the origin of $\breve{U}_{x_i} \subset \breve{\mathcal{U}}_{x_i} = \mathbb{A}^N$. It is straightforward to see that X is quasi-smooth at p if and only if $\breve{U}_{x_i}$ is smooth at $\breve{\mathsf{p}}$. Suppose that X is quasi-smooth at p. A system of local coordinates of U_{x_i} at $\breve{\mathsf{p}}$ is called a system of *local orbifold coordinates* of X at p. In this case, $\mathsf{p} \in X$ is a cyclic quotient singularity of index a_i and $\rho_{x_i} \colon \breve{U}_{x_i} \to U_{x_i}$ can be identified with (or analytically equivalent to) the quotient morphism q_{p} of $\mathsf{p} \in X$ after shrinking U_{x_i} and then $\breve{U}_{x_i}$. Moreover, if X is quasi-smooth, then $\breve{U}_i$ is smooth for any i.

Remark 2.16. When we work with $\mathbb{P} = \mathbb{P}(a,b,c,d,e)_{x,y,z,t,w}$ and its closed subscheme $X \subset \mathbb{P}$, then $\mathcal{U}_v = \mathbb{A}^5_{\breve{x},\dots,\hat{\breve{v}},\dots,\breve{w}}$, $\rho_v \colon \breve{\mathcal{U}}_v \to \mathcal{U}_v$, $\breve{U}_v = \rho_v^{-1}(U_v) \subset \mathcal{U}_v$ and $\rho_v \colon \breve{U}_v \to U_v$ are similarly defined.

2.2.d. Weighted hypersurfaces and quasi-tangent divisors

As in the previous subsections, we work with $\mathbb{P} = \mathbb{P}(a_0,\dots,a_N)_{x_0,\dots,x_N}$.

Definition 2.17. A *quasi-linear polynomial* (or a *quasi-linear form*) in variables $x_0,\dots,x_{n+1}$ is a quasi-homogeneous polynomial $f = f(x_0,\dots,x_{n+1})$ such that $x_i \in f$ for some $i = 0,\dots,n+1$.

Definition 2.18. We say that a subvariety $S \subset \mathbb{P}$ is a *quasi-linear subspace* of $\mathbb{P}$ if it is a complete intersection in $\mathbb{P}$ defined by quasi-linear equations of the form

$$\ell_1 + f_1 = \ell_2 + f_2 = \dots = \ell_k + f_k = 0,$$

where $\ell_1, \ell_2, \dots, \ell_k$ are linearly independent linear forms in variables $x_0,\dots,x_{n+1}$ and $f_1,\dots,f_k \in \mathbb{C}[x_0,\dots,x_{n+1}]$ are quasi-homogeneous polynomials which are not quasi-linear. A quasi-linear subspace of $\mathbb{P}$ of codimension 1 (resp. dimension 1) is called a *quasi-hyperplane* (resp. *quasi-line*) of $\mathbb{P}$.

It is clear that a quasi-linear subspace of $\mathbb{P}$ is isomorphic to a weighted projective space. In particular, a quasi-line is isomorphic to $\mathbb{P}^1$.

Let X be a hypersurface in $\mathbb{P} = \mathbb{P}(a_0,\dots,a_N)$ defined by a quasi-homogeneous polynomial of degree d. We often denote it as $X = X_d \subset \mathbb{P}(a_0,\dots,a_N)$. Suppose that X is quasi-smooth at a point $\mathsf{p} = \mathsf{p}_{x_i}$. Then the defining polynomial $F = F(x_0,\dots,x_N)$ of X can be written as

$$F = x_i^m f + x_i^{m-1} g_{m-1} + \dots + x_i g_1 + g_0, \tag{2.3}$$

where $m \ge 0$, $f = f(x_0,\dots,x_N)$ is a quasi-homogeneous polynomial of degree $d - ma_i$ which is quasi-linear and $g_k = g_k(x_0,\dots,\hat{x}_i,\dots,x_N)$ is a quasi-homogeneous polynomial of degree $d - ka_i$ which is not quasi-linear for $0 \le k \le m-1$. Note that the expression (2.3) is uniquely determined once the homogeneous coordinates of $\mathbb{P}$ are fixed.

Definition 2.19. Under the notation and assumptions as above, we call f the *quasi-tangent polynomial* of X at p and the divisor $(f=0)_X$ on X is called the *quasi-tangent divisor* of X at p. When $f = x_j$ for some j, then we also call x_j as the *quasi-tangent coordinate* of X at p.

Remark 2.20. Let $X = X_7 \subset \mathbb{P}(1,1,1,2,3)_{x,y,z,t,w}$ be a weighted hypersurface of degree 7. Suppose that its defining polynomial is of the form

$$F = t^3 x + t^2 w + t g_5 + g_7,$$

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where $g_5, g_7 \in \mathbb{C}[x, y, z, w]$ are quasi-homogeneous polynomials of degree 5, 7, respectively. In this case X is quasi-smooth at $\mathsf{p} = \mathsf{p}_t$. The quasi-tangent polynomial of X at p is $tx + w$. Note that x is not a quasi-tangent coordinate of X at p because of the presence of $t^2 w \in F$.

Lemma 2.21. *Let $X \subset \mathbb{P}$ be a weighted hypersurface of degree d. Assume that X is quasi-smooth at a point $\mathsf{p} = \mathsf{p}_{x_i}$ for some $i = 0, 1, \dots, N$, and let x_j be a homogeneous coordinate such that $x_j \in f$, where f is the quasi-tangent polynomial of X at p. Then, after a suitable choice of homogeneous coordinates $x_0, \dots, x_N$, the defining polynomial F of X can be written as*

$$F = x_i^m x_j + x_i^{m-1} g_{m-1} + \cdots + x_i g_1 + g_0,$$

where $g_k = g_k(x_0, \dots, \hat{x}_i, \dots, x_N)$ is a quasi-homogeneous polynomial of degree $d - k a_i$ which is not quasi-linear.

Proof. We can write $F = x_i^m f + g$, where $m \geq 0$, $f = f(x_0, \dots, x_N) \ni x_j$ is the quasi-tangent polynomial and g is a quasi-homogeneous polynomial of degree d which does not involve a monomial divisible by x_i^m and which is contained in the ideal $(x_0, \dots, \hat{x}_i, \dots, x_N)^2$. We write $g = x_j^e h_e + \cdots + x_j h_1 + h_0$, where $e \geq 0$ and h_k is a quasi-homogeneous polynomial of degree $d - k a_j$ which does not involve the variables x_j. By rescaling x_j, we may assume $f = x_j - \tilde{f}$, where $x_j \notin \tilde{f}$, and we write $\tilde{f} = x_i^n \tilde{f}_n + \cdots + x_i \tilde{f}_1 + \tilde{f}_0$, where $\tilde{f}_k$ is a quasi-homogeneous polynomial of degree $d - (m+k) a_i$ which does not involve the variable x_i. We consider the coordinate change $x_j \mapsto x_j + \tilde{f}$. Then the new defining polynomial can be written as

$$\begin{aligned} F &= x_i^m x_j + (x_j + x_i^n \tilde{f}_n + \cdots)^e h_e + \cdots + (x_j + x_i^n \tilde{f}_n + \cdots) h_1 + h_0 \\ &= (x_i^{ne-m} \tilde{f}_n^e + \cdots + x_j) x_i^m + \cdots \end{aligned}$$

It follows that the new quasi-tangent polynomial is $x_i^{ne-m} \tilde{f}_n^e + \cdots + x_j$. We claim that $ne - m < n$. We have $d = m a_i + a_j$, $d = e a_j + \deg h_e \geq e a_j$ and $a_j = n a_i + \deg \tilde{f}_n > n a_i$, which implies $ne - m < n$. Thus, repeating the above coordinate change, we can drop the degree of the quasi-tangent coordinate with respect to x_i, and we may assume $F = x_i^m f + x_i^{m-1} g_{m-1} + \cdots + x_i g + g_0$, where $f \ni x_j$ and g_k are quasi-homogeneous polynomials of degree a_j and $d - k a_i$, respectively, which do not involve the variable x_i. Moreover, g_k is not quasi-linear for $0 \leq k \leq m - 1$. Finally, replacing x_j, we may assume $f = x_j$ and this completes the proof. □

Remark 2.22. Suppose that a weighted hypersurface $X \subset \mathbb{P}$ is quasi-smooth at $\mathsf{p} = \mathsf{p}_{x_i}$. Then $\operatorname{omult}_{\mathsf{p}}((f = 0)_X) > 1$ for the quasi-tangent polynomial f of X at p. Moreover, x_j is a quasi-tangent coordinate of X at p if and only if $\operatorname{omult}_{\mathsf{p}}(H_{x_j}) > 1$.

2.3. The 95 families

2.3.a. Definition of the families

As it is explained in Section 1.2.b, quasi-smooth Fano 3-fold weighted hypersurfaces of index 1 are classified and they form 95 families. According to the classification, the minimum of the weights of an ambient space is 1. Hence a family is determined by a quadruple (a_1, a_2, a_3, a_4), which means that the family corresponding to a quadruple (a_1, a_2, a_3, a_4) is the family of weighted hypersurfaces of degree $d = a_1 + a_2 + a_3 + a_4$ in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$. The 95 families are numbered in the lexicographical order on (d, a_1, a_2, a_3, a_4), and each family is referred to as family No. i for $\mathsf{i} \in \{1, 2, \dots, 95\}$. Families No. 1 and 3 are the families consisting of quartic 3-folds and degree 6 hypersurfaces in $\mathbb{P}(1, 1, 1, 1, 3)$, respectively, and for any smooth member of these two families, K-stability (and hence the existence of KE metrics) is known.

Definition 2.23. We set

$$\mathsf{I} := \{1, 2, \dots, 95\} \setminus \{1, 3\},$$

and, for $\mathsf{i} \in \mathsf{I}$, we denote by $\mathcal{F}_{\mathsf{i}}$ the family consisting of the quasi-smooth members of family No. i.

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The main objects of this article is thus the members of $\mathcal{F}_{\mathsf{i}}$ for $\mathsf{i} \in \mathsf{I}$.

We set

$$\mathsf{I}_1 := \{2, 4, 5, 6, 8, 10, 14\}.$$

The set I_1 is characterized as follows: Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$, $a_1 \le a_2 \le a_3 \le a_4$, be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. Then $\mathsf{i} \in \mathsf{I}_1$ if and only if $a_2 = 1$. The computations of alpha invariants will be done in a relatively systematic way for families $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ (see Sections 4 and 5), while the computations will be done separately for families $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}_1$ (see Section 6).

We explain notation and conventions concerning the main objects of this article. Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) =: \mathbb{P}$ be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$.

- Unless otherwise specified, we assume $a_1 \le a_2 \le a_3 \le a_4$.
- In many situations (especially when we treat a specific family), we denote by x, y, z, t, w the homogeneous coordinates of $\mathbb{P}$ of degree, respectively, $1, a_1, a_2, a_3, a_4$.
- We denote by F the polynomial defining X in $\mathbb{P}$, which is quasi-homogeneous of degree $d = a_1 + a_2 + a_3 + a_4$.
- We set $A = -K_X$, which is the positive generator of of $\mathrm{Cl}(X) \cong \mathbb{Z}$. Note that we have

$$(-K_X)^3 = (A^3) = \frac{d}{a_1 a_2 a_3 a_4} = \frac{a_1 + a_2 + a_3 + a_4}{a_1 a_2 a_3 a_4}.$$

2.3.b. Definitions of QI and EI centers and birational (super)rigidity

In this subsection, let

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x,y,z,t,w}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$, where $a_1 \le a_2 \le a_3 \le a_4$. We give definitions of QI and EI centers, which are particular singular points on X and are important for understanding birational (super)rigidity of X. For EI centers, we only give an ad hoc definition (see [CPR00, Section 4.10] and [CP17, Section 4.2] for more detailed treatments).

Definition 2.24. Let $\mathsf{p} \in X$ be a singular point. We say that $\mathsf{p} \in X$ is an *EI center* if the upper script EI is given in the fourth column of Table 7, or equivalently if i and p belong to one of the following.

- $\mathsf{i} = 7$ and p is of type $\frac{1}{2}(1, 1, 1)$.
- $\mathsf{i} \in \{23, 40, 44, 61, 76\}$ and $\mathsf{p} = \mathsf{p}_t$.
- $\mathsf{i} \in \{20, 36\}$ and $\mathsf{p} = \mathsf{p}_z$.

We say that $\mathsf{p} \in X$ is a *QI center* if there are distinct j and k such that $d = 2a_k + a_j$ and the index of the cyclic quotient singularity $\mathsf{p} \in X$ coincides with a_k.

We say that $\mathsf{p} \in X$ is a *birational involution (BI) center* if it is either an EI center or a QI center.

Remark 2.25. Let X be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. Then the following are proved in [CP17].

1. No smooth point on X is a maximal center.
2. A singular point $\mathsf{p} \in X$ is a maximal center only if either p is a BI center or X is a member of $\mathcal{F}_{23}$ and $\mathsf{p} = \mathsf{p}_z$ is of type $\frac{1}{3}(1, 1, 2)$.

Note that a BI center $\mathsf{p} \in X$ is not always a maximal center (see Section 5.3, especially Remark 5.10, for the complete analysis for QI centers). Note also that the $\frac{1}{3}(1, 1, 2)$ point p_z on a member X of $\mathcal{F}_{23}$ is not a maximal center if X is general. However, $\mathsf{p}_z \in X$ can be a maximal center and in that case there is a birational involution of X (called an invisible involution) with center p_z (see [CP17, Section 4.3]).

Definition 2.26. We define the subset $\mathsf{I}_{\mathrm{BSR}} \subset \mathsf{I}$ as follows: $\mathsf{i} \in \mathsf{I}_{\mathrm{BSR}}$ if and only if a member X of $\mathcal{F}_{\mathsf{i}}$ does not admit a BI center. We then define $\mathsf{I}_{\mathrm{BR}} = \mathsf{I} \setminus \mathsf{I}_{\mathrm{BSR}}$.

Note that $|\mathsf{I}_{\mathrm{BSR}}| = 48$ and $|\mathsf{I}_{\mathrm{BR}}| = 45$. The following is a more precise version of Theorem 1.2.

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Theorem 2.27 [CP17]. *Let X be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$.*

1. *If $\mathsf{i} \in \mathsf{I}_{\mathrm{BSR}}$, then any member of $\mathcal{F}_{\mathsf{i}}$ is birationally superrigid.*
2. *If $\mathsf{i} \in \mathsf{I}_{\mathrm{BR}}$, then any member of $\mathcal{F}_{\mathsf{i}}$ is birationally rigid while its general member is not birationally superrigid.*

We emphasize that a family $\mathcal{F}_{\mathsf{i}}$, where $\mathsf{i} \in \mathsf{I}_{\mathrm{BR}}$, can contain (in fact does contain for most of $\mathsf{i} \in \mathsf{I}_{\mathrm{BR}}$) birationally superrigid Fano 3-folds as special members.

2.3.c. Numerics on weights and degrees

Let

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_{\mathsf{i}}$. Throughout the subsection, we assume that $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ and that $a_1 \le a_2 \le a_3 \le a_4$. We collect some elementary numerical results on weights $a_1, \dots, a_4$, the degree $d = a_1 + a_2 + a_3 + a_4$ of the defining polynomial $F = F(x, y, z, t, w)$ of X, and the anticanonical degree (A^3) of X which will be repeatedly used in the rest of this article.

Lemma 2.28. *One of the following happens.*

1. $d = 2a_4$.
2. $d = 3a_4$.
3. $d = 2a_4 + a_j$ *for some* $j \in \{1, 2, 3\}$.

Proof. We see that either $w^n \in F$ for some $n \ge 2$ or $x^n v \in F$ for some $n \ge 1$ and $v \in \{x, y, z, t\}$ by the quasi-smoothness of X.

Suppose $w^n \in F$ for some $n \ge 2$. Then we have

$$d = na_4 = a_1 + a_2 + a_3 + a_4 < 4a_4.$$

Hence, $n = 2, 3$ and we are in case (1) or (2). Suppose $w^n v \in F$ for some $n \ge 1$ and $v \in \{y, z, t\}$. Then we have $d = na_4 + a_j$ and moreover we have

$$a_4 + a_j < d = a_1 + a_2 + a_3 + a_4 < 3a_4 + a_j.$$

This shows $n = 2$, that is, $d = 2a_4 + a_j$.

If $a_1 = 1$, then the proof is completed. It remains to show that the case $d = 2a_4 + 1$ does not take place assuming $a_1 \ge 2$. Suppose $d = 2a_4 + 1$ and $a_1 \ge 2$. Then $w^2 x \in F$ and the singularity of $\mathsf{p}_w \in X$ is of type $\frac{1}{a_4}(a_1, a_2, a_3)$. There exist distinct $i, j \in \{1, 2, 3\}$ such that $a_i + a_j$ is divisible by a_4 since $\mathsf{p}_w \in X$ is terminal. We have $a_i + a_j = a_4$ since $0 < a_i + a_j < 2a_4$. Let $k \in \{1, 2, 3\}$ be such that $\{i, j, k\} = \{1, 2, 3\}$. Then

$$d = a_1 + a_2 + a_3 + a_4 = a_k + 2a_4.$$

Combining this with $d = 2a_4 + 1$, we have $a_k = 1$. This is a contradiction since $a_k \ge a_1 \ge 2$. □

Lemma 2.29.

1. *We have* $\mathsf{i} \in \{9, 17\}$ *if and only if* $d = 3a_4$ *and* $a_1 = 1$.
2. *We have* $a_1 a_2 a_3 (A^3) \le 3$ *and the equality holding if and only if* $d = 3a_4$.
3. *If* $a_1 < a_2$, *then we have* $a_1 (A^3) < 1$.
4. *If* $1 < a_1 < a_2$, *then* $a_1 a_3 (A^3) \le 1$.
5. *If* $a_1 < a_2$ *and* $d > 2a_4$, *then* $a_1 a_4 (A^3) \le 2$.
6. *If d is divisible by* a_4 *and* $\mathsf{i} \notin \{9, 17\}$, *then* $a_2 a_3 (A^3) \le 2$.
7. *If d is not divisible by* a_4 *and* $a_1 \ge 2$, *then* $a_2 a_4 (A^3) \le 2$.

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Proof. We prove (1). The 'only if' part is obvious. Suppose $d = 3a_4$ and $a_1 = 1$. Then we have $2a_4 = 1 + a_2 + a_3$. This implies $a_2 = a_4 - 1$ and $a_3 = a_4$ since $a_2 \le a_3 \le a_4$. Then, by setting $a = a_2 \ge 2$, X is a weighted hypersurface in $\mathbb{P}(1, 1, a, a+1, a+1)$ of degree $3(a+1)$. Suppose $\mathsf{p}_z \notin X$. Then some power of z is contained in F and this implies that $3(a+1)$ is divisible by a. In particular, we have $a = 3$ and this case corresponds to $\mathsf{i} = 17$. Suppose $\mathsf{p}_z \notin X$, then either $3(a+1) \equiv 1 (\mathrm{mod}\ a)$ or $3(a+1) \equiv a + 1 (\mathrm{mod}\ a)$ by the quasi-smoothness of X. In both cases, we have $a = 2$, ad hence $\mathsf{i} = 9$. Thus, (1) is proved.

The assertion (2) follows immediately since we have

$$a_1a_2a_3(A^3) = \frac{d}{a_4} \le 3$$

and $d \le 3a_4$ by Lemma 2.28.

We prove (3). Note that $2 \le a_2 \le a_3 \le a_4$. Note also that $a_4 > a_2$ because otherwise X has nonisolated singularity along L_{xy} which is impossible. In particular, we have $a_1 + \cdots + a_4 < 4a_4$ and $a_2a_3 \ge 4$ and we have

$$a_1(A^3) = \frac{a_1(a_1 + a_2 + a_3 + a_4)}{a_1a_2a_3a_4} < \frac{4}{a_2a_3} \le 1,$$

which proves (3).

We prove (4). We have $a_2 \ge 3$ since $a_2 > a_1 > 1$ and thus

$$a_1a_3(A^3) = \frac{d}{a_2a_4} \le \frac{3}{a_2} \le 1.$$

We prove (5). We have $d > 2a_4$ by assumption. Then, by Lemma 2.28, we have $d = 2a_4 + a_j$ for some $j \in \{1, 2, 3, 4\}$, and combining this with $d = a_1 + a_2 + a_3 + a_4$, we have

$$a_4 = a_1 + a_2 + a_3 - a_j \le a_2 + a_3.$$

If $a_1 > 1$, then we have $a_2, a_3 \ge 3$ and thus

$$a_1a_4(A^3) = \frac{a_1 + a_2 + a_3 + a_4}{a_2a_3} < \frac{3a_2 + 2a_3}{a_2a_3} = \frac{3}{a_3} + \frac{2}{a_2} \le \frac{5}{3}.$$

Suppose $a_1 = 1$. In this case $2 \le a_2 \le a_3$. If $a_3 \ge 3$, then

$$a_4(A^3) = \frac{1 + a_2 + a_3 + a_4}{a_2a_3} \le \frac{1 + 2a_2 + 2a_3}{a_2a_3} = \frac{1}{a_2a_3} + \frac{2}{a_3} + \frac{2}{a_2} \le \frac{11}{6}.$$

Suppose $a_3 = 2$, that is, $a_2 = a_3 = 2$. Then we have $a_4 = 3$ and $d = 8$ since $d = 5 + a_4 > 2a_4$ and a_4 is odd. In this case, we have $a_4(A^3) = 2$. This proves (5).

We prove (6). By Lemma 2.28 and (1), either $d = 2a_4$ or $d = 3a_4$ and $a_1 \ge 2$. If $d = 2a_4$ (resp. $d = 3a_4$ and $a_1 \ge 2$), then

$$a_2a_3(A^3) = \frac{2}{a_1} \le 2 \quad (\text{resp. } a_2a_3(A^3) = \frac{3}{a_1} \le 2).$$

This proves (6).

We prove (7). By Lemma 2.28, we have $d = 2a_4 + a_j$ for some $j \in \{1, 2, 3\}$. Then we have $a_4 = a_1 + a_2 + a_3 - a_j \le a_2 + a_3$. If $a_1 \ge 3$, then

$$a_2a_4(A^3) = \frac{a_1 + a_2 + a_3 + a_4}{a_1a_3} \le \frac{a_1 + 4a_3}{a_1a_3} = \frac{1}{a_3} + \frac{4}{a_1} \le \frac{5}{3}.$$

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We continue the proof assuming $a_1 = 2$. If in addition $a_2 < a_3$, then

$$a_2 a_4 (A^3) = \frac{2 + a_2 + a_3 + a_4}{2a_3} \leq \frac{2 + 2a_2 + 2a_3}{2a_3} \leq \frac{4a_3}{2a_3} = 2.$$

We continue the proof assuming $a_1 = 2$ and $a_2 = a_3$. In this case, by setting $a = a_2 = a_3$ and $b = a_4$, X is a weighted hypersurface of degree d in $\mathbb{P}(1, 2, a, a, b)$ and either $d = 2b + 2$ or $d = 2b + a$. If $d = 2b+2$, then $b = 2a$ but this is impossible since X has only terminal singularities. Hence, $d = 2b+a$. In this case $b = a + 2$ and $d = 3a + 4$. By the quasi-smoothness of X, we see that $d = 3a + 4$ is divisible by a. This implies that $a \in \{2, 4\}$. This is impossible since X has only terminal singularities. Therefore, (7) is proved. □

2.3.d. How to compute alpha invariants?

Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. For the proof of Theorem 1.8, it is necessary to show $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any point $\mathsf{p} \in X$. Let $\mathsf{p} \in X$ be a point. We briefly explain the most typical method of bounding $\alpha_{\mathsf{p}}(X)$ from below, which goes as follows.

1. Choose and fix a divisor S on X which vanishes at p to a relatively large (orbifold) multiplicity $m = \operatorname{omult}_{\mathsf{p}}(S) > 0$. In some cases, $S = H_x$ (when $\mathsf{p} \in H_x$), and in other cases, S is the quasi-tangent divisor of X at p. Let a be the positive integer such that $S \sim aA$.
2. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{a}S$. Then $D \cdot S$ is an effective 1-cycle on X.
3. Find a $\mathbb{Q}$-divisor $T \in |eA|_{\mathbb{Q}}$ for some $e \in \mathbb{Z}_{>0}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of $D \cdot S$. We will find such a $\mathbb{Q}$-divisor T by considering p-isolating set or class which will be explained in Section 3.1.c.
4. Let $q = q_{\mathsf{p}}$ be the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ be the preimage of p via q. By the above choices, $\operatorname{Supp}(q^*D) \cap \operatorname{Supp}(q^*S) \cap \operatorname{Supp}(q^*T)$ is a finite set of points including $\check{\mathsf{p}}$, and hence the local intersection number $(q^*D \cdot q^*S \cdot q^*T)_{\check{\mathsf{p}}}$ is defined (see Section 3.1.a). Then we have the inequalities

$$m \operatorname{omult}_{\mathsf{p}}(D) \leq (q_{\mathsf{p}}^*D \cdot q_{\mathsf{p}}^*S \cdot q_{\mathsf{p}}^*T)_{\check{\mathsf{p}}} \leq r(D \cdot S \cdot T) = rae(A^3),$$

   where r is the index of the cyclic quotient singularity $\mathsf{p} \in X$. Note that q is the identity morphism and $r = 1$ when $\mathsf{p} \in X$ is a smooth point. By Lemma 3.2 which will be explained below, we have

$$\operatorname{lct}_{\mathsf{p}}(X; D) \geq \frac{rae(A^3)}{m}$$

   for any D as in (2).
5. As a conclusion, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{\operatorname{lct}_{\mathsf{p}}(X; S), \ \frac{rae(A^3)}{m}\right\}.$$

6. It remains to bound $\operatorname{lct}_{\mathsf{p}}(X; S)$ from below. This is easy when S is quasi-smooth at p because in that case we have $\operatorname{lct}_{\mathsf{p}}(X; S) = 1$. The computation gets involved when S is the quasi-tangent divisor but will be done by considering suitable weighted blowups which will be explained in Section 3.2.b.

We need to consider variants of the above explained method or other methods especially for points in special positions. These will be explained in Section 3.

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3. Methods of computing log canonical thresholds

3.1. *Auxiliary results*

3.1.a. Some results on multiplicities and log canonicity

Let V be an n-dimensional variety. For effective Cartier divisors $D_1, \dots, D_n$ on V and a point $\mathsf{p} \in V$ which is an isolated component of $\operatorname{Supp}(D_1) \cap \cdots \cap \operatorname{Supp}(D_n)$, the *intersection multiplicity*

$$i(\mathsf{p}, D_1 \cdots D_n; V)$$

is defined (see [Ful98, Example 7.1.10]). Suppose that V is $\mathbb{Q}$-factorial. Then this definition is naturally generalized to effective $\mathbb{Q}$-divisors $D_1, \dots, D_n$ as follows:

$$i(\mathsf{p}, D_1, \cdots, D_n; V) := \frac{1}{d^n} i(\mathsf{p}, dD_1, \cdots, dD_n; V),$$

where d is a positive integer such that dD_i is a Cartier divisor for any i. In this paper, we set

$$(D_1 \cdots D_n)_{\mathsf{p}} := i(\mathsf{p}, D_1 \cdots D_n; V)$$

and call it the *local intersection number* of $D_1, \dots, D_n$ at p.

Remark 3.1. If $\mathsf{p} \in V$ is a smooth point, $D_1, \dots, D_n$ are effective divisors defined by $f_1, \dots, f_n \in \mathcal{O}_{V,\mathsf{p}}$ around p, and p is an isolated component of $\operatorname{Supp}(D_1) \cap \cdots \cap \operatorname{Supp}(D_n)$, then

$$(D_1 \cdots D_n)_{\mathsf{p}} = \dim_{\mathbb{C}} \mathcal{O}_{V,P}/(f_1, \dots, f_n).$$

If $X \subset \mathbb{P}(a_0, \dots, a_N)$ is an n-dimensional subvariety which is quasi-smooth at $\mathsf{p} = \mathsf{p}_{x_i} \in V$, $D_1 = (G_1 = 0)_X, \dots, D_n = (G_n = 0)_X$ are effective Weil divisors such that p is an isolated component of $D_1 \cap \cdots \cap D_n$, where $G_i = G_i(x_0, \dots, x_N)$ is a quasi-homogeneous polynomial of degree d_i, then

$$(D_1 \cdots D_n)_{\mathsf{p}} = \frac{1}{a_i} (\rho^* D_1 \cdots \rho^* D_n)_{\check{\mathsf{p}}} = \frac{1}{a_i} \dim_{\mathbb{C}} \mathcal{O}_{\check{U}_{\mathsf{p}}, \check{\mathsf{p}}}/(g_1, \dots, g_n),$$

where $\rho = \rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}} := X \cap \mathcal{U}_{\mathsf{p}}$ is the orbifold chart with $\check{\mathsf{p}} \in \check{U}_{\mathsf{p}}$ the preimage of p and $g_i = G(\check{x}_0, \dots, 1, \dots, \check{x}_N)$ with $\check{x}_j = x_j / x_i^{a_j/a_i}$ for $j \ne i$.

We will frequently use the following property of local intersection numbers. Let $D_1, \dots, D_n$ be effective $\mathbb{Q}$-divisors on X and $\mathsf{p} \in X$ be a smooth point. If p is an isolated component of $\operatorname{Supp}(D_1) \cap \cdots \cap \operatorname{Supp}(D_n)$, then

$$(D_1 \cdot \ldots \cdot D_n)_{\mathsf{p}} \ge \prod_{i=1}^{n} \operatorname{mult}_{\mathsf{p}}(D_i).$$

We refer readers to [Ful98, Corollary 12.4] for a proof. Although the following results are well-known to experts, we include their proofs for readers' convenience.

Lemma 3.2. *Let $\mathsf{p} \in X$ be either a germ of a smooth variety or a germ of a cyclic quotient singular point, and let D be an effective $\mathbb{Q}$-divisor on X. Then the inequality*

$$\frac{1}{\operatorname{omult}_{\mathsf{p}}(D)} \le \operatorname{lct}_{\mathsf{p}}(X, D)$$

holds.

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Proof. Let $q = q_{\mathsf{p}} \colon \check{X} \to X$ be the quotient morphism of $\mathsf{p} \in X$, which is étale in codimension 1, and let $\check{\mathsf{p}} \in \check{X}$ be the preimage of p. By [Kol+92, 20.4 Corollary], we have $\mathrm{lct}_{\mathsf{p}}(X; D) = \mathrm{lct}_{\check{\mathsf{p}}}(\check{X}; q^*D)$. Note that $\check{\mathsf{p}} \in \check{X}$ is smooth. Hence, by [Kol97, 8.10 Lemma], we have

$$\frac{1}{\mathrm{omult}_{\mathsf{p}}(D)} = \frac{1}{\mathrm{mult}_{\check{\mathsf{p}}}(q^*D)} \le \mathrm{lct}_{\check{\mathsf{p}}}(\check{X}; q^*D),$$

and the proof is completed. □

Lemma 3.3 ($2n$-inequality, cf. [Cor00, Corollary 3.5]). *Let $\mathsf{p} \in X$ be a germ of a smooth 3-fold, D an effective $\mathbb{Q}$-divisor on X, $n > 0$ a rational number, and let $\varphi \colon Y \to X$ be the blowup of X at p with exceptional divisor E. If $(X, \frac{1}{n}D)$ is not canonical at p, then there exists a line $L \subset E \cong \mathbb{P}^2$ with the following property.*

- *For any prime divisor T on X such that T is smooth at p and that its proper transform $\tilde{T}$ contains L, we have $\mathrm{mult}_{\mathsf{p}}(D|_T) > 2n$.*

Proof. We set $m = \mathrm{mult}_{\mathsf{p}}(D)$. By [Cor00, Corollary 3.5], one of the following holds.

1. $m > 2n$.
2. There is a line $L \subset E$ such that the pair
$$\left(Y, \left(\frac{m}{n} - 1\right)E + \frac{1}{n}\tilde{D}\right)$$
is not log canonical at the generic point of L.

Note that in [Cor00, Corollary 3.5] the boundary is a movable linear system $\mathcal{H}$, but the same argument applies if we replace $\mathcal{H}$ by an effective $\mathbb{Q}$-divisor D. We may assume $m \le 2n$ because otherwise $\mathrm{mult}_{\mathsf{p}}(D|_T) > 2n$ for any prime divisor T which is smooth at p and the assertion follows by choosing any line on E. Thus, the option (2) takes place. Let T be a prime divisor on X such that T is smooth at p and $\tilde{T} \supset L$. We have

$$K_Y + \left(\frac{m}{n} - 1\right)E + \frac{1}{n}\tilde{D} + \tilde{T} = \varphi^*\left(K_X + \frac{1}{n}D + T\right).$$

Note that $E|_{\tilde{T}} = L$, and we can write $\tilde{D}|_{\tilde{T}} = \alpha L + G$, where $\alpha \ge 0$ is a rational number and G is an effective $\mathbb{Q}$-divisor on $\tilde{T}$. Thus, by restricting the above equation to $\tilde{T}$, we have

$$K_{\tilde{T}} + \left(\frac{m}{n} - 1 + \alpha\right)L + G = \varphi^*\left(K_T + \frac{1}{n}D|_T\right),$$

and the pair

$$\left(\tilde{T}, \left(\frac{m}{n} - 1 + \alpha\right)L + G\right)$$

is not log canonical at the generic point of L. This implies $\frac{m}{n} - 1 + \alpha > 1$, and we have

$$\frac{1}{n}\mathrm{mult}_{\mathsf{p}}(D|_T) = \left(\frac{m}{n} - 1 + \alpha\right) + 1 > 2.$$

Thus, $\mathrm{mult}_{\mathsf{p}}(D|_T) > 2n$ and the proof is completed. □

Lemma 3.4. *Let $D \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ be a divisor on $\mathbb{P}^2$ which is not a triple line. Then $\mathrm{lct}(\mathbb{P}^2; D) \ge 1/2$.*

Proof. We have the following possibilities for D.

1. D is irreducible and reduced.
2. $D = Q + L$, where Q is an irreducible conic and L is a line.

3. $D = L_1 + 2L_2$, where L_1, L_2 are distinct lines.
4. $D = L_1 + L_2 + L_3$, where L_1, L_2, L_3 are mutually distinct lines.

If we are in one of the cases (1), (2) and (3), then $\mathrm{mult}_{\mathsf{p}}(D) \le 2$ for any point $\mathsf{p} \in D$ and thus $(\mathbb{P}^2, \frac{1}{2}D)$ is log canonical. If we are in case (3), then it is obvious that the pair $(\mathbb{P}^2, \frac{1}{2}D) = (\mathbb{P}^2, \frac{1}{2}L_1 + L_2)$ is log canonical. □

Lemma 3.5. *Let X be a Fano 3-fold of Picard number one, and let $\mathsf{p} \in X$ be a cyclic quotient terminal singular point (which is not a smooth point). If $\mathsf{p} \in X$ is not a maximal center, then there is at most one irreducible $\mathbb{Q}$-divisor $D \in |-K_X|_{\mathbb{Q}}$ such that (X, D) is not canonical at p.*

Proof. Suppose that there are two distinct irreducible $\mathbb{Q}$-divisors $D_i \sim_{\mathbb{Q}} -K_X$ such that (X, D_i) is not canonical at p for $i = 1, 2$. Let $r > 1$ be the index of the singularity $\mathsf{p} \in X$, and let $\varphi\colon Y \to X$ be the Kawamata blowup at p with exceptional divisor E. By [Kaw96], we have $\mathrm{ord}_E(D_i) > 1/r$. Take a positive integer n such that nD_1, nD_2 are both integral and $nD_1 \sim nD_2$. Then the pencil $\mathcal{M} \sim -nK_X$ generated by nD_1 and nD_2 is a movable linear system and we have $\mathrm{ord}_E(\mathcal{M}) \ge n/r$. It follows that the pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at p. This is a contradiction since $\mathsf{p} \in X$ is not a maximal center. □

Lemma 3.6. *Let*

$$X = X_d \subset \mathbb{P}(1, b_1, b_2, b_3, b_4)_{x, y_1, y_2, y_3, y_4}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. Let $i \in \{1, 2, 3, 4\}$ be such that $b_i > 1$ and $\mathsf{p} := \mathsf{p}_{y_i} \in X$. If H_x is the quasi-tangent divisor of X at p, then the pair (X, H_x) is not canonical at p.

Proof. Note that the point $\mathsf{p} \in X$ is of type $\frac{1}{b_i}(b_j, b_k, b_l)$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and it is a terminal singularity. Let $\varphi\colon Y \to X$ be the Kawamata blowup with exceptional divisor E. Since H_x is the quasi-tangent divisor of X at p, we have

$$\mathrm{ord}_E(H_x) > \frac{1}{b_i}.$$

Combining this with

$$K_Y = \varphi^* K_X + \frac{1}{b_i} E,$$

we see that the discrepancy of the pair (X, H_x) along E is negative. This completes the proof. □

3.1.b. Some results on singularities of weighted hypersurfaces

Lemma 3.7. *Let X be a quasi-smooth weighted hypersurface in $\mathbb{P}(b_0, \dots, b_4)$. Assume that $\mathbb{P}(b_0, \dots, b_4)$ is well-formed and X has at most isolated singularities. Then any quasi-hyperplane section on X is a normal surface.*

Proof. Let $x_0, \dots, x_4$ be the homogeneous coordinates of $\mathbb{P} = \mathbb{P}(b_0, \dots, b_4)$ of degree $b_0, \dots, b_4$, respectively. Let $F = F(x_0, \dots, x_4)$ be the defining polynomial of X, and let S be a quasi-hyperplane section on X. After replacing homogeneous coordinates, we may assume $S = (x_4 = 0)_X = (x_4 = F = 0) \subset \mathbb{P}$. It is enough to show that the singular locus $\mathrm{Sing}(S)$ of S is a finite set of points.

We write $F = x_4 G + \bar{F}$, where $G = G(x_0, \dots, x_4)$ and $\bar{F} = \bar{F}(x_0, \dots, x_3)$ are quasi-homogeneous polynomials. We set $\bar{\mathbb{P}} = \mathbb{P}(b_0, \dots, b_3)$. We claim that $\bar{\mathbb{P}}$ is well-formed. Suppose it is not. Then $\mathrm{Sing}(\mathbb{P})$ contains a two-dimensional stratum. We have $\mathrm{Sing}(X) = \mathrm{Sing}(\mathbb{P}) \cap X$ since a quasi-smooth weighted hypersurface is well-formed ([IF00, Theorem 6.17]). It follows that $\mathrm{Sing}(X)$ cannot be a finite set of points. This is a contradiction, and the claim is proved. The surface S is identified with the hypersurface $(\bar{F} = 0) \subset \bar{\mathbb{P}}$, and we have

$$\mathrm{Sing}(S) = (S \setminus \mathrm{QSm}(S)) \cup (\mathrm{Sing}(\bar{\mathbb{P}}) \cap S).$$

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We claim that $\mathrm{Sing}(\bar{\mathbb{P}}) \cap S$ is a finite set of points. Suppose not. Then $\mathrm{Sing}(\bar{\mathbb{P}}) \cap S$ contains a curve and so does $\mathrm{Sing}(\mathbb{P}) \cap S$. In particular, $\mathrm{Sing}(X) = \mathrm{Sing}(\mathbb{P}) \cap X$ contains a curve. This is impossible since $\mathrm{Sing}(X)$ is a finite set of points, and the claim is proved.

It remains to show that the closed subset $\Sigma := S \setminus \mathrm{QSm}(S) \subset \mathbb{P}$ is a finite set of points. Let $\Pi\colon \mathbb{A}^5 \setminus \{o\} \to \mathbb{P}$ be the natural quotient morphism. Then $\Sigma = \Pi(\mathrm{Sing}(C_S^*))$, where $C_S^* \subset \mathbb{A}^5 \setminus \{o\}$ is the punctured affine quasi-cone of S. We have $\mathrm{Sing}(C_X) \cap C_S = \mathrm{Sing}(C_S) \cap (G = 0)$. By the quasi-smoothness of X, we have $\mathrm{Sing}(C_X) = \{o\} \subset \mathbb{A}^5$. This implies $\Sigma \cap (G = 0) = \emptyset$. Since $(G = 0)$ is an ample divisor on $\mathbb{P}$, we see that Σ is a finite set of points. This completes the proof. □

Lemma 3.8. *Let S be a weighted hypersurface in $\mathbb{P}(b_0, b_1, b_2, b_3)$, and let $T \subset \mathbb{P}(b_0, b_1, b_2, b_3)$ be a quasi-hyperplane. If the scheme-theoretic intersection $S \cap T$ is quasi-smooth at a point p, then S is quasi-smooth at p.*

Proof. Let x_0, x_1, x_2, x_3 be the homogeneous coordinates of $\mathbb{P} = \mathbb{P}(b_0, b_1, b_2, b_3)$ of degree b_0, b_1, b_2, b_3, respectively, and let $F = F(x_0, x_1, x_2, x_3)$ be the defining polynomial of S. We may assume $T = H_{x_3} \subset \mathbb{P}$, and we write $F = x_3 G + \bar{F}$, where $G = G(x_0, x_1, x_2, x_3)$ and $\bar{F} = \bar{F}(x_0, x_1, x_2)$ are quasi-homogeneous polynomials. Then $S \cap T$ is the closed subscheme in $\mathbb{P}(b_0, b_1, b_2, b_3)$ defined by $x_3 = \bar{F} = 0$. By the quasi-smoothness of $S \cap T$ at p, there exists $i \in \{0, 1, 2\}$ such that

$$\frac{\partial \bar{F}}{\partial x_i}(\mathsf{p}) \neq 0.$$

It follows that

$$\frac{\partial F}{\partial x_i}(\mathsf{p}) = \frac{\partial \bar{F}}{\partial x_i}(\mathsf{p}) \neq 0$$

since $\mathsf{p} \in H_{x_3}$. Thus, S is quasi-smooth at p. □

Lemma 3.9. *Let S be a normal weighted hypersurface in a well-formed weighted projective 3-space $\mathbb{P}(b_0, \dots, b_3)$ and $T \subset \mathbb{P}(b_0, \dots, b_3)$ a quasi-hyperplane such that $T \neq S$. Let Γ be an irreducible component of $S \cap T$, and we assume that*

$$T|_S = \Gamma + \Delta,$$

where Δ is an effective divisor on S such that $\Gamma \not\subset \mathrm{Supp}(\Delta)$. If Γ is a smooth weighted complete intersection curve and S is quasi-smooth at each point of $\Gamma \cap \mathrm{Supp}(\Delta)$, then S is quasi-smooth along Γ and the pair (S, Γ) is purely log terminal (plt) along Γ.

Proof. We set $\Xi = \Gamma \cap \mathrm{Supp}(\Delta)$. By [IF00, Theorem 12.1], Γ is quasi-smooth. We have $(S \cap T) \setminus \Xi = \Gamma \setminus \Xi$. It follows that $S \cap T$ is quasi-smooth along $\Gamma \setminus \Xi$. By Lemma 3.8, S is quasi-smooth along $\Gamma \setminus \Xi$. Therefore, S is quasi-smooth along Γ.

For $i = 0, 1, 2, 3$, let $S_i = (x_i \neq 0) \cap S$ be the standard open set of S and let $\rho_i\colon \breve{S}_i \to S_i$ be the orbifold chart. Note that ρ_i is a finite surjective morphism of degree b_i which is étale in codimension 1. By the quasi-smoothness of S, the affine varieties $\breve{S}_i$ and $\rho_i^*(\Gamma \cap S_i)$ are smooth. Hence, the pair $(\breve{S}_i, \rho_i^*(\Gamma \cap S_i))$ is plt along $\rho_i^*(\Gamma \cap S_i)$. By [Kol+92, Corollary 20.4], the pair $(S_i, \Gamma \cap S_i)$ is plt along $\Gamma \cap S_i$. This completes the proof. □

Remark 3.10. Let S, T and Γ be as in Lemma 3.9. We assume in addition that Γ is rational, that is, $\Gamma \cong \mathbb{P}^1$. Let $\mathrm{Sing}_\Gamma(S) = \{\mathsf{p}_1, \dots, \mathsf{p}_n\}$ be the set of singular points of S along Γ, and let m_i be the index of the quotient singular point $\mathsf{p}_i \in S$. Then, since the pair (S, Γ) is plt along Γ, we can apply [Kol+92, Proposition 16.6] and we have

$$(K_S + \Gamma)|_\Gamma = K_\Gamma + \sum_{i=1}^{n} \frac{m_i - 1}{m_i} \mathsf{p}_i.$$

Thus, we have

$$(\Gamma^2)_S = -(K_S \cdot \Gamma)_S - 2 + \sum_{i=1}^{n} \frac{m_i - 1}{m_i}.$$

3.1.c. Isolating set and class

We recall the definitions of isolating set and class which are introduced by Corti, Pukhlikov and Reid [CPR00] as well as their basic properties.

Let V be a normal projective variety embedded in a weighted projective space $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$ with homogeneous coordinates $x_0, \ldots, x_N$ with $\deg x_i = a_i$, and let A be a Weil divisor on V such that $\mathcal{O}_V(A) \cong \mathcal{O}_V(1)$. We do not assume that $a_0 \leq \cdots \leq a_N$.

Definition 3.11. Let $\mathsf{p} \in V$ be a point. We say that a set $\{g_1, \ldots, g_m\}$ of quasi-homogeneous polynomials $g_1, \ldots, g_m \in \mathbb{C}[x_0, \ldots, x_N]$ *isolates* p or is a p-*isolating set* if p is an isolated component of the set

$$(g_1 = \cdots = g_m = 0) \cap V.$$

Definition 3.12. Let $\mathsf{p} \in V$ be a smooth point, and let L be a Weil divisor class on V. For positive integers k and l, we define $|\mathcal{I}_{\mathsf{p}}^k(lL)|$ to be the linear subsystem of $|lL|$ consisting of divisors vanishing at p with multiplicity at least k. We say that L *isolates* p or is a p-*isolating class* if p is an isolated component of the base locus of $|\mathcal{I}_{\mathsf{p}}^k(kL)|$.

Lemma 3.13 [CPR00, Lemma 5.6.4]. *Let $\mathsf{p} \in V$ be a smooth point. If $\{g_1, \ldots, g_m\}$ is a p-isolating class, then lA is a p-isolating class, where*

$$l = \max\{\, \deg g_i \mid i = 1, 2, \ldots, m \,\}.$$

Lemma 3.14. *Let $\mathsf{p} \in V$ be a point, $Z_1, \ldots, Z_k$ irreducible closed subsets of V such that* $\dim Z_i > 0$ *for any i, and let $g_1, \ldots, g_n \in \mathbb{C}[x_0, \ldots, x_N]$ be quasi-homogeneous polynomials. Suppose that V is quasi-smooth at p and that $\{g_1, \ldots, g_n\}$ isolates p. We set $G_i = (g_i = 0)_V$, and we set*

$$\mu := \min\left\{ \frac{\operatorname{omult}_{\mathsf{p}}(G_i)}{\deg g_i} \,\middle|\, i = 1, \ldots, n \right\}.$$

Then there exists an effective $\mathbb{Q}$-divisor $T \sim_{\mathbb{Q}} A$ such that $\operatorname{omult}_{\mathsf{p}}(T) \geq \mu$ *and* $\operatorname{Supp}(T)$ *does not contain any Z_i.*

Proof. Let d be the least common multiple of $\deg g_1, \ldots, \deg g_n$, and we set $e_i = d/\deg g_i$. Consider the linear system $\Lambda \subset |dA|$ on V generated by $g_1^{e_1}, \ldots, g_n^{e_n}$. We see that p is an isolating component of $\operatorname{Bs}\Lambda$ since $\{g_1, \ldots, g_n\}$ isolates p. Hence, a general $D \in \Lambda$ does not contain any Z_i in its support. Moreover, for any $D \in \Lambda$, we have

$$\operatorname{omult}_{\mathsf{p}}(D) \geq \min\{\, e_i \operatorname{omult}_{\mathsf{p}}(D_i) \mid i = 1, \ldots, n \,\} = d\mu.$$

Thus, the assertion follows by setting $T = \frac{1}{d} D \sim_{\mathbb{Q}} A$ for a general $D \in \Lambda$. □

Remark 3.15. Lemma 3.14 will be frequently applied in the following way: under the same notation and assumptions as in Lemma 3.14, there exists an effective $\mathbb{Q}$-divisor $T \sim_{\mathbb{Q}} eA$, where

$$e = \max\{\, \deg g_i \mid i = 1, \ldots, n \,\}$$

such that $\operatorname{omult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any Z_i.

Lemma 3.16. *Let $X = X_d \subset \mathbb{P}(1, a_1, \ldots, a_4)_{x,y,z,t,w}$ be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$, where we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$, and let $\mathsf{p} \in H_x \setminus L_{xy}$. Then $a_1 a_4 A$ isolates p. If w^k appears in the defining polynomial of X with nonzero coefficient, then $a_1 a_3 A$ isolates p.*

Proof. We can write $\mathsf{p} = (0\!:\!1\!:\!\alpha_2\!:\!\alpha_3\!:\!\alpha_4)$ for some $\alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}$. Then it is easy to see that the set

$$\{x, z^{a_1} - \alpha_2^{a_1} y^{a_2}, t^{a_1} - \alpha_3^{a_3} y^{a_3}, w^{a_1} - \alpha_4^{a_4} y^{a_4}\}$$

isolates p, and thus $a_1 a_4 A$ isolates p.

Suppose that w^k appears in the defining polynomial of X. Then the natural projection $\mathbb{P}(1, a_1, \dots, a_4) \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ restricts to a finite morphism $\pi \colon X \to \mathbb{P}(1, a_1, a_2, a_3)$. The common zero locus (in X) of the sections contained in the set

$$\{x, z^{a_1} - \alpha_2^{a_1} y^{a_2}, t^{a_1} - \alpha_3^{a_3} y^{a_3}\}$$

coincides with the set $\pi^{-1}(\mathsf{q})$, where $\mathsf{q} = (0\!:\!1\!:\!\alpha_2\!:\!\alpha_3) \in \mathbb{P}(1, a_1, a_2, a_3)$. It follows that the above set isolates p since $\pi^{-1}(\mathsf{q})$ is a finite set containing p. Thus, $a_1 a_3 A$ isolates p. □

3.2. *Methods*

3.2.a. Computations by intersecting two divisors

We recall methods of computing log canonical thresholds (LCTs) and consider their generalizations for some of them.

Lemma 3.17 (cf. [KOW18, Lemma 2.5]). *Let X be a normal projective $\mathbb{Q}$-factorial 3-fold with nef and big anticanonical divisor, and let $\mathsf{p} \in X$ be either a smooth point or a terminal quotient singular point of index r (below we set $r = 1$ when $\mathsf{p} \in X$ is a smooth point). Suppose that there are prime divisors $S \sim_{\mathbb{Q}} -aK_X$ and $T \sim_{\mathbb{Q}} -bK_X$ with $a, b \in \mathbb{Q}$ such that $S \cap T$ is irreducible and $q^*S \cdot q^*T = m\check{\Gamma}$, where $q = q_{\mathsf{p}} \colon \check{U} \to U$ is the quotient morphism of an analytic neighborhood $\mathsf{p} \in U$ of $\mathsf{p} \in X$, $\check{\mathsf{p}}$ is the preimage of p via q, m is a positive integer and $\check{\Gamma}$ is an irreducible and reduced curve on $\check{U}$. Then we have*

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{\operatorname{lct}_{\mathsf{p}}(X; \tfrac{1}{a}S), \ \frac{b}{m \operatorname{mult}_{\check{\mathsf{p}}}(\check{\Gamma})}, \ \frac{1}{rab(-K_X)^3}\right\}.$$

Proof. We set

$$c := \min\left\{\operatorname{lct}_{\mathsf{p}}(X; \tfrac{1}{a}S), \ \frac{b}{m \operatorname{mult}_{\check{\mathsf{p}}}(\check{\Gamma})}, \ \frac{1}{rab(-K_X)^3}\right\}.$$

We will derive a contradiction assuming $\alpha_{\mathsf{p}}(X) < c$. By the assumption, there is an irreducible $\mathbb{Q}$-divisor $D \in |-K_X|_{\mathbb{Q}}$ such that (X, cD) is not log canonical at p. Then the pair $(\check{U}, c\rho^*D)$ is not log canonical at $\check{\mathsf{p}}$ and we have

$$\operatorname{mult}_{\check{\mathsf{p}}}(q^*D) > \frac{1}{c}. \tag{3.1}$$

Since $q^*S \cdot q^*T = m\check{\Gamma}$ and $S \cap T$ is irreducible, we have $S \cdot T = m\Gamma$, where Γ is an irreducible and reduced curve such that $\check{\Gamma} = q^*\Gamma$. We have

$$(-K_X \cdot \Gamma) = \frac{1}{m}(-K_X \cdot S \cdot T) = \frac{ab(-K_X)^3}{m}. \tag{3.2}$$

This in particular implies

$$(T \cdot \Gamma) = b(-K_X \cdot \Gamma) = \frac{ab^2(-K_X)^3}{m}. \tag{3.3}$$

We have $\operatorname{Supp}(D) \ne S$ since $\operatorname{lct}_{\mathsf{p}}(X; S) \ge c$, and thus $q^*D \cdot q^*S$ is an effective 1-cycle on $\check{U}$. We write $q^*D \cdot q^*S = \gamma\check{\Gamma} + \check{\Delta}$, where $\gamma \ge 0$ and $\check{\Delta}$ is an effective 1-cycle on $\check{U}$ such that $\check{\Gamma} \not\subset \operatorname{Supp}(\check{\Delta})$.

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Then $D \cdot S = \gamma\Gamma + \Delta + \Xi$, where $\Delta = \frac{1}{r} q_* \check{\Delta}$ and Ξ is an effective 1-cycle such that $\Gamma \not\subset \mathrm{Supp}(\Xi)$. By equation (3.2), we have

$$a(-K_X)^3 = (-K_X \cdot D \cdot S) \geq \gamma(-K_X \cdot \Gamma) = \frac{ab(-K_X)^3}{m}\gamma,$$

where the inequality holds since $-K_X$ is nef. Note that $(-K_X)^3 > 0$ since $-K_X$ is nef and big. Hence,

$$\gamma \leq \frac{m}{b}. \tag{3.4}$$

By equations (3.1) and (3.3), we have

$$\begin{aligned} r\left(ab(-K_X)^3 - \frac{ab^2(-K_X)^3}{m}\gamma\right) &= r(T \cdot (D \cdot S - \gamma\Gamma)) \\ &= r(T \cdot (\Delta + \Xi)) \geq r(T \cdot \Delta) \\ &\geq r(T \cdot \Delta)_{\mathsf{p}} = (q^* T \cdot \check{\Delta})_{\check{\mathsf{p}}} \\ &\geq \mathrm{mult}_{\check{\mathsf{p}}}(\check{\Delta}) \\ &> \frac{1}{c} - \gamma\, \mathrm{mult}_{\check{\mathsf{p}}}(\check{\Gamma}), \end{aligned}$$

where $(\,\cdot\,)_{\mathsf{p}}$ and $(\,\cdot\,)_{\check{\mathsf{p}}}$ denote the local intersection numbers at p and $\check{\mathsf{p}}$, respectively. It follows that

$$\left(\mathrm{mult}_{\check{\mathsf{p}}}(\check{\Gamma}) - \frac{rab^2(-K_X)^3}{m}\right)\gamma > \frac{1}{c} - rab(-K_X)^3. \tag{3.5}$$

We have $\mathrm{mult}_{\check{\mathsf{p}}}(\check{\Gamma}) - rab^2(-K_X)^3/m > 0$ since $1/c - rab(-K_X)^3 \geq 0$ by the definition of c. Combining equations (3.4) and (3.5), we have

$$c > \frac{b}{m\, \mathrm{mult}_{\check{\mathsf{p}}}(\check{\Gamma})}.$$

This contradicts the definition of c and the proof is completed. □

Lemma 3.17 is very useful in computing alpha invariants but works only when $S \cap T$ is irreducible. We consider its generalization that can be applied when $S \cap T$ is reducible.

Definition 3.18. Let $M = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbb{R}$, where $n \geq 2$. For a nonempty subset $I \subset \{1, 2, \dots, n\}$, we denote by M_I the submatrix of M consisting of ith rows and columns for $i \in I$. We say that M satisfies the *condition* $(\star)$ if the following are satisfied.

- $(-1)^{|I|} \det M_I \geq 0$ for any nonempty proper subset $I \subset \{1, 2, \dots, n\}$.
- $(-1)^{n-1} \det M > 0$.
- $a_{ij} > 0$ for any i, j with $i \neq j$.

For $v = {}^t(v_1, \dots, v_n), w = {}^t(w_1, \dots, w_n) \in \mathbb{R}^n$, the expression $v \leq w$ means $v_i \leq w_i$ for any i.

Lemma 3.19. *Let $M = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbb{R}$ satisfying the condition $(\star)$, and let $v, w \in \mathbb{R}^n$. Then $Mv \leq Mw$ implies $v \leq w$.*

Proof. It is enough to show that $v \leq 0$ assuming $Mv \leq 0$ for $v \in \mathbb{R}^n$. We prove this assertion by induction on $n \geq 2$. The case $n = 2$ is easily done, and we omit it.

Assume $n \geq 3$. Suppose that there is a diagonal entry a_{kk} such that $a_{kk} = 0$. Then we have $\det M_{\{k,l\}} < 0$ since $a_{kl}, a_{lk} > 0$. By the condition $(\star)$, this is impossible since $n \geq 3$.

In the following, we may assume that $a_{ii} \neq 0$ for any i. By the condition $(\star)$, we have $a_{ii} = \det M_{\{i\}} \leq 0$ and hence $a_{ii} < 0$ for any i. Let M' be the matrix obtained by adding the first row multiplied by the

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positive integer $-a_{i1}/a_{11}$ to the ith row, for $i = 2, \dots, n$. Then we obtain the inequality $M'v \leq 0$ and we can write

$$M' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & N' & \\ 0 & & & \end{pmatrix},$$

where N' is an $(n-1) \times (n-1)$ matrix. It is straightforward to check that N satisfies the condition $(\star)$. Since $N'^t(v_2 \ \dots \ v_n) \leq 0$, we have $v_2, \dots, v_n \leq 0$ by induction hypothesis. Next, let M'' be the matrix obtained by adding the nth row multiplied by the positive integer $-a_{in}/a_{nn}$ to the ith row, for $i = 1, 2, \dots, n-1$. Then we have $M''v \leq 0$ and, by repeating the similar argument as above, we conclude $v_1, \dots, v_{n-1} \leq 0$ by induction. This completes the proof. □

Definition 3.20. Let S be a normal projective surface, and let $\Gamma_1, \dots, \Gamma_k$ be irreducible and reduced curves on S. Then the $k \times k$ matrix

$$M(\Gamma_1, \dots, \Gamma_k) = ((\Gamma_i \cdot \Gamma_j)_S)_{1 \leq i,j \leq k}$$

is called the *intersection matrix* of curves $\Gamma_1, \dots, \Gamma_k$ on S.

Lemma 3.21. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. Let $S \in |-aK_X|$ be a normal surface on X, $T \in |-bK_X|$ an effective divisor and $\mathsf{p} \in S$ a point, where $a, b > 0$. We set $r = 1$ when $\mathsf{p} \in X$ is a smooth point, and otherwise we denote by r the index of the cyclic quotient singularity $\mathsf{p} \in X$. Suppose that*

$$T|_S = m_1\Gamma_1 + m_2\Gamma_2 + \cdots + m_k\Gamma_k,$$

where $\Gamma_1, \dots, \Gamma_k$ are distinct irreducible and reduced curves on S and $m_1, \dots, m_k$ are positive integers, and the following properties are satisfied.

- $rb \deg \Gamma_1 \leq m_1$.
- $\mathsf{p} \in \Gamma_1 \setminus (\cup_{i \geq 2}\Gamma_i)$*, and S, Γ_1 are both quasi-smooth at* p.
- *The intersection matrix $M(\Gamma_1, \dots, \Gamma_k)$ satisfies the condition $(\star)$.*

Then we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{a, \ \frac{m_1}{rab(-K_X)^3 + \frac{m_1^2}{b} - rm_1 \deg \Gamma_1}\right\}.$$

Proof. Let $D \in |-K_X|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor. If $\mathrm{Supp}(D) = S$, then $D = \frac{1}{a}S$ and we have $\mathrm{lct}_{\mathsf{p}}(X, D) \geq a$ since S is quasi-smooth at p. We assume $\mathrm{Supp}(D) \neq S$. It is enough to prove the inequality

$$\mathrm{lct}_{\mathsf{p}}(X; D) \geq \frac{m_1}{rab(-K_X)^3 + \frac{m_1^2}{b} - rm_1 \deg \Gamma_1}. \tag{3.6}$$

We can write

$$D|_S = \gamma_1\Gamma_1 + \cdots + \gamma_k\Gamma_k + \Delta,$$

where $\gamma_1, \dots, \gamma_k \geq 0$ and Δ is an effective $\mathbb{Q}$-divisor on S such that $\Gamma_i \not\subset \mathrm{Supp}(\Delta)$ for $i = 1, \dots, k$. We set $\sigma_i = (\Gamma_i^2)_S$ and $\chi_{i,j} = (\Gamma_i \cdot \Gamma_j)_S$. For $i = 1, \dots, k$, we have

$$\begin{aligned} b \deg \Gamma_i &= (T|_S \cdot \Gamma_i)_S \\ &= m_1\chi_{1,i} + \cdots + m_{i-1}\chi_{i-1,i} + m_i\sigma_i + m_{i+1}\chi_{i+1,i} + \cdots + m_k\chi_{k,i}, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\deg \Gamma_i &= (D|_S \cdot \Gamma_i)_S \\ &\geq \gamma_1 \chi_{1,i} + \cdots + \gamma_{i-1}\chi_{i-1} + \gamma_i \sigma_i + \gamma_{i+1}\chi_{i+1,i} + \cdots + \gamma_k \chi_{k,i+1}.\end{aligned} \tag{3.8}$$

We set $M = M(\Gamma_1, \dots, \Gamma_k)$. Combining inequalities (3.7) and (3.8), we have

$$M \begin{pmatrix} b\gamma_1 \\ \vdots \\ b\gamma_k \end{pmatrix} \leq M \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix}.$$

By Lemma 3.19, this implies $\gamma_i \leq m_i/b$ for any i.

When p is a singular point, then we set $\rho = \rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}}$, which is the orbifold chart of X containing p. When p is a smooth point of X, then we set $\check{U} = U = X$ and $\rho \colon \check{U} \to U$ is assumed to be the identity morphism. Moreover, we set $\check{S} := q^*(S \cap U)$ and $\rho_S = \rho|_{\check{S}} \colon \check{S} \to S \cap U$. We see that $\check{S}$ is smooth at the preimage $\check{\mathsf{p}}$ of p since S is quasi-smooth at p, and

$$\rho^* D|_{\check{S}} = \gamma_1 \rho_S^* \Gamma_1 + \cdots \gamma_k \rho_S^* \Gamma_k + \rho_S^* \Delta.$$

This implies

$$\operatorname{mult}_{\check{\mathsf{p}}}(\rho_S^* \Delta) \geq \operatorname{omult}_{\mathsf{p}}(D) - \gamma_1$$

since $\rho_S^* \Gamma_i$ does not pass through $\check{\mathsf{p}}$ for $i \geq 2$ and $\rho_S^* \Gamma_1$ is smooth at $\check{\mathsf{p}}$ by the quasi-smoothness of Γ_1 at p. We have

$$\begin{aligned} r(ab(-K_X)^3 - b\gamma_1 \deg \Gamma_1) &\geq r(T|_S \cdot (D|_S - \gamma_1 \Gamma_1 - \cdots - \gamma_k \Gamma_k))_S \\ &= r(T|_S \cdot \Delta)_S \\ &\geq m_1 r(\Gamma_1 \cdot \Delta)_S \\ &\geq m_1 (\rho_S^* \Gamma_1 \cdot \rho_S^* \Delta)_{\check{\mathsf{p}}} \\ &\geq m_1 \operatorname{mult}_{\check{\mathsf{p}}}(\rho_S^* \Delta) \\ &\geq m_1 (\operatorname{omult}_{\mathsf{p}}(D) - \gamma_1). \end{aligned}$$

Since $m_1 - rb \deg \Gamma_1 \geq 0$ and $\gamma_1 \leq m_1/b$, we have

$$\begin{aligned} \operatorname{omult}_{\mathsf{p}}(D) &\leq \frac{1}{m_1}(rab(-K_X)^3 + (m_1 - rb \deg \Gamma_1)\gamma_1) \\ &\leq \frac{1}{m_1}\left(rab(-K_X)^3 + \frac{m_1^2}{b} - rm_1 \deg \Gamma_1\right). \end{aligned}$$

. This implies equation (3.6), and the proof is completed. □

The following is a version of Lemma 3.21, which may be effective when S is singular at p.

Lemma 3.22. *Let X be a normal projective $\mathbb{Q}$-factorial 3-fold. Let $S \sim_{\mathbb{Q}} -aK_X$ be a normal surface on X, $T \sim_{\mathbb{Q}} -bK_X$ an effective divisor and $\mathsf{p} \in S$ a point, where a, b are positive rational numbers. Suppose that*

$$T|_S = m_1 \Gamma_1 + m_2 \Gamma_2 + \cdots + m_k \Gamma_k,$$

where $\Gamma_1, \dots, \Gamma_k$ are distinct irreducible and reduced curves on S and $m_1, \dots, m_k$ are positive integers, and the following properties are satisfied.

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- $b \deg \Gamma_1 \leq \operatorname{mult}_{\mathsf{p}}(\Gamma_1)$.
- $\mathsf{p} \in \Gamma_1 \setminus (\cup_{i \geq 2}\Gamma_i)$, *and* X *is smooth at* p.
- *The intersection matrix* $M(\Gamma_1, \ldots, \Gamma_k)$ *satisfies the condition* $(\star)$.

Then we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{\frac{a}{\operatorname{mult}_{\mathsf{p}}(S)}, \ \frac{\operatorname{mult}_{\mathsf{p}}(S)}{ab(-K_X)^3 + \frac{m_1}{b}\operatorname{mult}_{\mathsf{p}}(\Gamma_1) - m_1 \deg \Gamma_1}\right\}.$$

Proof. Let $D \in |-K_X|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor. If $\operatorname{Supp}(D) = S$, then $D = \frac{1}{a}S$ and we have $\operatorname{lct}_{\mathsf{p}}(X, D) \geq a/\operatorname{mult}_{\mathsf{p}}(S)$. We assume $\operatorname{Supp}(D) \neq S$. It is enough to show that

$$\operatorname{lct}_{\mathsf{p}}(X; D) \geq \frac{\operatorname{mult}_{\mathsf{p}}(S)}{ab(-K_X)^3 + \frac{m_1}{b}\operatorname{mult}_{\mathsf{p}}(\Gamma_1) - m_1 \deg \Gamma_1}. \tag{3.9}$$

We write

$$D|_S = \gamma_1\Gamma_1 + \cdots + \gamma_k\Gamma_k + \Delta,$$

where $\gamma_1, \ldots, \gamma_k \geq 0$ and Δ is an effective divisor on S such that $\Gamma_i \not\subset \operatorname{Supp}(\Delta)$ for $i = 1, \ldots, k$. By the same argument as in the proof of Lemma 3.21, we have $\gamma_i \leq m_i/b$ for any i. We consider the 1-cycle $D \cdot S = \gamma_1\Gamma_1 + \cdots + \gamma_k\Gamma_k$ on X, and we have

$$\begin{aligned} ab(-K_X)^3 - b\gamma_1 \deg \Gamma_1 &\geq (T \cdot (D \cdot S - \gamma_1\Gamma_1 - \cdots - \gamma_k\Gamma_k))_X \\ &= (T \cdot \Delta)_X \\ &\geq \operatorname{mult}_{\mathsf{p}}(\Delta) \\ &\geq (\operatorname{mult}_{\mathsf{p}}(S))(\operatorname{mult}_{\mathsf{p}}(D)) - \gamma_1 \operatorname{mult}_{\mathsf{p}}(\Gamma_1). \end{aligned}$$

Since $\operatorname{mult}_{\mathsf{p}}(\Gamma_1) - b \deg \Gamma_1 \geq 0$ and $\gamma_1 \leq m_1/b$, we have

$$\begin{aligned} \operatorname{mult}_{\mathsf{p}}(D) &\leq \frac{1}{\operatorname{mult}_{\mathsf{p}}(S)}(ab(-K_X)^3 + (\operatorname{mult}_{\mathsf{p}}(\Gamma_1) - b \deg \Gamma_1)\gamma_1) \\ &\leq \frac{1}{\operatorname{mult}_{\mathsf{p}}(S)}\left(ab(-K_X)^3 + \frac{m_1}{b}\operatorname{mult}_{\mathsf{p}}(\Gamma_1) - m_1 \deg \Gamma_1\right). \end{aligned}$$

. This implies equation (3.9), and the proof is completed. □

Lemma 3.23. *Let* X *be a normal projective* $\mathbb{Q}$*-factorial* 3*-fold. Let* $S \sim_{\mathbb{Q}} -aK_X$ *be a normal surface on* X*,* $T \sim_{\mathbb{Q}} -bK_X$ *an effective divisor and* $\mathsf{p} \in X$ *a point, where* a, b *be positive rational numbers. Suppose that*

$$T|_S = \Gamma_1 + \Gamma_2,$$

where Γ_1, Γ_2 *are distinct irreducible and reduced curves on* S*, and the following properties are satisfied.*

- $\deg \Gamma_i \leq 2/b$ *for* $i = 1, 2$.
- $\mathsf{p} \in \Gamma_1 \cap \Gamma_2$ *and all the* X*,* S*,* Γ_1 *and* Γ_2 *are smooth at* p.
- *The intersection matrix* $M(\Gamma_1, \Gamma_2)$ *satisfies the condition* $(\star)$.

Then we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{a, \ \frac{b}{2}\right\}.$$

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Proof. We have $\operatorname{lct}_{\mathsf{p}}(X, \frac{1}{a}S) \geq a$ since S is smooth at p by assumption. Let $D \in |-K_X|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X such that $\operatorname{Supp}(D) \neq S$. It is enough to prove the inequality $\operatorname{lct}_{\mathsf{p}}(X; D) \geq b/2$. We write

$$D|_S = \gamma_1 \Gamma_1 + \gamma_2 \Gamma_2 + \Delta,$$

where $\gamma_1, \gamma_2 \geq 0$ and Δ is an effective divisor on S with $\Gamma_1, \Gamma_2 \not\subset \operatorname{Supp}(\Delta)$. By the proof of Lemma 3.21, we have $\gamma_1, \gamma_2 \leq 1/b$. We have

$$ab(-K_X)^3 = (-K_X|_S \cdot T|_S)_S = \deg \Gamma_1 + \deg \Gamma_2. \tag{3.10}$$

Since $\operatorname{mult}_{\mathsf{p}}(T|_S) = 2$ and $\operatorname{mult}_{\mathsf{p}}(\Delta) \geq \operatorname{mult}_{\mathsf{p}}(D) - \gamma_1 - \gamma_2$, we have

$$\begin{aligned} ab(-K_X)^3 - b\gamma_1 \deg \Gamma_1 - b\gamma_2 \deg \Gamma_2 &= (T|_S \cdot (D|_S - \gamma_1 \Gamma_1 - \gamma_2 \Gamma_2))_S \\ &= (T|_S \cdot \Delta)_S \\ &\geq 2(\operatorname{mult}_{\mathsf{p}}(D) - \gamma_1 - \gamma_2). \end{aligned}$$

By equation (3.10), the assumption $\deg \Gamma_1, \deg \Gamma_2 \leq 2/b$ and $\gamma_1, \gamma_2 \leq 1/b$, we have

$$\begin{aligned} \operatorname{mult}_{\mathsf{p}}(D) &\leq \frac{1}{2}(ab(-K_X)^3 + (2 - b \deg \Gamma_1)\gamma_1 + (2 - b \deg \Gamma_2)\gamma_2) \\ &\leq \frac{2}{b}. \end{aligned}$$

This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq b/2$ and thus $\alpha_{\mathsf{p}}(X) \geq \min\{a, b/2\}$. □

3.2.b. Computations by weighted blowups

We explain methods of computing LCTs via suitable weighted blowups.

Let $\mathsf{p} \in X$ be a germ of a smooth variety of dimension n with a system of local coordinates $\{x_1, \dots, x_n\}$ at p, and let D be an effective $\mathbb{Q}$-divisor on X. Let $\varphi \colon Y \to X$ be the weighted blowup at p with weight $\operatorname{wt}(x_1, \dots, x_n) = (c_1, \dots, c_n)$, where $\underline{c} = (c_1, \dots, c_n)$ is a tuple of positive integers such that $\gcd\{c_1, \dots, c_n\} = 1$. Let $E \cong \mathbb{P}(\underline{c}) = \mathbb{P}(c_1, \dots, c_n)$ be the exceptional divisor of φ. Note that Y can be singular along a divisor on E (see Remark 3.24 below) so that we cannot expect the usual adjunction $(K_Y + E)|_E = K_E$. In general, we need a correction term and we have

$$(K_Y + E)|_E = K_E + \operatorname{Diff}$$

where the correction term Diff is a $\mathbb{Q}$-divisor on E which is called the *different* (see [Kol+92, Chapter 16]).

Remark 3.24. We give a concrete description of Diff. Let $\mathbb{P}(\underline{c})^{\mathrm{wf}}$ be the well-formed model of $\mathbb{P}(\underline{c})$, and we identify E with $\mathbb{P}(\underline{c})^{\mathrm{wf}}$. For $i = 0, 1, \dots, n$, let

$$H_i^{\mathrm{wf}} = (\tilde{x}_i = 0) \subset E \cong \mathbb{P}(\underline{c})^{\mathrm{wf}}$$

be the quasi-hyperplane of $\mathbb{P}(\underline{c})^{\mathrm{wf}}_{\tilde{x}_1, \dots, \tilde{x}_n}$, and we set $m_i = \gcd\{c_0, \dots, \hat{c}_i, \dots, c_n\}$. We see that Y is singular at the generic point of H_i^{wf} if and only if $m_i > 1$, and if this is the case, then the singularity of Y along H_i^{wf} is a cyclic quotient singularity of index m_i. It follows from [Kol+92, Proposition 16.6] that

$$\operatorname{Diff} = \sum_{i=1}^{n} \frac{m_i - 1}{m_i} H_i^{\mathrm{wf}}$$

under the identification $E \cong \mathbb{P}(\underline{c})^{\mathrm{wf}}$.

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Lemma 3.25. *Let the notation and assumption as above. Then we have*

$$\operatorname{lct}_{\mathsf{p}}(X;D) \ge \min\left\{\frac{c_1+\cdots+c_n}{\operatorname{ord}_E(D)},\ \operatorname{lct}(E,\operatorname{Diff};\tilde{D}|_E)\right\}, \tag{3.11}$$

where $\tilde{D}$ is the proper transform of D. If in addition the inequality

$$\frac{c_1+\cdots+c_n}{\operatorname{ord}_E(D)} \le \operatorname{lct}(E,\operatorname{Diff}_E;\tilde{D}|_E) \tag{3.12}$$

holds, then we have

$$\operatorname{lct}_{\mathsf{p}}(X;D) = \frac{c_1+\cdots+c_n}{\operatorname{ord}_E(D)}. \tag{3.13}$$

Proof. We set $c = c_1+\cdots+c_n$ and let λ be any rational number such that

$$0 < \lambda \le \min\left\{\frac{c}{\operatorname{ord}_E(D)},\ \operatorname{lct}(E,\operatorname{Diff};\tilde{D}|_E)\right\}.$$

We will show that the pair $(X,\lambda D)$ is log canonical at p, which will prove the inequality (3.11).

We assume that the pair $(X,\lambda D)$ is not log canonical at p. We have

$$K_Y + \lambda\tilde{D} + (\lambda\operatorname{ord}_E(D) - c + 1)E = \varphi^*(K_X + \lambda D), \tag{3.14}$$

and the pair $(Y,\lambda\tilde{D}+(\lambda\operatorname{ord}_E(D)-c+1)E)$ is not log canonical along E. Since $\lambda \le c/\operatorname{ord}_E(D)$, we have

$$\lambda\operatorname{ord}_E(D) - c + 1 \le 1,$$

which implies that the pair $(Y,\lambda\tilde{D}+E)$ is not log canonical along E. Thus, the pair $(E,\operatorname{Diff}+\lambda\tilde{D}|_E)$ is not log canonical. This is impossible since $\lambda \le \operatorname{lct}(E,\operatorname{Diff};\tilde{D}|_E)$. Therefore, the pair $(X,\lambda D)$ is log canonical at p, and the inequality (3.11) is proved.

By considering the coefficient of E in equation (3.14), it is easy to see that

$$\operatorname{lct}_{\mathsf{p}}(X;D) \le \frac{c}{\operatorname{ord}_E(D)}.$$

Under the assumption (3.12), this shows the equality (3.13). □

We consider Lemma 3.25 in more details in a concrete setting.

Definition 3.26. Let $\underline{c} = (c_1,\dots,c_n)$ be an n-tuple of positive integers such that $\gcd\{c_1,\dots,c_n\} = 1$, and we set

$$m_i = \gcd\{c_1,\dots,\hat{c}_i,\dots,c_n\}$$

for $i = 1,\dots,n$. Let $f = f(x_1,\dots,x_n)$ be a polynomial which is quasi-homogeneous with respect to $\operatorname{wt}(x_1,\dots,x_n) = \underline{c}$.

If f is irreducible and $f \ne x_i$ for $i = 1,\dots,n$, then there exists an irreducible polynomial $f^{\mathrm{wf}} = f^{\mathrm{wf}}(\tilde{x}_1,\dots,\tilde{x}_n)$ (in new variables $\tilde{x}_1,\dots,\tilde{x}_n$) such that

$$f^{\mathrm{wf}}(x_1^{m_1},\dots,x_n^{m_n}) = f(x_1,\dots,x_n).$$

We call f^{wf} the *well-formed model* of f (with respect to the weight $\operatorname{wt}(x_1,\dots,x_n) = \underline{c}$).

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In general, we have a decomposition

$$f = x_1^{\lambda_1} \cdots x_n^{\lambda_n} f_1^{\mu_1} \cdots f_k^{\mu_k},$$

where $k, \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_k$ are nonnegative integers and $f_1, \dots, f_k$ are irreducible polynomials in variables $x_1, \dots, x_n$ which are quasi-homogeneous with respect to $\mathrm{wt}(x_1, \dots, x_n) = \underline{c}$ and which are not x_i for any i. We define

$$f^{\mathrm{wf}} := f(\tilde{x}_1^{1/m_1}, \dots, \tilde{x}_n^{1/m_n}) = \tilde{x}^{\lambda_1/m_1} \cdots \tilde{x}_n^{\lambda_n/m_n} (f_1^{\mathrm{wf}})^{\mu_1} \cdots (f_k^{\mathrm{wf}})^{\mu_k}$$

and call it the *well-formed model* of *f*. Note that f^{wf} is in general not a polynomial since λ_i/m_i need not be an integer. In this case, the effective $\mathbb{Q}$-divisor

$$\mathcal{D}_f^{\mathrm{wf}} := \sum_{i=1}^{n} \frac{\lambda_i}{m_i} H_i^{\mathrm{wf}} + \sum_{j=1}^{k} \mu_j (f_j^{\mathrm{wf}} = 0)$$

on the well-formed model $\mathbb{P}(\underline{c})^{\mathrm{wf}}_{\tilde{x}_1,\dots,\tilde{x}_n}$ of $\mathbb{P}(\underline{c})$ is called the *effective $\mathbb{Q}$-divisor on $\mathbb{P}(\underline{c})^{\mathrm{wf}}$ associated to f*, where H_i^{wf} is the quasi-hyperplane on $\mathbb{P}(\underline{c})^{\mathrm{wf}}$ defined by $\tilde{x}_i = 0$.

Lemma 3.27. *Let $\mathbb{P}(\underline{b}) := \mathbb{P}(b_0, \dots, b_{n+1})_{x_0,\dots,x_{n+1}}$ be a well-formed weighted projective space, and let $X \subset \mathbb{P}(\underline{b})$ be a normal weighted hypersurface with defining polynomial $F = F(x_0, \dots, x_{n+1})$. Let $\underline{c} = (c_1, \dots, c_n)$ be a tuple of positive integers such that $\gcd\{c_1, \dots, c_n\} = 1$. Assume that*

$$F = x_0^e x_{n+1} + \sum_{i=1}^{e} x_0^{e-i} f_i,$$

where $e \in \mathbb{Z}_{>0}$ and $f_i = f_i(x_1, \dots, x_n, x_{n+1})$ is a quasi-homogeneous polynomial of degree $ib_0 + b_{n+1}$. Let $G = G(x_1, \dots x_n)$ be the lowest weight part of $\bar{F} := F(1, x_1, \dots, x_n, 0)$ with respect to $\mathrm{wt}(x_1, \dots, x_n) = \underline{c}$ Then, for the point $\mathsf{p} = \mathsf{p}_{x_0} = (1:0:\cdots:0) \in X$, we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_{x_{n+1}}) \geq \min\left\{\frac{c_1 + \cdots + c_n}{\mathrm{wt}_{\underline{c}}(\bar{F})},\ \mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_G^{\mathrm{wf}})\right\},$$

where $\mathrm{wt}_{\underline{c}}(\bar{F})$, Diff and $\mathcal{D}_G^{\mathrm{wf}}$ are as follows.

- *$\mathrm{wt}_{\underline{c}}(\bar{F})$ is the weight of $\bar{F}$ with respect to $\mathrm{wt}(x_1, \dots, x_n) = \underline{c}$.*
- *$\mathrm{Diff} = \sum_{i=1}^{n} \frac{m_i - 1}{m_i} H_i^{\mathrm{wf}}$, where $H_i^{\mathrm{wf}} = (\tilde{x}_i = 0)$ is the ith coordinate quasi-hyperplane of $\mathbb{P}(\underline{c})^{\mathrm{wf}}_{\tilde{x}_1,\dots,\tilde{x}_n}$ and $m_i = \gcd\{c_1, \dots, \hat{c}_i, \dots, c_n\}$ for $i = 1, \dots, n$.*
- *$\mathcal{D}_G^{\mathrm{wf}}$ is the effective $\mathbb{Q}$-divisor on $\mathbb{P}(\underline{c})^{\mathrm{wf}}$ associated to G.*

If in addition the inequality

$$\frac{c_1 + \cdots + c_n}{\mathrm{wt}(\bar{F})} \leq \mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_G^{\mathrm{wf}})$$

holds, then we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_{x_{n+1}}) = \frac{c_1 + \cdots + c_n}{\mathrm{wt}(\bar{F})}.$$

Proof. Let $\rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}} \subset X$, where $U_{\mathsf{p}} = U_{x_0}$, be the orbifold chart containing p, and we set $\rho = \rho_{\mathsf{p}}, \check{U} = \check{U}_{\mathsf{p}}$ and $U = U_{\mathsf{p}}$. We set $H = H_{x_{n+1}}$ and $\check{H} = \rho^* H$. We have $\mathrm{lct}_{\mathsf{p}}(X; H) = \mathrm{lct}_{\check{\mathsf{p}}}(\check{U}; \check{H})$.

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The variety $\breve{U}$ is the hypersurface in $\mathcal{U}_{x_0} = \mathbb{A}^{n+1}_{\breve{x}_1,\dots,\breve{x}_{n+1}}$ defined by the equation

$$F(1, \breve{x}_1, \dots, \breve{x}_{n+1}) = \breve{x}_{n+1} + \sum_{i=1}^{e} \breve{f}_i = 0, \tag{3.15}$$

where $\breve{f}_i = f_i(\breve{x}_1, \dots, \breve{x}_{n+1})$, and $\breve{\mathsf{p}}$ corresponds to the origin. We see that $\{\breve{x}_1, \dots, \breve{x}_n\}$ is a system of local coordinates of $\breve{U}$ at $\breve{\mathsf{p}}$. Let $\varphi\colon Y \to \breve{U}$ be the weighted blowup at $\breve{\mathsf{p}}$ with $\mathrm{wt}(\breve{x}_1, \dots, \breve{x}_n) = (c_1, \dots, c_n)$. We can identify the φ-exceptional divisor E with $\mathbb{P}(\underline{c})_{\breve{x}_1,\dots,\breve{x}_n}$. Filtering off terms divisible by $\breve{x}_{n+1}$ in equation (3.15), we have

$$(-1 + \cdots)\breve{x}_{n+1} = F(1, \breve{x}_1, \dots, \breve{x}_n, 0)$$

on $\breve{U}$, where the omitted term in the left-hand side is a polynomial vanishing at $\breve{\mathsf{p}}$. Since $\breve{H}$ is the divisor on $\breve{U}$ defined by $\breve{x}_{n+1} = 0$, we see that $\mathrm{ord}_E(\breve{H}) = \mathrm{wt}_{\underline{c}}(\bar{F})$ and the divisor $\tilde{H}|_E$ corresponds to the divisor $\mathcal{D}^{\mathrm{wf}}_G$ on $E \cong \mathbb{P}(\underline{c})^{\mathrm{wf}}$, where $\tilde{H}$ is the proper transform of $\breve{H}$ on Y. Therefore, the proof is completed by Lemma 3.25 and Remark 3.24. $\square$

Lemma 3.28. *Let $X \subset \mathbb{P}(a, b_1, b_2, b_3, r)_{x,y_1,y_2,y_3,z}$ be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$ with defining polynomial $F = F(x, y_1, y_2, y_3, z)$. Assume that F can be written as*

$$F = z^k x + z^{k-1} f_{r+a} + z^{k-2} f_{2r+a} + \cdots + f_{kr+a},$$

where $f_i \in \mathbb{C}[x, y_1, y_2, y_3]$ is a quasi-homogeneous polynomial of degree i, and we set

$$\bar{F} := F(0, y_1, y_2, y_2, 1) \in \mathbb{C}[y_1, y_2, y_3].$$

If either $\bar{F} \in (y_1, y_2, y_3)^2 \setminus (y_1, y_2, y_3)^3$ or $\bar{F} \in (y_1, y_2, y_3)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form in y_1, y_2, y_3, then for the point $\mathsf{p} := \mathsf{p}_z \in X$, we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq \frac{1}{2}.$$

If in addition $a = 1$, $r > 1$ and $\mathsf{p} \in X$ is not a maximal center, then

$$\alpha_{\mathsf{p}}(X) = \min\{1, \mathrm{lct}_{\mathsf{p}}(X; H_x)\} \geq \frac{1}{2}.$$

Proof. Let $\rho_{\mathsf{p}}\colon \breve{U}_{\mathsf{p}} \to U_{\mathsf{p}}$ be the orbifold chart of X containing p. We see that Let $\breve{U}_{\mathsf{p}}$ be the hypersurface in $\breve{\mathcal{U}}_{\mathsf{p}} = \mathbb{A}^4_{\breve{x},\breve{y}_1,\breve{y}_2,\breve{y}_3}$ defined by the equation

$$F(\breve{x}, \breve{y}_1, \breve{y}_2, \breve{y}_3, 1) = 0.$$

We see that $\breve{U}_{\mathsf{p}}$ is smooth and the morphism ρ_{p} can be identified with the quotient morphism of the singularity $\mathsf{p} \in X$ over a suitable analytic neighborhood of p. We denote by $\breve{\mathsf{p}} \in \breve{U}$ the origin of $\breve{\mathcal{U}}_{\mathsf{p}} = \mathbb{A}^4$ which is the preimage of p via ρ_{p}. Filtering off terms divisible by x in $F(x, y_1, y_2, y_3, 1)$, we have

$$(-1 + \cdots)\breve{x} = F(0, \breve{y}_1, \breve{y}_2, \breve{y}_3, 1) = \bar{F}(\breve{y}_1, \breve{y}_2, \breve{y}_3) =: \breve{F}$$

on $\breve{U}_{\mathsf{p}}$. Note that we can choose $\{\breve{y}_1, \breve{y}_2, \breve{y}_3\}$ as a system of local coordinates of $\breve{U}_{\mathsf{p}}$ at $\breve{\mathsf{p}}$. If $\bar{F} \in (y_1, y_2, y_3)^2 \setminus (y_1, y_2, y_3)^3$, then $\mathrm{omult}_{\mathsf{p}}(H_x) = \mathrm{mult}_{\breve{\mathsf{p}}}(\rho_{\mathsf{p}}^* H_x) = 2$ and hence $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$.

Suppose that $\bar{F} \in (y_1, y_2, y_3)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form in y_1, y_2, y_3. Let $\varphi\colon V \to \breve{U}$ be the blowup of $\breve{U}$ at $\breve{\mathsf{p}}$ with exceptional divisor $E \cong \mathbb{P}^2$. We set $D = \rho_{\mathsf{p}}^* H_x$. Since

$\mathrm{mult}_{\check{\mathsf{p}}}(D) = 3$, we have

$$K_V + \frac{1}{2}\tilde{D} = \varphi^*\left(K_{\check{U}} + \frac{1}{2}D\right) + \frac{1}{2}E,$$

where $\tilde{D}$ is the proper transform of D on V. The divisor $\tilde{D}|_E$ on E is isomorphic to the hypersurface in $\mathbb{P}^2_{\check{y}_1,\check{y}_2,\check{y}_3}$ defined by the cubic part of $\bar{F}(\check{y}_1,\check{y}_2,\check{y}_3)$, and the pair $(E, \frac{1}{2}\tilde{D}|_E)$ is log canonical by Lemma 3.4. It then follows that the pair $(V, \frac{1}{2}\tilde{D})$ is log canonical along E. This shows that the pair $(\check{U}, \frac{1}{2}D)$ is log canonical at $\check{\mathsf{p}}$, and hence $\mathrm{lct}_{\mathsf{p}}(X; H_x) \ge 1/2$ as desired.

Suppose in addition that $r > 1$ and $\mathsf{p} \in X$ is not a maximal center. By Lemma 3.6, the pair (X, H_x) is not canonical at $\mathsf{p} = \mathsf{p}_z$ and thus we have $\alpha_{\mathsf{p}}(X) = \min\{1, \mathrm{lct}_{\mathsf{p}}; (X; H_x)\}$ by Lemma 3.5. This proves the latter assertion. □

3.2.c. Computations by $2n$-inequality

Lemma 3.29. *Let $X \subset \mathbb{P}(b_1, b_2, b_3, c, r)_{x_1,x_2,x_3,y,z}$ be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$ with defining polynomial $F = F(x_1, x_2, x_3, y, z)$, and suppose $\mathsf{p} := \mathsf{p}_z \in X$. We assume that $b_1 \le b_2 \le b_3$ and that we can choose y as a quasi-tangent coordinate of X at p. Then*

$$\alpha_{\mathsf{p}}(X) \ge \frac{2}{rb_2b_3(A^3)}.$$

In particular, if $rb_2b_3(A^3) \le 4$, then $\alpha_{\mathsf{p}}(X) \ge 1/2$.

Proof. Let $\rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}}$ be the orbifold chart of X containing $\mathsf{p} = \mathsf{p}_z$. We set $\rho = \rho_{\mathsf{p}}$, $\check{U} = \check{U}_{\mathsf{p}}$ and $U = U_{\mathsf{p}}$. We see that $\check{U}$ is the hypersurface in $\check{\mathcal{U}}_{\mathsf{p}} = \mathbb{A}^4_{\check{x}_1,\check{x}_2,\check{x}_3,\check{y})}$ defined by the equation

$$F(\check{x}_1, \check{x}_2, \check{x}_3, \check{y}, 1) = 0.$$

We see that $\check{U}$ is smooth, and the morphism ρ can be identified with the quotient morphism of $\mathsf{p} \in X$ over a suitable analytic neighborhood of p. We denote by $\check{\mathsf{p}} \in \check{U}$ the origin which is the preimage of p via ρ. By the assumption, we can choose $\check{x}_1, \check{x}_2, \check{x}_3$ as a system of local coordinates of $\check{U}$ at $\check{\mathsf{p}}$.

We set

$$\lambda := \frac{2}{rb_2b_3(A^3)}$$

and assume that $\alpha_{\mathsf{p}}(X) < \lambda$. Then there exists an irreducible $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} A$ such that the pair $(X, \lambda D)$ is not log canonical at p. In particular, the pair $(\check{U}, \lambda\rho^* D)$ is not log canonical at $\check{\mathsf{p}}$. Let $\varphi \colon V \to \check{U}$ be the blowup of $\check{U}$ at $\check{\mathsf{p}}$ with exceptional divisor $E \cong \mathbb{P}^2$. By Lemma 3.3, there exists a line $L \subset E$ with the property that for any prime divisor T on $\check{U}$ such that T is smooth at $\check{\mathsf{p}}$ and that its proper transform $\tilde{T}$ contains L, we have $\mathrm{mult}_{\check{\mathsf{p}}}(D|_T) > 2/\lambda$. By a slight abuse of notation, we have an isomorphism $E \cong \mathbb{P}^2_{\check{x}_1,\check{x}_2,\check{x}_3}$. The line $L \subset E$ is isomorphic to $(\alpha_1\check{x}_1 + \alpha_2\check{x}_2 + \alpha_3\check{x}_3 = 0) \subset \mathbb{P}^2$, for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ with $(\alpha_1, \alpha_2, \alpha_3) \ne (0, 0, 0)$. We set

$$\check{T} := (\alpha_1\check{x}_1 + \alpha_2\check{x}_2 + \alpha_3\check{x}_3 = 0) \subset \check{U}.$$

Then $\check{T}$ is smooth at $\check{\mathsf{p}}$ and its proper transform on V contains L. It follows that $\mathrm{mult}_{\check{\mathsf{p}}}(\rho^* D|_{\check{T}}) > 2/\lambda$. Set $k := \max\{\, i \mid \alpha_i \ne 0 \,\}$. We have $r\check{T} = \rho^* G$ for some effective Weil divisor $G \sim rb_k A$. Let $j \in \{1, 2, 3\}$ be such that

$$b_j = \max\{\, b_i \mid 1 \le i \le 3, i \ne k \,\}.$$

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Then, since $\{x_1, x_2, x_3\}$ isolates p, we can take an effective $\mathbb{Q}$-divisor $S \sim_{\mathbb{Q}} A$ such that $\operatorname{omult}_{\mathsf{p}}(S) \geq 1/b_j$ and $\rho^* S$ does not contain any component of $\rho^* D|_{\check{T}}$. Hence, we have

$$rb_k(A^3) = (D \cdot G \cdot S) \geq (\rho^* D \cdot \check{T} \cdot \rho^* S)_{\check{\mathsf{p}}} > \frac{2}{b_j \lambda} = \frac{rb_2 b_3 (A^3)}{b_j}.$$

This is a contradiction since $b_j b_k \leq b_2 b_3$, and the proof is completed. □

3.2.d. Computations by $\overline{\mathrm{NE}}$

Let X be a quasi-smooth Fano 3-fold weighted hypersurface of index 1. Let $\mathsf{p} \in X$ be a singular point, and we denote by $\varphi \colon Y \to X$ the Kawamata blowup at p. In [CP17], the assertion $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$ is verified in many cases, where $\overline{\mathrm{NE}}(Y)$ is the cone of effective curves on Y. Thus, the following result is very useful.

Lemma 3.30 [KOW18, Lemma 2.8]. *Let $\mathsf{p} \in X$ be a terminal quotient singular point and $\varphi \colon Y \to X$ the Kawamata blowup at p. Suppose that $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$ and there exists a prime divisor S on X such that $\tilde{S} \sim_{\mathbb{Q}} -mK_Y$ for some $m > 0$, where $\tilde{S}$ is the proper transform of S on Y. Then $\alpha_{\mathsf{p}}(X) \geq 1$.*

4. Smooth points

The aim of this section is to prove the following.

Theorem 4.1. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}$$

for any smooth point $\mathsf{p} \in X$.

We explain the organization of this section. Throughout the present section, let

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$, where we assume $a_1 \leq \cdots \leq a_4$. Note that $a_2 \geq 2$ since $\mathsf{i} \notin \mathsf{I}_1$. Recall that we denote by $F = F(x, y, z, t, w)$ the defining polynomial of X with $\deg F = d$, and we set $A := -K_X$. We set

$$U_1 := \bigcup_{v \in \{x,y,z,t,w\}, \deg v = 1} (v \neq 0) \cap X,$$

$$L_{xy} := H_x \cap H_y = (x = y = 0) \cap X.$$

Note that U_1 is an open subset of X contained in $\operatorname{Sm}(X)$, and L_{xy} is a one-dimensional closed subset of X. The proof of the inequality $\alpha_{\mathsf{p}}(X) \geq 1/2$ for $\mathsf{p} \in U_1$ will be done in Section 4.1. The proof for the other smooth points will be done as follows.

- If $1 < a_1 < a_2$, then $\operatorname{Sm}(X) \subset U_1 \sqcup (H_x \setminus L_{xy}) \sqcup L_{xy}$. In this case, the proof of $\alpha_{\mathsf{p}}(X) \geq 1/2$ for smooth point p of X contained in $H_x \setminus L_{xy}$ (resp. L_{xy}) will be done in Section 4.2 (resp. Sections 4.3 and 4.4), respectively.
- If $1 = a_1 < a_2$, then $\operatorname{Sm}(X) \subset U_1 \sqcup L_{xy}$. In this case, the proof of $\alpha_{\mathsf{p}}(X) \geq 1/2$ for smooth point p of X contained in L_{xy} will be done in Sections 4.3 and 4.4.
- If $1 < a_1 = a_2$, then $\operatorname{Sm}(X) \subset U_1 \sqcup H_x$. In this case, the proof of $\alpha_{\mathsf{p}}(X) \geq 1/2$ for smooth point p of X contained in H_x will be done in Section 4.5.

Therefore, Theorem 4.7 will follow from Propositions 4.7, 4.8, 4.10, 4.11 and 4.19, which are the main results of Sections 4.1, 4.2, 4.3, 4.4 and 4.5, respectively.

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4.1. Smooth points on U_1 for families indexed by $\mathsf{I} \setminus \mathsf{I}_1$

Lemma 4.2. *We have*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{a_2 a_4 (A^3)}$$

for any point $\mathsf{p} \in U_1$.

Proof. Let $\mathsf{p} \in U_1$ be a point. We may assume $\mathsf{p} = \mathsf{p}_x$ by a change of coordinates. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor. The linear system $|\mathcal{I}_{\mathsf{p}}(a_2 A)|$ is movable, and let $S \in |\mathcal{I}_{\mathsf{p}}(a_2 A)|$ be a general member so that $\mathrm{Supp}(S) \neq \mathrm{Supp}(D)$. The set $\{y, z, t, w\}$ isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_4 A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ (see Remark 3.15). Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = a_2 a_4 (A^4).$$

This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1/a_2 a_4 (A^3)$, and the proof is completed. □

Lemma 4.3. *Suppose that d is divisible by a_4. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{a_2 a_3 (A^3)}$$

for any point $\mathsf{p} \in U_1$.

Proof. Let $\mathsf{p} \in U_1$ be a point. We may assume $\mathsf{p} = \mathsf{p}_x$. We can choose coordinates so that $w^{d/a_4} \in F$. Indeed, if $a_4 > a_3$, then $w^{d/a_4} \in F$ by the quasi-smoothness of X. If $a_4 = a_3$, then there is a monomial of degree d consisting of t, w by the quasi-smoothness of X and we can choose coordinates t, w so that $w^{d/a_4} \in F$. Under the above choice of coordinates, we see that $\{y, z, t\}$ isolates p.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor. Let S be a general member of the movable linear system $|\mathcal{I}_{\mathsf{p}}(a_2 A)|$ so that $\mathrm{Supp}(S)$ does not contain $\mathrm{Supp}(D)$. We can take a $\mathbb{Q}$-divisor $T \in |a_3 A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ since $\{y, z, t\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = a_2 a_3 (A^3).$$

This shows $\alpha_{\mathsf{p}}(D) \geq 1/a_2 a_3 (A^3)$ and the proof is completed. □

Remark 4.4. The objects of Section 4 are members of families $\mathcal{F}_{\mathsf{i}}$ for $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$, and the inequality $a_2 \geq 2$ is assumed throughout the present section. It is, however, noted that in Lemmas 4.2 and 4.3, the assumption $a_2 \geq 2$ is not required, and the statement holds for members of $\mathcal{F}_{\mathsf{i}}$ for any $\mathsf{i} \in \mathsf{I}$.

Lemma 4.5. *Suppose that d is not divisible by a_4, and assume $a_1 = 1$. Then*

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{\mathrm{lct}_{\mathsf{p}}(X; S_{\mathsf{p}}), \ \frac{1}{a_4 (A^3)}\right\} \geq \frac{1}{2}$$

for any $\mathsf{p} \in U_1$*, where S_{p} is the unique member of* $|\mathcal{I}_{\mathsf{p}}(A)|$.

Proof. Let $\mathsf{p} \in U_1$ be a point. We may assume $\mathsf{p} = \mathsf{p}_x$. Note that we have $a_2 > 1$ and thus the linear system $|\mathcal{I}_{\mathsf{p}}(A)|$ indeed consists of a unique member S_{p}. In this case, $S_{\mathsf{p}} = H_y$.

We first prove $\mathrm{lct}_{\mathsf{p}}(X; S_{\mathsf{p}}) \geq 1/2$, that is, $(X, \frac{1}{2} H_y)$ is log canonical at p. Assume to the contrary that $(X, \frac{1}{2} H_y)$ is not log canonical at p. Then $\mathrm{mult}_{\mathsf{p}}(H_y) \geq 3$. Suppose $\mathrm{mult}_{\mathsf{p}}(H_y) = 3$. Then, by Lemma 3.4, the degree 3 part of $F(1, 0, z, t, w)$ with respect to $\deg(z, t, w) = (1, 1, 1)$ is a cube of a linear form, that is, it can be written as $(\alpha z + \beta t + \gamma w)^3$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. By Lemma 2.28, we have $d < 3a_4$ since d is not divisible by a_4. From this, we deduce $\gamma = 0$. Then we can write

$$F = x^{d-1}y + x^{d-3a_3}(\alpha x^{a_3-a_2}z + \beta t)^3 + g + yh,$$

where $g = g(x,z,t,w) \in (z,t,w)^4 \subset \mathbb{C}[x,z,t,w]$ and $h = h(x,y,z,t,w)$. By the inequality $d < 3a_4$, no monomial in g can be divisible by w^3 so that $g \in (z,t)^2$. But then X is not quasi-smooth along the nonempty subset

$$(y = x^{d-1} + h = z = t = 0) \subset X.$$

This is impossible, and we have $\mathrm{mult}_{\mathsf{p}}(H_y) \geq 4$. By the same argument as above, we can write

$$F = x^{d-1}y + g + yh,$$

where $g \in (z,t)^2$, which implies that X is not quasi-smooth. This is a contradiction and thus $\mathrm{lct}_{\mathsf{p}}(X; S_{\mathsf{p}}) \geq 1/2$.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $S_{\mathsf{p}} = H_y$. Note that $D \cdot H_y$ is an effective 1-cycle. The set $\{y,z,t,w\}$ isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_4 A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of $D \cdot H_y$. We have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_y \cdot T) \leq a_4(A^3) \leq 2,$$

where the last inequality follows from (5) of Lemma 2.29 since $a_1 = 1$. Thus, $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1/a_4(A^3) \geq 1/2$, and the proof is completed. □

Lemma 4.6. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{9, 17\}$. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}$$

for any point $\mathsf{p} \in U_1$.

Proof. In this case,

$$X = X_{3a+3} \subset \mathbb{P}(1,1,a,a+1,a+1)_{x,y,z,t,w},$$

where $a = 2, 3$ if $\mathsf{i} = 9, 17$, respectively. Let $\mathsf{p} \in U_1$ be a point. We may assume $\mathsf{p} = \mathsf{p}_x$.

We show that $(X, \frac{1}{2}H_y)$ is log canonical at p. Assume to the contrary that it is not. Then $\mathrm{mult}_{\mathsf{p}}(H_y) \geq 3$ and, by Lemma 3.4, we can write

$$F = x^{3a+2}y + x^3(\alpha z x + \beta t + \gamma w)^3 + g + yh,$$

where $g = g(x,z,t,w) \in (z,t,w)^4 \subset \mathbb{C}[x,y,t,w]$ and $h = h(x,y,z,t,w)$. By degree reason, any monomial in $g \in (z,t,w)^4$ is divisible by z^3. It follows that X is not quasi-smooth along the nonempty subset

$$(y = x^{3a+2} + h = z = \beta t + \gamma w = 0) \subset X.$$

This is a contradiction, and the pair $(X, \frac{1}{2}H_y)$ is log canonical at p.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_y. The set $\{y,z,t,w\}$ clearly isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |(a+1)A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_y$. Then we have

$$2\operatorname{mult}_{\mathsf{p}}(D) \le (D \cdot H_y \cdot T)_{\mathsf{p}} \le (D \cdot H_y \cdot T) = \frac{3}{a} \le \frac{3}{2}$$

since $\operatorname{mult}_{\mathsf{p}}(H_y) \ge 2$. This shows $\operatorname{lct}_{\mathsf{p}}(X, D) \ge 4/3$ and the proof is completed. □

Proposition 4.7. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$. Then*

$$\alpha_{\mathsf{p}}(X) \ge 1$$

for any point $\mathsf{p} \in U_1$.

Proof. Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$, where we assume $a_1 \le \cdots \le a_4$.

- If d is not divisible by a_4 and $a_1 \ge 2$, then $a_2 a_4 (A^3) \le 2$ and the assertion follows from Lemma 4.2.
- If d is not divisible by a_4 and $a_1 = 1$, then the assertion follows from Lemma 4.5.
- If d is divisible by a_4 and $\mathsf{i} \notin \{9, 17\}$, then $a_2 a_3 (A^3) \le 2$ by Lemma 2.29 and the assertion follows from Lemma 4.3.
- If $\mathsf{i} \in \{9, 17\}$, then the assertion follows from Lemma 4.6.

This completes the proof. □

4.2. *Smooth points on* $H_x \setminus L_{xy}$ *for families with* $1 < a_1 < a_2$

Let

$$X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)_{x,y,z,t,w}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ satisfying $1 < a_1 < a_2 \le a_3 \le a_4$. In this section, we set $\bar{F} = F(0, y, z, t, w)$. Then H_x is isomorphic to the weighted hypersurface in $\mathbb{P}(a_1, \dots, a_4)$ defined by $\bar{F} = 0$. We note that if a smooth point $\mathsf{p} \in X$ contained in H_x satisfies $\operatorname{mult}_{\mathsf{p}}(H_x) > 2$, then p belongs to the subset

$$\bigcap_{v_1, v_2 \in \{y,z,t,w\}} \left(\frac{\partial^2 \bar{F}}{\partial v_1 \partial v_2} = 0 \right) \cap X.$$

The following is the main result of this section.

Proposition 4.8. *Let $X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$, $a_1 \le \cdots \le a_4$, be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ satisfying $1 < a_1 < a_2$. Then*

$$\alpha_{\mathsf{p}}(X) \ge 1$$

for any smooth point p of X contained in $H_x \setminus L_{xy}$.

The rest of this section is entirely devoted to the proof of Proposition 4.8 which will be done by division into several cases.

4.2.a. Case: $1 < a_1 < a_2$ and $d = 2a_4$

We will prove Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 2a_4$.

Let $\mathsf{p} \in X$ be a smooth point contained in $H_x \setminus L_{xy}$. We have $\operatorname{mult}_{\mathsf{p}}(H_x) \le 2$ since $w^2 \in F$, and thus $\operatorname{lct}_{\mathsf{p}}(X; H_x) \ge 1/2$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. By Lemma 3.16,

a_1a_3A isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_1a_3A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_x \cdot T)_{\mathsf{p}} \leq (D \cdot H_x \cdot T) = a_1a_3(A^3) \leq 1,$$

where the last inequality follows from Lemma 2.29. This shows $\mathrm{lct}_{\mathsf{p}}(X;D) \geq 1$ and the proof is completed.

4.2.b. Case: $1 < a_1 < a_2$ **and** $d = 2a_4 + a_1$

We will prove Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 2a_4 + a_1$.

Let $\mathsf{p} \in X$ be a smooth point contained in $H_x \setminus L_{xy}$. We can write

$$F = w^2y + wf + g,$$

where $f, g \in \mathbb{C}[x,y,z,t]$ are quasi-homogeneous polynomials of degree $d - a_4$ and d, respectively. Since $\partial^2\bar{F}/\partial w^2 = y$ and y does not vanish at $\mathsf{p} \in H_x \setminus L_{xy}$, we have $\mathrm{mult}_{\mathsf{p}}(H_x) \leq 2$ and thus $\mathrm{lct}_{\mathsf{p}}(X;H_x) \geq 1/2$.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. By Lemma 3.16, a_1a_4A isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_1a_4A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_x \cdot T)_{\mathsf{p}} \leq (D \cdot H_x \cdot T) = a_1a_4(A^3) \leq 2,$$

where the last inequality from Lemma 2.29. This shows $\mathrm{lct}_{\mathsf{p}}(X;D) \geq 1/2$ and the proof is completed.

4.2.c. Case: $1 < a_1 < a_2$ **and** $d = 2a_4 + a_2$

We will prove Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 2a_4 + a_2$.

Let $\mathsf{p} \in X$ be a smooth point contained in $H_x \setminus L_{xy}$. We can write

$$F = w^2z + w(zf + g) + h,$$

where $f, h \in \mathbb{C}[x,y,z,t]$ and $g \in \mathbb{C}[x,y,t]$ are quasi-homogeneous polynomials of degrees $a_4, a_2 + a_4$ and d, respectively.

Claim 1. $\mathrm{lct}_{\mathsf{p}}(X;H_x) \geq 1/2$.

Proof of Claim 1. We prove $\mathrm{mult}_{\mathsf{p}}(H_x) \leq 2$. Assume to the contrary that $\mathrm{mult}_{\mathsf{p}}(H_x) > 2$. Since

$$\frac{\partial^2\bar{F}}{\partial w^2} = z, \quad \frac{\partial^2\bar{F}}{\partial w \partial z} = 2w + f,$$

the point p is contained in $H_x \cap H_z \cap (2w + f = 0)$. Suppose in addition that $\mathsf{p} \in H_t$. Note that we have $a_4 = a_1 + a_3$ since $d = a_1 + a_2 + a_3 + a_4$ and $d = 2a_4 + a_2$. We see that a_1 does not divide $a_4 = a_1 + a_3$ because otherwise $\gcd\{a_1, a_3, a_4\} > 1$ and X has a nonisolated singularity which is clearly worse than terminal, a contradiction. It follows that f does not contain a power of y, that is, $f(0,y,0) = 0$, and

$$\mathsf{p} \in H_x \cap H_z \cap H_t \cap (2w + f = 0) = H_x \cap H_z \cap H_t \cap H_w = \{\mathsf{p}_y\}.$$

This is impossible since p_y is a singular point of X. Thus, $\mathsf{p} \notin H_t$. Since $a_4 = a_1 + a_3$, we may assume that $\mathsf{p} \in H_w$ after replacing w by $w - \xi yt$ for some $\xi \in \mathbb{C}$. We can write $\mathsf{p} = (0\!:\!1\!:\!0\!:\!\lambda\!:\!0)$ for some nonzero $\lambda \in \mathbb{C}$. The set $\{x, z, w, t^{a_1} - \lambda^{a_1}y^{a_3}\}$ isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_1a_3A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of $H_x \cdot H_z$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(H_x) \leq (H_x \cdot H_z \cdot T)_{\mathsf{p}} \leq (H_x \cdot H_z \cdot T) = a_1a_2a_3(A^3) < 3,$$

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where the last inequality follows from Lemma 2.29. This shows $\mathrm{mult}_{\mathsf{p}}(H_x) \le 2$, and thus $\mathrm{lct}_{\mathsf{p}}(X; H_x) \ge 1/2$. □

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. By Lemma 3.16, $a_1 a_4 A$ isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_1 a_4 A|_{mbQ}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \ge 1$ and $\mathrm{Supp}(T)$ does not contain any component of $D \cdot H_x$. We have

$$\mathrm{mult}_{\mathsf{p}}(D) \le (D \cdot H_x \cdot T)_{\mathsf{p}} \le (D \cdot H_x \cdot T) = a_1 a_4 (A^3) < 2,$$

where the last inequality follows from Lemma 2.29. Thus, $\mathrm{lct}_{\mathsf{p}}(X; D) > 1/2$ and we conclude $\alpha_{\mathsf{p}}(X) \ge 1/2$.

4.2.d. Case: $1 < a_1 < a_2$, $d = 2a_4 + a_3$ and $a_3 \ne a_4$

The proof of Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 2a_4 + a_3$ is completely parallel to the one given in Section 4.2.c. Indeed, the same proof applies after interchanging the role of z and t (and hence a_2 and a_3). Thus, we omit the proof.

4.2.e. Case: $1 < a_1 < a_2$ and $d = 3a_4$

We will prove Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 3a_4$.

Let $\mathsf{p} \in X$ be a smooth point contained in $H_x \setminus L_{xy}$. We first prove $\alpha_{\mathsf{p}}(X) \ge 1/2$ assuming that the inequality $\mathrm{mult}_{\mathsf{p}}(H_x) \le 2$ holds. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. By Lemma 3.16, $a_1 a_4 A$ isolates p, and hence we can take a $\mathbb{Q}$-divisor $T \in |a_1 a_4 A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \ge 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \le (D \cdot H_x \cdot T)_{\mathsf{p}} \le (D \cdot H_x \cdot T) = a_1 a_4 (A^3) \le 2,$$

where the last inequality follows from Lemma 2.29. This shows $\mathrm{mult}_{\mathsf{p}}(D) \le 2$, and we have $\mathrm{lct}_{\mathsf{p}}(X; D) \ge 1/2$.

Therefore, the proof of Proposition 4.8 under the assumption of $1 < a_1 < a_2$ and $d = 3a_4$ is reduced to the following.

Claim 2. *We have* $\mathrm{mult}_{\mathsf{p}}(H_x) \le 2$ *for any smooth point* $\mathsf{p} \in X$ *contained in* $H_x \setminus L_{xy}$.

The rest of this subsection is devoted to the proof of Claim 2, which will be done by considering each family individually. The families satisfying $1 < a_1 < a_2$ and $d = 3a_4$ are families $\mathcal{F}_{\mathsf{i}}$, where

$$\mathsf{i} \in \{19, 27, 39, 49, 59, 66, 84\}.$$

In the following, for a polynomial $f(x, y, z, \dots)$ in variables $x, y, z, \dots$, we set $\bar{f} = \bar{f}(y, z, \dots) := f(0, y, z, \dots)$. We first consider the family $\mathcal{F}_{27}$, which is the unique family satisfying $d = 3a_3 = 3a_4$ We then consider the rest of the families which satisfy $d = 3a_4 > 3a_3$.

4.2.e.1. The family $\mathcal{F}_{27}$

We can choose w and t so that

$$F = w^2 t + w t^2 + w t b_5 + w c_{10} + t d_{10} + e_{15},$$

where $b_5, c_{10}, d_{10}, e_{15} \in \mathbb{C}[x, y, z]$ are quasi-homogeneous polynomials of indicated degrees. Let $\mathsf{p} \in H_x \setminus L_{xy}$ be a smooth point of X, and we assume $\mathrm{mult}_{\mathsf{p}}(H_x) > 2$. Since

$$\frac{\partial^2 \bar{F}}{\partial w^2} = 2t, \quad \frac{\partial^2 \bar{F}}{\partial t^2} = 2w,$$

we have $\mathsf{p} \in H_t \cap H_w$. Then we can write $\mathsf{p} = (0\!:\!1\!:\!\lambda\!:\!0\!:\!0)$ for some nonzero $\lambda \in \mathbb{C}$ since $\mathsf{p} \notin H_y$ and $\mathsf{p} \neq \mathsf{p}_y$. We can write $\bar{e}_{15} = z(z^2 - \lambda^2 y^3)(z^2 - \mu y^3)$ for some $\mu \in \mathbb{C}$. We have

$$\frac{\partial^2 \bar{F}}{\partial z^2}(\mathsf{p}) = 2\lambda(7\lambda^2 - 3\mu) = 0,$$
$$\frac{\partial^2 \bar{F}}{\partial z \partial y}(\mathsf{p}) = -3\lambda^2(3\lambda^2 + \mu) = 0,$$

which implies $\lambda = 0$. This is a contradiction, and Claim 2 is proved for the family $\mathcal{F}_{27}$.

We consider the rest of the families, which satisfies $d = 3a_4 > 3a_3$. Replacing w if necessary, we can write

$$F = w^3 + w g_{2a_4} + h_{3a_4},$$

where $g_{2a_4}, h_{3a_4} \in \mathbb{C}[x, y, z, t]$ are quasi-homogeneous polynomials of degree $2a_4, 3a_4$, respectively. Let $\mathsf{p} \in H_x \setminus L_{xy}$ be a smooth point of X, and we assume $\mathrm{mult}_{\mathsf{p}}(H_x) > 2$. Since $\partial^2 \bar{F}/\partial w^2 = 6w$, we have $\mathsf{p} \in H_w$ so that $\mathsf{p} \in H_x \cap H_w$ and $\mathsf{p} \notin H_y$. In the following, we will derive a contradiction by considering each family separately.

4.2.e.2. The family $\mathcal{F}_{19}$

Replacing $t \mapsto t - \xi z$ for a suitable $\xi \in \mathbb{C}$, we may assume $\mathsf{p} \in H_t$. Since $\mathsf{p} \in H_x \cap H_t \cap H_w$, $\mathsf{p} \notin H_y$ and $\mathsf{p} \neq \mathsf{p}_y$, we have $\mathsf{p} = (0\!:\!1\!:\!\lambda\!:\!0\!:\!0)$ for a nonzero $\lambda \in \mathbb{C}$. We can write $\bar{h}_{12} = (z^2 - \lambda^2 y^3)(z^2 - \mu y^3) + t e_9$ for some $\mu \in \mathbb{C}$ and $e_9 = e_9(y, z, t)$. It is then straightforward to check that

$$\frac{\partial^2 \bar{F}}{\partial z^2}(\mathsf{p}) = \frac{\partial^2 \bar{F}}{\partial z \partial y}(\mathsf{p}) = 0$$

is impossible, and this is a contradiction.

4.2.e.3. The family $\mathcal{F}_{39}$

We have $g_{2a_4} = g_{12}$, $h_{3a_4} = h_{18}$, and we can write

$$g_{12}(0, y, z, t) = \alpha t z y + \lambda z^3 + \mu y^3,$$

where $\alpha, \lambda, \mu \in \mathbb{C}$. By the quasi-smoothness of X, we have $\lambda \neq 0$ and $\mu \neq 0$. We have

$$\frac{\partial^2 \bar{F}}{\partial w \partial t} = \alpha z y, \quad \frac{\partial^2 \bar{F}}{\partial w \partial z} = \alpha t y + 3\lambda z^2, \quad \frac{\partial^2 \bar{F}}{\partial w \partial y} = \alpha t z + 3\mu y^2.$$

Suppose that $\alpha \neq 0$, then, since both $\alpha z y$ and $\alpha t y + 3\lambda z^2$ vanish at p and y does not vanish at p, we have $\mathsf{p} \in H_z \cap H_t$. It follows that $\mathsf{p} = \mathsf{p}_y$. This is impossible since p_y is a singular point of X. Thus, $\alpha = 0$. Then both $3\lambda z^2$ and $\alpha t z + 3\mu y^2$ vanish at p, which implies that y vanishes at p. This is a contradiction.

4.2.e.4. The family $\mathcal{F}_{49}$

We have $g_{2a_4} = g_{14}$, $h_{3a_4} = h_{21}$, and we can write

$$h_{24}(0, y, z, t) = \lambda t^3 y + \alpha t^2 y^3 + \beta t z^3 + \gamma t y^5 + \delta z^3 y^2 + \varepsilon y^7,$$

where $\lambda, \alpha, \beta, \dots, \varepsilon \in \mathbb{C}$. By the quasi-smoothness of X, we have $\lambda \neq 0$, and by replacing t, we can assume that $\alpha = 0$. Since $\mathsf{p} \in H_w$, $\mathsf{p} \notin H_y$ and

$$\frac{\partial^2 \bar{F}}{\partial t^2} = w \frac{\partial^2 g_{14}(0, y, z, t)}{\partial t^2} + 6\lambda t y,$$

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we have $\mathsf{p} \in H_t$. Then $\mathsf{p} \notin H_z$ because otherwise $\mathsf{p} = \mathsf{p}_y$ is a singular point and this is impossible. We have

$$\frac{\partial^2 \bar{F}}{\partial z \partial y} = w \frac{\partial g_{14}(0, y, z, t)}{\partial z \partial y} + 6\delta z^2 y,$$

which implies $\delta = 0$. Then, by the quasi-smoothness of X, we have $\varepsilon \neq 0$. But the polynomial

$$\frac{\partial^2 \bar{F}}{\partial y^2} = w \frac{\partial^2 g_{14}(0, y, z, t)}{\partial y^2} + 20\gamma t y^3 + 42\varepsilon y^5$$

does not vanish at p. This is a contradiction.

4.2.e.5. The family $\mathcal{F}_{59}$

We have $g_{2a_4} = g_{16}$, $h_{3a_4} = h_{24}$, and we can write

$$h_{24}(0, y, z, t) = \lambda t^3 y + \mu z^4 + \alpha z^3 y^2 + \beta z^2 y^4 + \gamma z y^6 + \delta y^8,$$

where $\lambda, \mu, \alpha, \beta, \gamma, \delta \in \mathbb{C}$. By the quasi-smoothness of X, we have $\lambda \neq 0$ and $\mu \neq 0$. Since $\mathsf{p} \notin H_y$, $\lambda \neq 0$ and

$$\frac{\partial^2 \bar{F}}{\partial t^2} = w \frac{\partial^2 g_{16}(0, y, z, t)}{\partial t^2} + 6\lambda t y,$$

we have $\mathsf{p} \in H_t$. This is a contradiction since $\mathsf{p} \in H_x \cap H_t \cap H_w$ but $H_x \cap H_t \cap H_w$ consists of singular points.

4.2.e.6. The family $\mathcal{F}_{66}$

We have $h_{3a_4} = h_{27}$, and we can write

$$h_{27}(0, y, z, t) = \lambda t^3 z + \mu t y^4 + \alpha z^2 y^3,$$

where $\alpha, \lambda, \mu \in \mathbb{C}$. By the quasi-smoothness of X, we have $\lambda \neq 0$ and $\mu \neq 0$. We have

$$\frac{\partial^2 \bar{F}}{\partial t \partial y} = w \frac{\partial^2 g_{18}(0, y, z, t)}{\partial t \partial y} + 4\mu y^3.$$

This is a contradiction since $\mathsf{p} \in H_w$, $\mathsf{p} \notin H_y$ and $\mu \neq 0$.

4.2.e.7. The family $\mathcal{F}_{84}$

We have

$$g_{24}(0, y, z, t) = \alpha t z y + \lambda z^3, \quad h_{36}(0, y, z, t) = \mu t^4 + \beta z y^4,$$

where $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. By the quasi-smoothness of X, we have $\lambda \neq 0$ and $\mu \neq 0$. We have

$$\frac{\partial^2 \bar{F}}{\partial t^2} = w \frac{\partial^2 g_{24}(0, y, z, t)}{\partial t^2} + 12\mu t^2,$$

which implies $\mathsf{p} \in H_t$ since $\mathsf{p} \in H_w$ and $\mu \neq 0$. We have

$$\frac{\partial^2 \bar{F}}{\partial w \partial z} = \frac{g_{24}(0, y, z, t)}{\partial z} = \alpha t y + 3\lambda z^2,$$

which implies $\mathsf{p} \in H_z$ since $\mathsf{p} \in H_t$ and $\lambda \neq 0$. This shows $\mathsf{p} = \mathsf{p}_y$, and this is a contradiction since p_y is a singular point.

Table 1. *L_{xy}: Irreducible and smooth case.*

No.	Equation	No.	Equation
11	$w^2+h(z,t)$	55	$w^2+t^3z+z^8$
15	$w^2+t^4+z^6$	57	$w^2+t^4z+z^6$
16	$w^2z+h(z,t), t^3 \in h$	66	$w^3+wz^3+t^3z$
17	$c(t,w)+z^4$	68	$w^2+t^4+z^7$
19	$w^3+h(z,t)$	70	$w^2+t^3+tz^5$
21	$w^2+h(z,t)$	71	$w^2+t^3z+z^5$
26	$w^2z+z^5+t^3$	72	$w^2+t^3+z^{10}$
27	$c(t,w)+z^5$	74	$w^2z+t^3+tz^5$
34	$w^2+h(z,t)$	75	$w^2+t^5+z^6$
35	$w^2+t^3z+z^6$	76	$w^2t+t^3z+z^5$
36	$w^2z+t^3+tz^3$	80	$w^2+t^3z+tz^6$
41	$w^2+t^4+z^5$	84	$w^3+wz^3+t^4$
45	$w^2z+t^4+z^5$	86	$w^2+t^4z+tz^5$
48	$w^2z+t^3+z^7$	88	$w^2+t^3+z^7$
51	$w^2+t^3z+tz^4$	90	$w^2+t^3+tz^7$
53	$w^2+t^3+z^8$	93	$w^2+t^5+tz^5$
54	$w^2z+t^3+z^4$	95	$w^2+t^3+z^{11}$

Therefore, we derive a contradiction for all families and the proof of Claim 2 is completed.

4.3. *Smooth points on L_{xy} for families with $a_1 < a_2$, Part 1*

In this section and the next sections, we consider a member

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x,y,,t,w}$$

of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ satisfying $a_1 < a_2 \le a_3 \le a_4$ and prove $\alpha_{\mathsf{p}}(X) \ge 1/2$ for a smooth point p of X contained in the one-dimensional scheme $L_{xy} := (x = y = 0) \cap X$. We divide families indexed by $\mathsf{I} \setminus \mathsf{I}_1$ into two types:

- Families $\mathcal{F}_{\mathsf{i}}$ such that $L_{xy} := (x = y = 0) \cap X$ is irreducible and reduced for any member X. These families are treated in the current Section 4.3.
- Families $\mathcal{F}_{\mathsf{i}}$ such that L_{xy} is either reducible or nonreduced for some member X (see equation (4.1) for specific families). These families will be treated in Section 4.4.

The objects of this section are members X such that the one-dimensional scheme $L_{xy} = (x = y = 0) \cap X$ is irreducible and reduced.

Lemma 4.9. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$ which satisfies $a_1 < a_2$ and which is listed in Table 1 (resp. Table 2). Then L_{xy} is an irreducible smooth curve (resp. irreducible and reduced curve which is smooth along $L_{xy} \cap \mathrm{Sm}(X)$).*

Proof. Let F be the defining polynomial of X, and set $d = \deg F$. The scheme L_{xy} is isomorphic to the hypersurface in $\mathbb{P}(a_2, a_3, a_4)_{z,t,w}$ defined by the polynomial $f := F(0, 0, z, t, w)$. We explain that f can be transformed into the polynomial given in Tables 1 and 2 by a suitable change of homogeneous coordinates.

Suppose $\mathsf{i} \notin \{11, 15, 16, 17, 21, 27, 34\}$. Then there are only a few monomials of degree d in variables z, t, w. We first simply express f as a linear combination of those monomials and then consider the following coordinate change.

- Suppose $d = 2a_4$. In this case, f is quadratic with respect to w and we eliminate the term of the form $wg(z, t)$ by replacing w suitably. Then we may assume $f = w^2 + h(z, t)$ for some quasi-homogeneous polynomial $h = h(z, t)$ of degree d.

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Table 2. *L_{xy}: Irreducible and singular case.*

No.	Equation	Sing.	No.	Equation	Sing.
43	t^4+z^5	p_w	77	w^2+t^3z	p_z
44	w^2t+z^4	p_t	78	w^2+tz^5	p_t
46	t^3+z^7	p_w	79	w^2z+t^3	p_z
47	w^2z+t^3	p_z	81	w^2+t^4z	p_z
56	t^3+z^8	p_w	82	w^2+t^3	p_z
59	w^3+z^4	p_t	83	w^2+z^9	p_t
61	w^2t+z^5	p_t	85	w^2+t^3z	p_z
62	w^2+t^3z	p_z	87	w^2+t^5	p_z
65	w^2z+t^3	p_z	89	w^2+t^3	p_z
67	w^2+z^7	p_t	91	w^2+t^3z	p_z
69	w^2z+t^4	p_z	92	w^2+t^3	p_z
73	w^2+z^5	p_t	94	w^2+t^3	p_z

- Suppose $d = 3a_4$. In this case, f is cubic with respect to w and we eliminate the term of the form $w^2g(z,t)$ by replacing w suitably. Then we may assume $f = w^3 + wh_1(z,t) + h_2(z,t)$ for some quasi-homogeneous polynomials $h_1 = h_1(z,t), h = h_2(z,t)$ of degrees $d - a_4 = 2a_4$ and $d = 3a_4$, respectively.

After the above coordinate change, we observe that f is a linear combination of at most three distinct monomials and it is possible to make those coefficients 1 by rescaling z, t, w. The resulting polynomial is the one given in Tables 1 and 2. Once an explicit form of the defining polynomial is given, it is then easy to show that L_{xy} is entirely smooth or is smooth along $L_{xy} \cap \mathrm{Sm}(X)$.

For $\mathsf{i} = \{11, 15, 16, 17, 21, 27, 34\}$, the description of f is explained as follows.

- Suppose $\mathsf{i} = 11$. In this case, $f = w^2 + h(z,t)$, where h is a quintic form in z, t. The solutions of the equation $h = 0$ correspond to the five singular points of type $\frac{1}{2}(1,1,1)$. Thus, h does not have a multiple component and in particular L_{xy} is smooth.
- Suppose $\mathsf{i} = 15$. In this case, $f = w^2 + \alpha t^4 + \beta t^2z^3 + \gamma z^6$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. We have $\alpha \neq 0$ (resp. $\gamma \neq 0$) because otherwise X cannot be quasi-smooth at p_t (resp. p_z). Replacing t and rescaling z, we may assume $\alpha = 1, \beta = 0$ and $\gamma = 1$, and we obtain the desired form $f = w^2 + t^4 + z^6$. It is easy to see that L_{xy} is smooth.
- Suppose $\mathsf{i} = 16$. In this case, $f = w^2z + h(z,t)$, where $h = \alpha t^3 + \beta t^2z^2 + \gamma tz^4 + \delta z^6$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. By the quasi-smoothness of X, we have $\alpha \neq 0$. Moreover, the solutions of $h(z,t) = 0$ correspond to three singular points of type $\frac{1}{2}(1,1,1)$. Thus, $h(z,t)$ does not have a multiple component and in particular L_{xy} is smooth.
- Suppose $\mathsf{i} = 17$. In this case, $f = c(t,w) + \alpha z^4$, where $\alpha \in \mathbb{C}$ and $c(t,w)$ is a cubic form in t, w. By the quasi-smoothness of X, we have $\alpha \neq 0$ and we may assume $\alpha = 1$ by rescaling z. Moreover, the solutions of $c(t,w) = 0$ correspond to three singular points of type $\frac{1}{4}(1,1,3)$. Thus, $c(t,w)$ does not have a multiple component and in particular L_{xy} is smooth.
- Suppose $\mathsf{i} = 21$. In this case, $f = w^2 + h(z,t)$, where $h = \alpha t^3z + \beta t^2z^3 + \gamma tz^5 + \delta z^7$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. By the quasi-smoothness of X at p_t, we have $t^3z \in F$, that is, $\alpha \neq 0$. Moreover, the solutions of $\alpha t^3 + \beta t^2z^2 + \gamma tz^4 + \delta z^6 = 0$ correspond to the three singular points of type $\frac{1}{2}(1,1,1)$. Thus, h does not have a multiple component and in particular L_{xy} is smooth.
- Suppose $\mathsf{i} = 27$. In this case, $f = c(t,w) + \alpha z^5$, where $\alpha \in \mathbb{C}$ and $c(t,w)$ is a cubic form in t, w. By the same arguments as in the case of $\mathsf{i} = 17$, $c(t,w)$ does not have a multiple component and we may assume $\alpha = 1$. Thus, L_{xy} is smooth.
- Suppose $\mathsf{i} = 34$. In this case, $f = w^2 + h(z,t)$, where $h = \alpha t^3 + \beta t^2z^3 + \gamma tz^6 + \delta z^9$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. By the quasi-smoothness of X, we have $t^3 \in F$, that is, $\alpha \neq 0$. Moreover, the solutions of $h = 0$ correspond to three singular points of type $\frac{1}{2}(1,1,1)$. Thus, h does not have a multiple component and in particular L_{xy} is smooth.

This completes the proof. □

Proposition 4.10. *Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$, $a_1 \le a_2 \le a_3 \le a_4$ be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ which satisfies $a_1 < a_2$ and which is listed in Tables 1 and 2. Then*

$$\alpha_{\mathsf{p}}(X) \ge 1$$

for any smooth point p of X contained in L_{xy}.

Proof. Take a point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$. Let $S \in |A|$ and $T \in |a_1 A|$ be general members. By Lemma 4.9, L_{xy} is an irreducible and reduced curve, and we have $S \cdot T = L_{xy}$. Note that L_{xy} is quasi-smooth at p, and we have $\mathrm{mult}_{\mathsf{p}}(L_{xy}) = 1$. By Lemma 3.8, S is quasi-smooth at p. It follows that $\mathrm{lct}_{\mathsf{p}}(X; S) = 1$. By Lemma 3.17, we have

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{\mathrm{lct}_{\mathsf{p}}(X; S),\ \frac{a_1}{\mathrm{mult}_{\mathsf{p}}(L_{xy})},\ \frac{1}{a_1(A^3)}\right\} = 1$$

since $1/a_1(A^3) > 1$ by Lemma 2.29. □

4.4. *Smooth points on L_{xy} for families with $a_1 < a_2$, Part 2*

In this section, we consider families $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ such that L_{xy} is either irreducible or reduced for some member X. Specifically, these families consist of families $\mathcal{F}_{\mathsf{i}}$ with

$$\begin{aligned}\mathsf{i} \in \{&7, 9, 12, 13, 15, 20, 23, 24, 25, 29, 30, 31, 32,\\ &33, 37, 38, 39, 40, 42, 49, 50, 52, 58, 60, 63, 64\},\end{aligned} \tag{4.1}$$

and the aim of this section is to prove the following.

Proposition 4.11. *Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$, $a_1 \le a_2 \le a_3 \le a_4$, be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ which satisfies $a_1 < a_2$ and which is not listed in Tables 1 and 2. Then*

$$\alpha_{\mathsf{p}}(X) \ge \frac{1}{2}$$

for any smooth point p of X contained in L_{xy}.

The proof of Proposition 4.11 will be completed in Section 4.4.b by considering each family separately and by case-by-case arguments. Those arguments form several patterns, and we describe them in Section 4.4.a.

4.4.a. General arguments

In this subsection, let

$$X = X_d \subset \mathbb{P}(1, a, b_1, b_2, b_3)_{x,y,z_1,z_2,z_3}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$. Throughout this subsection, we assume that $a < b_i$ for $i = 1, 2, 3$. Note that we do not assume $b_1 \le b_2 \le b_3$. As before, we denote by $F = F(x, y, z_1, z_2, z_3)$ the defining polynomial of X, and we set $A := -K_X$.

The following very elementary lemma will be used several times.

Lemma 4.12. *Let a, e_1, e_2, e_3 be positive integers such that $a < e_i$ for $i = 1, 2, 3$ and $\gcd\{e_1, e_2, e_3\} = 1$, and let $\lambda \ge 1$ be a number. Then the following inequalities hold.*

1. $\dfrac{1 + e_2 + e_3}{e_1 e_2 e_3} \le \dfrac{1}{2}$.
2. $\dfrac{a + e_2 + e_3}{e_1 e_2 e_3} + \dfrac{\lambda}{a} \le \dfrac{1}{2} + \lambda$.
3. $a(a + e_2 + e_3) < e_1 e_2 e_3$.

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Proof. In view of the assumption $\gcd\{e_1, e_2, e_3\} = 1$, it is easy to see that

$$\frac{1 + e_2 + e_3}{e_1 e_2 e_3} = \frac{1}{e_1 e_2 e_3} + \frac{1}{e_1 e_3} + \frac{1}{e_1 e_2}$$

attains its maximum when $(e_1, e_2, e_3) = (2, 2, 3)$, which implies (1).

It is also easy to see that

$$\frac{a + e_2 + e_3}{e_1 e_2 e_3} + \frac{\lambda}{a},$$

viewed as a function of a, attains its maximum when $a = 1$ since $1 \le a \le \sqrt{\lambda e_1 e_2 e_3}$. Combining this with (1), the inequality (2) follows.

By the assumption, we have $e_1 e_2 \ge (a + 1)^2$. Hence, we have

$$\begin{aligned} e_1 e_2 e_3 - a(a + e_2 + e_3) &= e_3(e_1 e_2 - a) - a^2 - a e_2 \\ &\ge e_3(a^2 + a + 1) - a^2 - a e_2 \\ &= a^2(e_3 - 1) + a(e_3 - e_2) + e_3 \\ &> 0. \end{aligned}$$

This proves (3). □

Lemma 4.13. *Suppose that $L_{xy} := (x = y = 0)_X$ is an irreducible and reduced curve which is smooth along $L_{xy} \cap \mathrm{Sm}(X)$. Then*

$$\alpha_{\mathsf{p}}(X) \ge 1$$

for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof. Let $S \in |A|$ and $T \in |aA|$ be general members. Then we have $S \cap T = L_{xy}$. Take any point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$. By Lemma 3.8, S is smooth at p. It follows that $\mathrm{mult}_{\mathsf{p}}(L_{xy}) = 1$ and $\mathrm{lct}_{\mathsf{p}}(X; S) = 1$. By Lemma 3.17, we have

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{\mathrm{lct}_{\mathsf{p}}(X; S),\ \frac{1}{\mathrm{mult}_{\mathsf{p}}(L_{xy})},\ \frac{1}{a(A^3)}\right\} = 1$$

since $1/a(A^3) > 1$ by Lemma 2.29. □

Lemma 4.14. *Let $S \in |A|$ and $T \in |aA|$ be general members. Suppose that the following assertions are satisfied.*

1. *$T|_S = \Gamma + \Delta$, where $\Gamma = (x = y = z_1 = 0)$ is a quasi-line and Δ is an irreducible and reduced curve which is quasi-smooth along $\Delta \cap \mathrm{Sm}(X)$.*
2. *S is quasi-smooth along $\Gamma \cap \Delta$.*
3. *$\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_{z_2}, \mathsf{p}_{z_3}\}$.*

Then

$$\alpha_{\mathsf{p}}(X) \ge \frac{1}{2}$$

for any point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof. By assumptions (1), (2) and Lemma 3.9, S is quasi-smooth along Γ.

Claim 3. *The intersection matrix $M = M(\Gamma, \Delta)$ satisfies the condition $(\star)$.*

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Proof of Claim 3. By the assumption (3) and the quasi-smoothness of S, we see that $\mathrm{Sing}_\Gamma(S) = \{\mathsf{p}_{z_2}, \mathsf{p}_{z_3}\}$ and $\mathsf{p}_{z_i} \in S$ is a cyclic quotient singularity of index b_i for $i = 2, 3$. By Remark 3.10, we have

$$(\Gamma^2)_S = -2 + \frac{b_2 - 1}{b_2} + \frac{b_3 - 1}{b_3} = -\frac{b_2 + b_3}{b_2 b_3} < 0.$$

By taking the intersection number of $T|_S = \Gamma + \Delta$ and Γ, we have

$$(\Gamma \cdot \Delta)_S = -(\Gamma^2)_S + (T \cdot \Gamma) = \frac{a + b_2 + b_3}{b_2 b_3} > 0.$$

Note that we have

$$(T \cdot \Delta) = (T^2 \cdot S) - (T \cdot \Gamma) = a^2(A^3) - \frac{a}{b_2 b_3} = \frac{a(a + b_2 + b_3)}{b_1 b_2 b_3},$$

and then by taking the intersection number of $T|_S = \Gamma + \Delta$ and Δ, we have

$$(\Delta^2)_S = (T \cdot \Delta) - (\Gamma \cdot \Delta)_S = -\frac{(b_1 - a)(a + b_2 + b_3)}{b_1 b_2 b_3} < 0.$$

Finally, we have

$$\begin{aligned} \det M &= \frac{b_2 + b_3}{b_2 b_3} \cdot \frac{(b_1 - a)(a + b_2 + b_3)}{b_1 b_2 b_3} - \frac{(a + b_2 + b_3)^2}{b_2^2 b_3^2} \\ &= -\frac{a(a + b_2 + b_3)(b_1 + b_2 + b_3)}{b_1 b_2^2 b_3^2} < 0. \end{aligned}$$

It follows that M satisfies the condition $(\star)$. □

Let $\mathsf{p} \in (\Gamma \setminus \Delta) \cap \mathrm{Sm}(X)$ be a point. By Lemma 3.7, S is a normal surface. It is easy to check that $a \deg \Gamma = a/(b_2 b_3) \le 1$ and that X, S and Γ are smooth at p. Thus, we can apply Lemma 3.21 and we conclude

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{1, \frac{1}{a(A^3) + \frac{1}{a} - \deg \Gamma}\right\} = \min\left\{1, \frac{1}{\frac{a+b_2+b_3}{b_1 b_2 b_3} + \frac{1}{a}}\right\} \ge \frac{2}{3},$$

where the last inequality follows from Lemma 4.12.

Let $\mathsf{p} \in (\Delta \setminus \Gamma) \cap \mathrm{Sm}(X)$ be a point. Note that Δ is smooth at p since it is quasi-smooth at p by the assumption (1). We have

$$a \deg \Delta = a\left(a(A^3) - \frac{1}{b_2 b_3}\right) = \frac{a(a + b_2 + b_3)}{b_1 b_2 b_3} < 1$$

by Lemma 4.12. Note that we have

$$a(A^3) + \frac{1}{a} - \deg \Delta = \frac{1}{a} + \frac{1}{b_2 b_3} \le 1 + \frac{1}{4} = \frac{5}{4}$$

since $1 \le a < b_i$. Thus, we can apply Lemma 3.21 and conclude

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{1, \frac{1}{a(A^3) + \frac{1}{a} - \deg \Delta}\right\} \ge \frac{4}{5}.$$

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Finally, let $\mathsf{p} \in (\Gamma \cap \Delta) \cap \mathrm{Sm}(X)$ be a point. Note that S is smooth at p by the assumption (2), and we have

$$\deg(\Gamma) = \frac{1}{b_2 b_3} < \frac{2}{a}, \quad \deg(\Delta) = \frac{a + b_2 + b_3}{b_1 b_2 b_3} < \frac{2}{a},$$

where the former inequality is obvious and the latter follows from Lemma 4.12. Thus, we can apply Lemma 3.23 and conclude that

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{a}{2}\right\} \geq \frac{1}{2}.$$

Therefore, the proof is completed. □

Remark 4.15. Let the notation and assumption as in Lemma 4.14. Assume in addition that $a \geq 2$ and $\Gamma \cap \Delta \subset \mathrm{Sing}(X)$. Then

$$\alpha_{\mathsf{p}}(X) \geq \frac{43}{54} > \frac{3}{4}$$

for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Indeed, since $\Gamma \cap \Delta \subset \mathrm{Sing}(X)$, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{1}{\frac{a+b_2+b_3}{b_1 b_2 b_3} + \frac{1}{a}}, \frac{4}{5}\right\}$$

by the proof of Lemma 4.14. Since $2 \leq a < \sqrt{b_1 b_2 b_3}$ and $a < b_i$ for $i = 1, 2, 3$, we have

$$\frac{a + b_2 + b_3}{b_1 b_2 b_3} + \frac{1}{a} \leq \frac{3a + 1}{(a + 1)^3} + \frac{1}{a} \leq \frac{43}{54}.$$

This proves the desired inequality.

Lemma 4.16. *Suppose that b_1, b_2, b_3 are mutually coprime and $a \in \{1, 2\}$. Suppose in addition that F can be written as*

$$F = f_1(z_1, z_2)x + f_2(z_1, z_2)y + z_3^m z_2 + g(x, y, z_1, z_2, z_3),$$

where $m \in \{2, 3\}$ and $f_1, f_2 \in \mathbb{C}[z_1, z_2], g \in [x, y, z_1, z_2, z_3]$ are quasi-homogeneous polynomials satisfying the following properties.

1. $\deg F = b_1 b_2 + a$.
2. *g is contained in the ideal $(x, y) \cap (x, y, z_3)^2 \subset \mathbb{C}[x, y, z_1, z_2, z_3]$.*

Then

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}$$

for any point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof. We have $d = m b_3 + b_2$ since $z_3^m z_2 \in F$, and combining this with $d = a + b_1 + b_2 + b_3$, we have

$$a + b_1 + b_3 = m b_3. \tag{4.2}$$

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Claim 4. *We can assume*

$$F = \begin{cases} z_2^{b_1}x - z_1^{b_2}y + z_3^m z_2 + g, & \text{if } a = 1, \\ z_1^k z_2^l x + (z_1^{b_2} - z_2^{b_1})y + z_3^m z_2 + g, & \text{if } a = 2 \end{cases} \tag{4.3}$$

after replacing x and y suitably, where k and l are nonnegative integers.

Proof of Claim 4. Suppose $a = 1$. Then, since $\deg f_1 = \deg f_2 = b_1 b_2$, we can write

$$f_1(z_1, z_2)x + f_2(z_1, z_2)y = z_2^{b_1}\ell_1(x, y) + z_1^{b_2}\ell_2(x, y),$$

where ℓ_1, ℓ_2 are linear forms in x, y. We see that ℓ_1, ℓ_2 are linearly independent because otherwise we can write $\ell_2 = \alpha\ell_1$ for some nonzero $\alpha \in \mathbb{C}$ and X is not quasi-smooth along

$$(x = y = w = \ell_1 = z_2^{b_2} + \alpha z_1^{b_1} = 0) \subset X.$$

This is a contradiction. Thus, ℓ_1, ℓ_2 are linearly independent and we may assume $\ell_1 = x$ and $\ell_2 = y$, as desired.

Suppose $a = 2$. We have $f_2 = \alpha z_1^{b_2} + \beta z_2^{b_1}$ for some $\alpha, \beta \in \mathbb{C}$ since $\deg f_2 = b_1 b_2$. By the quasi-smooth of X at $\mathsf{p}_z, \mathsf{p}_t$, we have $\alpha, \beta \neq 0$, and thus we may assume $\alpha = 1$, $\beta = -1$ by rescaling z_1, z_2. We see that the equation $f_1(z_1, z_2) = f_2(z_1, z_2) = 0$ on variables z_1, z_2 has only trivial solution because otherwise X is not quasi-smooth along the nonempty set

$$(x = y = w = f_1(z_1, z_2) = f_2(z_1, z_2) = 0) \subset X,$$

which is impossible. This implies that $f_1 \neq 0$ as a polynomial, and there exists a monomial $z_1^k z_2^l$ of degree $b_1 b_2 + 1$. Since b_1 is coprime to b_2, $z_1^k z_2^l$ is the unique monomial of degree $b_1 b_2 + 1$ in variables z_1, z_2. Thus, we have $f_2 = \gamma z_1^k z_2^l$ for some nonzero $\gamma \in \mathbb{C}$. Rescaling x, we may assume $\gamma = 1$, and the claim is proved. □

We continue the proof of Lemma 4.16. Let $S \in |A|$ and $T \in |aA|$ be general members. We have

$$T|_S = \Gamma + m\Delta,$$

where

$$\Gamma = (x = y = z_2 = 0), \quad \Delta = (x = y = z_3 = 0)$$

since $F(0, 0, z, t, w) = z_3^m z_2$. We see that Γ and Δ are quasi-lines of degree $1/b_1 b_3$ and $1/b_1 b_2$, respectively, and $\Gamma \cap \Delta = \{\mathsf{p}_{z_1}\} \subset \mathrm{Sing}(X)$. We see that S is quasi-smooth at p_{z_1} since $z_1^{b_2}y \in F$. By Lemma 3.9, S is quasi-smooth along Γ and the pair (S, Γ) is plt.

Claim 5. *The intersection matrix $M = M(\Gamma, \Delta)$ satisfies the condition* ($\star$).

Proof of Claim 5. We see that d is not divisible by b_1 or b_3 since $d = b_1 b_2 + a = m b_3 + b_2$, $a < b_1$ and b_2 is coprime to b_3. It follows that $\mathrm{Sing}_\Gamma(S) = \{\mathsf{p}_{z_1}, \mathsf{p}_{z_3}\}$ and $\mathsf{p}_{z_i} \in S$ is a cyclic quotient singularity of index b_i for $i = 1, 3$. By Remark 3.10, we have

$$(\Gamma^2)_S = -2 + \frac{b_1 - 1}{b_1} + \frac{b_3 - 1}{b_3} = -\frac{b_1 + b_3}{b_1 b_3} < 0.$$

By taking intersection number of $T|_S = \Gamma + m\Delta$ and Γ, we obtain

$$(\Gamma \cdot \Delta)_S = \frac{1}{m}(a \deg \Gamma - (\Gamma^2)_S) = \frac{a + b_1 + b_3}{m b_1 b_3} = \frac{1}{b_1} > 0.$$

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Similarly, by taking intersection number of $T|_S$ and Δ, we obtain

$$(\Delta^2)_S = \frac{1}{m}(a \deg \Delta - (\Gamma \cdot \Delta)_S) = -\frac{b_2 - a}{m b_1 b_2} < 0,$$

where the second equality follows from equation (4.2). Finally, we have

$$\det M = \frac{b_1 + b_3}{b_1 b_3} \cdot \frac{b_2 - a}{m b_1 b_2} - \frac{1}{b_1^2} = -\frac{a(b_1 + b_2 + b_3)}{b_1^2 b_2 b_3} < 0,$$

where the second equality follows from equation (4.2). It follows that M satisfies the condition $(\star)$. $\square$

Let $\mathsf{p} \in (\Gamma \setminus \Delta) \cap \operatorname{Sm}(X)$. We see that X, S and Γ are smooth at p, and it is easy to see $a \deg \Gamma = a/(b_1 b_3) < 1$. Hence, we can apply Lemma 3.21 and we have

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{1,\ \frac{1}{a(A^3) + \frac{1}{a} - \deg \Gamma}\right\} = \min\left\{1,\ \frac{1}{\frac{a+b_1+b_3}{b_1 b_2 b_3} + \frac{1}{a}}\right\} \ge \frac{2}{3},$$

where the last inequality follows from Lemma 4.12.

It remains to consider $\mathsf{p} \in (\Delta \setminus \Gamma) \cap \operatorname{Sm}(X)$ since $\Gamma \cap \Delta = \{\mathsf{p}_{z_1}\} \subset \operatorname{Sing}(X)$. We first consider the case when $a = 2$. In this case, $S = H_x$ is quasi-smooth along $\Delta \setminus \{\mathsf{q}\}$, where $\mathsf{q} = (0:0:1:1:0) \in (\Delta \setminus \Gamma) \cap \operatorname{Sm}(X)$, and S has a double point at q. We have $\operatorname{mult}_{\mathsf{p}}(\Delta) = 1$, and $a \deg \Delta = a/(b_1 b_2) < 1$. Thus, we can apply Lemma 3.22 and conclude

$$\begin{aligned}
\alpha_{\mathsf{p}}(X) &\ge \min\left\{\frac{2}{\operatorname{mult}_{\mathsf{p}}(S)},\ \frac{\operatorname{mult}_{\mathsf{p}}(S)}{2(A^3) + \frac{m}{2} - m \deg \Delta}\right\} \\
&= \min\left\{\frac{2}{\operatorname{mult}_{\mathsf{p}}(S)},\ \frac{\operatorname{mult}_{\mathsf{p}}(S)}{\frac{1}{b_1 b_3} + \frac{m}{2}}\right\} \\
&\ge \min\left\{1,\ \frac{1}{\frac{1}{12} + \frac{3}{2}}\right\} \\
&= \frac{12}{19}
\end{aligned}$$

since $1/(b_1 b_3) \le 1/12$, $m \in \{2, 3\}$ and $\operatorname{mult}_{\mathsf{p}}(S) \in \{1, 2\}$.

Suppose $a = 1$. We set $S' = (z_3 = 0) \cap X \in |b_3 A|$. For $\lambda \in \mathbb{C}$, we set $T'_\lambda = (y - \lambda x = 0) \cap X \in |A|$. We can write $g(x, \lambda x, z_1, z_2, 0) = x^2 h_\lambda$ for some $h_\lambda = h_\lambda(x, z_1, z_2)$ since $g \in (x, y, z_3)^2$. In view of equation (4.3), we have

$$F(x, \lambda x, z_1, z_2, 0) = x \phi_\lambda(x, z_1, z_2),$$

where

$$\phi_\lambda(x, z_1, z_2) = z_1^{b_2} - \lambda z_2^{b_1} + x h_\lambda.$$

The polynomial ϕ_λ is irreducible for any nonzero $\lambda \in \mathbb{C}$. We have

$$T'_\lambda|_{S'} = \Delta + \Xi_\lambda,$$

where

$$\Xi_\lambda = (y - \lambda x = z_3 = \phi_\lambda = 0)$$

is an irreducible and reduced curve. We have $\Delta \cap \Xi_\lambda = \{\mathsf{q}_\lambda\} \subset \mathrm{Sm}(X)$, where $\mathsf{q}_\lambda = (0\!:\!0\!:\!\sqrt[b_2]{\lambda}\!:\!1\!:\!0)$. It is easy to see that S' is quasi-smooth at q_λ. Hence, S' is quasi-smooth along Δ by Lemma 3.9.

Claim 6. *The intersection matrix $M' = M(\Delta, \Xi_\lambda)$ satisfies the condition $(\star)$.*

Proof of Claim 6. By Remark 3.10, we have

$$(\Delta^2)_S = -\frac{b_3 - 1}{b_1 b_2} - 2 + \frac{b_1 - 1}{b_1} + \frac{b_2 - 1}{b_2} = -\frac{b_1 + b_2 + b_3 - 1}{b_1 b_2} < 0.$$

By taking intersection number of $T'_\lambda|_{S'} = \Delta + \Xi_\lambda$ and Δ, we obtain

$$(\Delta \cdot \Xi_\lambda)_S = \frac{b_1 + b_2 + b_3}{b_1 b_2} > 0.$$

By taking intersection number of $T'_\lambda|_{S'}$ and Ξ_λ, we obtain

$$(\Xi_\lambda^2)_S = 0.$$

It is then obvious that $\det M' < 0$ and the proof is completed. □

Now, we take any point $\mathsf{p} \in (\Delta \setminus \Gamma) \cap \mathrm{Sm}(X)$, and then we can choose a nonzero $\lambda \in \mathbb{C}$ so that $\mathsf{p} \neq \mathsf{q}_\lambda$. By Lemma 3.8, S' is smooth at p since $S' \cap T'_\lambda$ is smooth at p. It is easy to see that $\deg \Delta = 1/(b_1 b_2) < 1$. Thus, we can apply Lemma 3.21 and conclude

$$\alpha_{\mathsf{p}}(X) \ge \min\left\{b_3, \ \frac{1}{b_3(A^3) + 1 - \deg \Delta}\right\} = \min\left\{b_3, \ \frac{1}{2}\right\} = \frac{1}{2},$$

where the first equality follows since

$$b_3(A^3) + 1 - \deg \Delta = \frac{d-1}{b_1 b_2} + 1 = 2.$$

This completes the proof. □

Lemma 4.17. *Suppose that b_1, b_2, b_3 are mutually coprime and $a \in \{1, 2, 3\}$. Suppose in addition that F can be written as*

$$F = z_3^m + z_1^{e_1} y - z_2^{e_2} x + g(x, y, z_1, z_2, z_3),$$

where $m \ge 2, e_1, e_2$ are positive integers and $g \in \mathbb{C}[x, y, z_1 z_2, z_3]$ is a homogeneous polynomial satisfying the following properties.

1. *If $a \ge 2$, then $m \le 2a$.*
2. *If $a = 1$, then $e_1 \le b_2$.*
3. *g is a homogeneous polynomial contained in the ideal $(x, y) \cap (x, y, z_3)^2 \subset \mathbb{C}[x, y, z_1, z_2, z_3]$.*

Then

$$\alpha_{\mathsf{p}}(X) \ge \frac{1}{2}$$

for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof. We first consider the case where $a \ge 2$. Let $S \in |A|$ and $T \in |aA|$ be general members. We have

$$S \cdot T = m\Gamma,$$

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where

$$\Gamma = (x = y = z_3 = 0)$$

is a quasi-line of degree $1/(b_1 b_2)$. Let $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$. It is straightforward to check that S is smooth at p, which implies $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{a}S) = a$. By Lemma 3.17, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{a, \frac{a}{m}, \frac{1}{a(A^3)}\right\} \geq \frac{1}{2}$$

since $a/m \geq 1/2$ and $1/(a(A^3)) > 1$ by the assumption (1) and Lemma 2.29, respectively.

In the following, we assume $a = 1$. We set $S' = (z_3 = 0) \cap X \in |b_3 A|$ and $\Gamma = (x = y = z_3 = 0) \subset S'$. We have $L_{xy} = \Gamma$ set-theoretically. For $\lambda \in \mathbb{C}$, we set $T'_\lambda = (y - \lambda x = 0) \cap X \in |aA|$. We can write

$$g(x, \lambda x, z_1, z_2, 0) = x^2 h_\lambda(x, z_1, z_2),$$

where h_λ is a quasi-homogeneous polynomial in variables x, z_1, z_2 since $g \in (x, y, z_3)^2$. We have

$$F(x, \lambda x, z_1, z_2, 0) = x(\lambda z_1^{e_1} - z_2^{e_2} + x h_\lambda).$$

Claim 7. *The quasi-homogeneous polynomial*

$$\phi_\lambda := \lambda z_1^{e_1} - z_2^{e_2} + x h_\lambda \in \mathbb{C}[x, z_1, z_2]$$

is irreducible for any $\lambda \in \mathbb{C} \setminus \{0\}$.

Proof of Claim 7. We assume $\lambda \neq 0$. If ϕ_λ is a reducible polynomial, then we can write

$$\phi_\lambda = -(z_2^{c_2} + \cdots + \alpha z_1^{c_1} + \cdots)(z_2^{e_2 - c_2} + \cdots + \beta z_1^{e_1 - c_1} + \cdots)$$

for some $c_1, c_2 \in \mathbb{Z}_{\geq 0}$ with $c_1 \leq e_1$ and $0 < c_2 < e_2$, and nonzero $\alpha, \beta \in \mathbb{C}$ such that $\alpha\beta = \lambda$. We have $c_2 b_2 = c_1 b_1$. Since b_1 is coprime to b_2, we see that c_1 is divisible by b_2. This implies $c_1 = e_1 = b_2$ since $c_1 \leq e_1 \leq b_2$. By the equality $e_2 b_2 = e_1 b_1$, we have $c_2 = e_2 = b_1$. This is a contradiction since $c_2 < e_2$. Therefore, ϕ_λ is irreducible for $\lambda \neq 0$. □

We continue the proof of Lemma 4.17. By Claim 7, we have

$$T'_\lambda|_{S'} = \Gamma + \Delta_\lambda,$$

where

$$\Delta_\lambda = (y - \lambda x = z_3 = \phi_\lambda = 0)$$

is an irreducible and reduced curve for any $\lambda \in \mathbb{C} \setminus \{0\}$. We have $\Gamma \cap \Delta_\lambda = \{\mathsf{q}_\lambda\}$, where

$$\mathsf{q}_\lambda = (0\!:\!0\!:\!1\!:\!\sqrt[e_2]{\lambda}\!:\!0).$$

It is easy to check that S' is quasi-smooth at q_λ. By Lemma 3.9, S' is quasi-smooth along Γ and the pair (S', Γ) is plt.

Claim 8. *The intersection matrix $M' = M(\Gamma, \Xi'_\lambda)$ satisfies the condition $(\star)$.*

Proof of Claim 8. We see that $\mathrm{Sing}_\Gamma(S') = \{\mathsf{p}_{z_1}, \mathsf{p}_{z_2}\}$ and $\mathsf{p}_{z_i} \in S'$ is a cyclic quotient singular point of index b_i for $i = 1, 2$. By the same computation as in Claim 6, we have

$$\begin{aligned}(\Gamma^2)_{S'} &= -\frac{b_1 + b_2 + b_3 - 1}{b_1 b_2} < 0,\\ (\Gamma \cdot \Delta_\lambda)_{S'} &= \frac{b_1 + b_2 + b_3}{b_1 b_2} > 0,\\ (\Delta_\lambda^2)_{S'} &= 0.\end{aligned}$$

It is then easy to see that $\det M' < 0$, which shows that M' satisfies $(\star)$. □

Now, take a point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X) = \Gamma \cap \mathrm{Sm}(X)$. We choose and fix a general $\lambda \in \mathbb{C}$ so that Δ_λ is irreducible and $\mathsf{q}_\lambda \neq \mathsf{p}$. Then $\mathsf{p} \in (\Gamma \setminus \Xi_\lambda) \cap \mathrm{Sm}(X)$. We see that X, S' and Γ are smooth at p, and $\deg \Gamma = 1/(b_1 b_2) < 1$. Thus, we can apply Lemma 3.21 and conclude

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{b_3,\ \frac{1}{b_3(A^3) + 1 - \deg \Gamma}\right\} = \min\left\{b_3,\ \frac{1}{\frac{e_1}{b_2} + 1}\right\} \geq \frac{1}{2},$$

where the last inequality follows from the assumption (2). This completes the proof. □

Lemma 4.18. *Let $S \in |A|$ and $T \in |aA|$ be general members. Suppose that*

$$S \cdot T = 2\Gamma,$$

where $\Gamma = (x = y = z_3 = 0)$. Then

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{\mathrm{lct}_{\mathsf{p}}(X; S),\ \frac{a}{2},\ \frac{1}{a(A^3)}\right\} \geq \frac{1}{2}$$

for any point $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof. Let $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$. We have $\mathrm{mult}_{\mathsf{p}}(\Gamma) = 1$, and the first inequality follows from Lemma 3.17. We have $\mathrm{mult}_{\mathsf{p}}(S) \leq \mathrm{mult}_{\mathsf{p}}(S \cdot T) = 2$, which implies $\mathrm{lct}_{\mathsf{p}}(X; S) \geq 1/2$. Thus, the second inequality in the statement follows since $1/(a(A^3)) > 1$ by Lemma 2.29. □

4.4.b. Proof of Proposition 4.11

This subsection is entirely devoted to the proof of Proposition 4.11.

Let $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$, $a_1 \leq \cdots \leq a_4$, be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ satisfying $a_1 < a_2$. Let $S \in |A|$ and $T \in |a_1 A|$ are general members so that their scheme-theoretic intersection $S \cap T$ coincides with L_{xy}. Note that S is a normal surface by Lemma 3.7 and T is a quasi-hyperplane section on X. We set

$$f := F(0, 0, z, t, w)$$

so that L_{xy} is isomorphic to the hypersurface in $\mathbb{P}(a_2, a_3, a_4)_{z,t,w}$ defined by $f = 0$.

4.4.b.1. The family $\mathcal{F}_7$

We have

$$f = w^2 \ell(z, t) + h(z, t),$$

where ℓ, h are linear and quadratic forms in z, t, respectively. By the quasi-smoothness of X, we have $\ell(z, t) \neq 0$, and $h(z, t)$ does not have a multiple component.

- Case (i): h is not divisible by ℓ. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): h is divisible by ℓ. Replacing z and t, we may assume $\ell(z,t) = z$. We can write $h = zc(z,t)$, where $c(z,t)$ is a cubic form in z,t. Note that $c(z,t)$ is not divisible by z since $h(z,t) = zc(z,t)$ does not have a multiple component, and we can assume $c(0,t) = -t^3$ by rescaling t. In this case, $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + c(z,t) = 0).$$

We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve since $c(z,t)$ does not have a multiple component. We have $\Gamma \cap \Delta = \{\mathsf{q}\}$, where $\mathsf{q} = (0\!:\!0\!:\!0\!:\!1\!:\!1) \in \mathrm{Sm}(X)$. We claim that S is quasi-smooth (and hence smooth) at q. We have $(\partial F/\partial z)(\mathsf{q}) = (\partial F/\partial t)(\mathsf{q}) = (\partial F/\partial w)(\mathsf{q}) = 0$. Hence, at least one of $(\partial F/\partial x)(\mathsf{q})$ and $(\partial F/\partial y)(\mathsf{q})$ is nonzero by the quasi-smoothness of X. By choosing x and y, we may assume that $S = H_x$ and $(\partial F/\partial y)(\mathsf{q}) \neq 0$. It then follows that S is quasi-smooth at q. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, the assumptions of Lemma 4.14 are satisfied and we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.2. The family $\mathcal{F}_9$

We have

$$f = c(t,w) + z^3 \ell(t,w),$$

where $\ell = \ell(t,w)$ and $c = c(t,w)$ are linear and cubic forms in t,w, respectively. By the quasi-smoothness of X, $c(t,w)$ does not have a multiple component.

- Case (i): $\ell(t,w) \neq 0$ and $c(t,w)$ is not divisible by $\ell(t,w)$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\ell(t,w) \neq 0$ and $c(t,w)$ is divisible by $\ell(t,w)$. We write $c(t,w) = \ell(t,w)q(t,w)$, where $q(t,w)$ is a quadratic form in t,w which is not divisible by $\ell(t,w)$. Replacing t and w, we may assume $\ell = t$, that is, $f = w(q(t,w) + z^3)$. We may also assume $q(0,w) = -w^2$ since $c(t,w)$ does not have a multiple component. In this case, $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = q + z^3 = 0).$$

We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve since $q(t,w)$ is not a square of a linear form. We have $\Gamma \cap \Delta = \{\mathsf{q}\}$, where $\mathsf{q} = (0:0:1:0:1) \in \mathrm{Sm}(X)$. By the similar argument as in Case (ii) of Section 4.4.b, we can conclude that S is quasi-smooth at q. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\ell(t,w) = 0$. In this case, $f = \ell_1 \ell_2 \ell_3$, where ℓ_1, ℓ_2, ℓ_3 are linear forms in t,w which are not mutually proportional, and $T|_S = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are as follows.
  - For $i = 1,2,3$, $\Gamma_i = (x = y = \ell_i = 0)$ is a quasi-line and $\mathrm{Sing}_{\Gamma_i} = \{1 \times \frac{1}{2}(1,1), 1 \times \frac{1}{3}(1,2)\}$.
  - $\Gamma_i \cap \Gamma_j = \{\mathsf{p}_z\} \subset \mathrm{Sing}(X)$ for $i \neq j$. Moreover, S is quasi-smooth at p_z since $S \in |A|$ is general.

  We can compute $(\Gamma_i^2)_S = -5/6$ by the method explained in Remark 3.10 and then we have $(\Gamma_i \cdot \Gamma_j)_S = 1/2$ for $i \neq j$ by considering $(\Gamma_l \cdot T|_S)_S$ for $l = 1,2,3$. Thus, the intersection matrix of $\Gamma_1, \Gamma_2, \Gamma_3$ is given by

$$((\Gamma_i \cdot \Gamma_j)_S) = \begin{pmatrix} -\frac{5}{6} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{5}{6} \end{pmatrix}$$

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and it satisfies the condition ($\star$). By Lemma 3.21, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{3}{4}\right\} = \frac{3}{4}$$

for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.3. The family $\mathcal{F}_{12}$

We have $w^2z \in F$ by the quasi-smoothness of X at p_w. Hence, rescaling w, we can write $f = w^2z + \alpha wt^2 + \lambda wz^3 + \beta t^2z^2 + \mu z^5$, where $\alpha, \beta, \lambda, \mu \in \mathbb{C}$. We can eliminate the monomial z^5 by replacing w and hence we assume $\mu = 0$. Then, by the quasi-smoothness of X at p_z, we have $\lambda \neq 0$ and we may assume $\lambda = -1$ by rescaling z. Thus, we can write

$$f = w^2z + \alpha wt^2 - wz^3 + \beta t^2z^2.$$

- Case (i): $\alpha \neq 0$ and $\beta \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha \neq 0$ and either $\beta = 0$ or $\beta = \alpha$. When $\beta = \alpha$, we replace $w \mapsto w - z^2$ and $z \mapsto -z$. After this replacement, we may assume $\beta = 0$. Moreover, we may assume $\alpha = 1$ by rescaling t. Then we have $f = w(wz + t^2 - z^3)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = wz + t^2 - z^3 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve. We have $\Gamma \cap \Delta = \{\mathsf{q}\}$, where $\mathsf{q} = (0:0:1:1:0) \in \mathrm{Sm}(X)$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at q. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\alpha = 0$ and $\beta \neq 0$. Rescaling t, we may assume $\beta = 1$. Then we have $f = z(w^2 + wz^2 + t^2z)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + wz^2 + t^2z = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$. By the similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at p_t. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iv): $\alpha = \beta = 0$. In this case, $f = zw(w + z^2)$ and $T|_S = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are as follows.
  - $\Gamma_1 = (x = y = z = 0)$ is a quasi-line of degree $1/12$ and $\mathrm{Sing}_{\Gamma_1}(S) = \{1 \times \frac{1}{3}(1,3), 1 \times \frac{1}{4}(1,3)\}$.
  - $\Gamma_2 = (x = y = w = 0)$ and $\Gamma_3 = (x = y = w + z^2 = 0)$ are quasi-lines of degree $1/6$ and $\mathrm{Sing}_{\Gamma_i}(S) = \{1 \times \frac{1}{2}(1,1), \frac{1}{3}(1,2)\}$ for $i = 2, 3$.
  - For $1 \leq i < j \leq 3$, we have $\Gamma_i \cap \Gamma_j = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$. Moreover, S is quasi-smooth at p_t since $S \in |A|$ is general.

  By the similar computation as in Case (iii) of Section 4.4.b, the intersection matrix of $\Gamma_1, \Gamma_2, \Gamma_3$ is given by

$$\begin{pmatrix} -\frac{7}{12} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{5}{6} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{5}{6} \end{pmatrix}$$

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and it satisfies the condition (★). By Lemma 3.21, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} 3/4, & \text{if } \mathsf{p} \in \Gamma_1 \cap \mathrm{Sm}(X), \\ 4/5, & \text{if } \mathsf{p} \in \Gamma_i \cap \mathrm{Sm}(X) \text{ for } i = 2, 3. \end{cases}$$

Thus, $\alpha_{\mathsf{p}}(X) \geq 3/4$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.4. The family $\mathcal{F}_{13}$

We have

$$f = \alpha w t^2 + \beta w z^3 + \gamma t^3 z + \delta t z^4,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Note that $(\alpha, \gamma) \neq (0, 0)$ since X is quasi-smooth at p_t.

- Case (i): $\alpha \neq 0$, $(\beta, \delta) \neq (0, 0)$ and (α, γ) is not proportional to (β, δ). In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_w \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha \neq 0$, $(\beta, \delta) \neq (0, 0)$ and (α, γ) is proportional to (β, δ). In this case, $f = (\alpha w + \gamma t z)(t^2 + \varepsilon z^3)$, where $\varepsilon := \beta/\alpha \in \mathbb{C}$ is nonzero. Replacing w and z, we may assume $f = w(t^2 - z^3)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = t^2 - z^3 = 0).$$

We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} = \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{q}\}$, where $\mathsf{q} = (0\!:\!0\!:\!1\!:\!1\!:\!0) \in \mathrm{Sm}(X)$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at q. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\alpha \neq 0$ and $(\beta, \delta) = (0, 0)$. In this case, $f = t^2(\alpha w + \gamma t z)$ and we may assume $f = t^2 w$ by replacing w. We can write

$$F = f_1(z, w) x + f_2(z, w) y + t^2 w + g(x, y, z, t, w),$$

where $f_1, f_2 \in \mathbb{C}[z, w]$ and $g \in \mathbb{C}[x, y, z, t, w]$ are homogeneous polynomials such that $g \in (x, y) \cap (x, y, t)^2$. By Lemma 4.16, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iv): $\alpha = 0$ and $\beta \neq 0$. Note that $\gamma \neq 0$. In this case, $f = z(\beta w z^2 + \gamma t^3 + \delta t z^3)$. Then $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = \beta w z^2 + \gamma t^3 + \delta t z^3 = 0).$$

We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\} \subset \mathrm{Sing}(X)$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at p_w. Finally, we have $\mathrm{Sing}_{\Gamma}(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Then, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (v): $\alpha = \beta = 0$ and $\delta \neq 0$. Note that $\gamma \neq 0$. In this case, $f = zt(\gamma t^2 + \delta z^3)$ and $T|_S = \Gamma_1 + \Gamma_2 + \Gamma_3$, where $\Gamma_1, \Gamma_2, \Gamma_3$ are as follows.
  - $\Gamma_1 = (x = y = z = 0)$ is a quasi-line of degree $1/15$ and $\mathrm{Sing}_{\Gamma_1}(S) = \{1 \times \frac{1}{3}(1, 2), 1 \times \frac{1}{5}(2, 3)\}$.
  - $\Gamma_2 = (x = y = t = 0)$ is a quasi-line of degree $1/10$ and $\mathrm{Sing}_{\Gamma_2}(S) = \{1 \times \frac{1}{2}(1, 1), 1 \times \frac{1}{5}(2, 3)\}$.
  - $\Gamma_3 = (x = y = \gamma t^2 + \delta z^3 = 0)$ is an irreducible smooth curve of degree $1/5$.
  - For $1 \leq i < j \leq 3$, we have $\Gamma_i \cap \Gamma_j = \{\mathsf{p}_w\} \subset \mathrm{Sing}(X)$. Moreover, S is quasi-smooth at p_w.

  We compute $(\Gamma_1^2)_S = -8/15$ and $(\Gamma_2^2)_S = -15/19$ by the method explained in Remark 3.10. We can choose z, t as orbifold coordinates of S at p_w. It follows that Γ_1, Γ_2 intersect transversally at the point over p_w on the orbifold chart of S at p_w, and we have $(\Gamma_1 \cdot \Gamma_2)_S = 1/5$. Then, by taking intersections

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with $T|_S = \Gamma_1 + \Gamma_2 + \Gamma_3$ with Γ_i for $i = 1, 2, 3$, we see that the intersection matrix of $\Gamma_1, \Gamma_2, \Gamma_3$ is given by

$$\begin{pmatrix} -\frac{8}{15} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{7}{10} & \frac{3}{5} \\ \frac{2}{5} & \frac{3}{5} & -1 \end{pmatrix}$$

and it satisfies the condition $(\star)$. By Lemma 3.21,

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} 10/13, & \text{if } \mathsf{p} \in \Gamma_1 \cap \mathrm{Sm}(X), \\ 15/19, & \text{if } \mathsf{p} \in \Gamma_2 \cap \mathrm{Sm}(X), \\ 6/7, & \text{if } \mathsf{p} \in \Gamma_3 \cap \mathrm{Sm}(X). \end{cases}$$

Thus, we have $\alpha_{\mathsf{p}}(X) \geq 10/13$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

- Case (vi): $\alpha = \beta = \delta = 0$. In this case, $\gamma \neq 0$ and we may assume that $f = t^3 z$ by rescaling z. We can write

$$F = f_1(z, w)x + f_2(z, w)y + t^3 z + g(x, y, z, t, w),$$

where $f_1, f_2 \in \mathbb{C}[z, w]$ and $g \in \mathbb{C}[x, y, z, t, w]$ are homogeneous polynomials such that $g \in (x, y) \cap (x, y, t)^2$. By Lemma 4.16, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.5. The family $\mathcal{F}_{15}$

We have $w^2 \in f$. Replacing w, we may assume that the coefficients of z^6 and t^4 are both 0. Then, by the quasi-smoothness of X at $\mathsf{p}_z, \mathsf{p}_t \in X$, we have $t^2 w, z^3 w \in F$. Hence, by rescaling w, t and z, we can write

$$f = w^2 + (t^2 - z^3)w + \alpha t^2 z^3$$

for some $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. Replacing w and rescaling z, we may assume $f = w(w + t^2 - z^3)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = w + t^2 - z^3 = 0).$$

We see that Γ and Δ are both quasi-lines. We have $\Gamma \cap \Delta = \{\mathsf{q}\}$, where $\mathsf{q} = (0\!:\!0\!:\!1\!:\!1\!:\!0) \in \mathrm{Sm}(X)$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at q. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.6. The family $\mathcal{F}_{20}$

We have $w^2 z \in F$ by the quasi-smoothness of X at p_w. Hence, we can write

$$f = w^2 z + \alpha w t^2 + \beta t z^3,$$

where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$ and $\beta \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

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- Case (ii): $\alpha \neq 0$ and $\beta = 0$. We have $f = w(wz + \alpha t^2)$ and thus $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = wz + \alpha t^2 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\} \subset \mathrm{Sing}(X)$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at p_z. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\alpha = 0$ and $\beta \neq 0$. We have $f = z(w^2 + \beta t z^2)$ and thus $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + \beta t z^2 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\}$. By a similar argument as in Case (ii) of Section 4.4.b, we conclude that S is quasi-smooth at p_t. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iv): $\alpha = \beta = 0$. In this case, $f = w^2 z$ and we can write

$$F = f_1(z,t)x + f_2(z,t)y + w^2 z + g(x,y,z,t,w),$$

  where $f_1, f_2 \in \mathbb{C}[z,t]$ and $g \in \mathbb{C}[x,y,z,t,w]$ are homogeneous polynomials such that $g \in (x,y) \cap (x,y,w)^2$. By Lemma 4.16, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.7. The family $\mathcal{F}_{23}$

We have $w^2 t \in F$ by the quasi-smoothness of X at p_w. Hence, we can write

$$f = w^2 t + \alpha w z^3 + \beta t^2 z^2,$$

where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$ and $\beta \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha \neq 0$ and $\beta = 0$. We have $f = w(wt + \alpha z^3)$ and thus $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = wt + \alpha z^3 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$, and S is quasi-smooth at p_t since $t^3 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\alpha = 0$ and $\beta \neq 0$. We have $f = t(w^2 + \beta t z^2)$ and thus $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = w^2 + \beta t z^2 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\}$, and S is quasi-smooth at p_z since $z^4 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iv): $\alpha = \beta = 0$. In this case, $f = w^2 t$ and we can write

$$F = f_1(z,t)x + f_2(z,t) + w^2 t + g(x,y,z,t,w),$$

  where $f_1, f_2 \in \mathbb{C}[z,t]$ and $g \in \mathbb{C}[x,y,z,t,w]$ are homogeneous polynomials such that $g \in (x,y) \cap (x,y,w)^2$. By Lemma 4.16, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

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4.4.b.8. The family $\mathcal{F}_{24}$

We have $t^3 \in F$ and, by rescaling t, we can write

$$f = \alpha w z^4 + t^3 + \beta t z^5,$$

where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_w \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$ and $\beta \neq 0$. By rescaling z, we may assume $f = t(t^2 + z^5)$. Then $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = t^2 + z^5 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} = \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\}$, and S is quasi-smooth at p_w by a similar argument as in Case (ii) of Section 4.4.b. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (iii): $\alpha = \beta = 0$. In this case, $f = t^3$ and the defining polynomial F of X can be written as

$$F = w^2 \ell_1(x, y) + t^3 + z^7 \ell_2(x, y) + w h_8(x, y, z, t) + h_{15}(x, y, z, t),$$

  where $h_8, h_{15} \in \mathbb{C}[x, y, z, t]$ are homogeneous polynomials of degrees 8, 15, respectively, such that $z^4 \notin h_8$ and $t^3, z^7 x, z^7 y \notin h_{15}$, and ℓ_1, ℓ_2 are linear forms in x, y. Note that $h_8, h_{15} \in (x, y) \cap (x, y, t)^2$. By the quasi-smoothness of X, we see that ℓ_1 and ℓ_2 are linearly independent. Replacing x, y, we can assume that

$$F = w^2 x + t^3 - z^7 y + g,$$

  where $h = w h_8 + h_{15} \in (x, y) \cap (x, y, t)^2$. Thus, by Lemma 4.17, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.9. The family $\mathcal{F}_{25}$

We have $z^5 \in F$ and, by rescaling z, we can write

$$f = \alpha w t^2 + \beta t^3 z + z^5,$$

where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_w \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. By the quasi-smoothness of X at p_t, we have $\beta \neq 0$, and hence we may assume $\beta = 1$. Then $f = z(t^3 + z^5)$ and we have $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = t^3 + z^5 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} = \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\}$, and S is quasi-smooth at p_w by a similar argument as in Case (ii) of Section 4.4.b. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.10. The family $\mathcal{F}_{29}$

We have $w^2 \in F$ and, by rescaling w, we can write $f = w^2 + \lambda w z^4 + \alpha t^2 z^3 + \mu z^8$, where $\alpha, \lambda, \mu \in \mathbb{C}$. By replacing w, we can eliminate the term μz^8, that is, we may assume $\mu = 0$. Then, by the quasi-smoothness

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of X at p_z, we have $wz^4 \in F$, that is, $\lambda \neq 0$. Thus, we can write

$$f = w^2 + wz^4 + \alpha t^2 z^3.$$

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. Then we have $f = w(w + z^4)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = w + z^4).$$

  We see that Γ and Δ are both quasi-lines. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\}$, and S is quasi-smooth at p_t by a similar argument as in Case (ii) of Section 4.4.b. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.11. The family $\mathcal{F}_{30}$

We have $w^2 \in F$ and, by rescaling w, we can write $f = w^2 + \lambda wt^2 + \mu t^4 + \alpha tz^4$, where $\alpha, \lambda, \mu \in \mathbb{C}$. We may assume $\mu = 0$ by replacing w, and then we have $wt^2 \in F$ by the quasi-smoothness of X at p_t. Thus, we can write

$$f = w^2 + wt^2 + \alpha tz^4.$$

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. Then we have $f = w(w + t^2)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = w + t^2 = 0).$$

  We see that Γ and Δ are both quasi-lines. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\}$, and S is quasi-smooth at p_z by a similar argument as in Case (ii) of Section 4.4.b. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.12. The family $\mathcal{F}_{31}$

We have $w^2 z \in F$ by the quasi-smoothness of X at p_w, and we have $z^4 \in F$. Rescaling w and z, we can write

$$f = w^2 z + \alpha wt^2 - z^4,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = z(w^2 - z^3)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 - z^3 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$, and S is quasi-smooth at p_t by a similar argument as in Case (ii) of Section 4.4.b. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.13. The family $\mathcal{F}_{32}$

We have $t^4 \in F$, and we can write

$$f = \alpha wz^3 + t^4 + \beta tz^4,$$

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where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_w \in \operatorname{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.
- Case (ii): $\alpha = 0$ and $\beta \neq 0$. Rescaling z, we may assume $\beta = 1$ and $f = t(t^3 + z^4)$. Then $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = t^3 + z^4 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} = \Delta \cap \operatorname{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\}$, and S is quasi-smooth at p_w since $w^2 y \in F$ and $S = H_x$. Finally, we have $\operatorname{Sing}_\Gamma = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.
- Case (iii): $\alpha = \beta = 0$. In this case, $f = t^4$. Since $wz^3, tz^4 \notin F$, we have $z^5 x \in F$, and we can write

$$F = w^2 y + t^4 + z^5 x + w h_9(x, y, z, t) + h_{16}(x, y, z, t),$$

  where $g_i \in \mathbb{C}[x, y, z, t]$ is a homogeneous polynomial of degree i such that $z^3 \notin h_9$ and $t^4, tz^4, z^5 x \notin h_{16}$. Note that $h_9, h_{16} \in (x, y) \cap (x, y, t)^2$. Thus, by Lemma 4.17, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.

4.4.b.14. The family $\mathcal{F}_{33}$

We have $w^2 z \in F$ by the quasi-smoothness of X at p_w, and we can write

$$f = w^2 z + \alpha w t^2 + \beta t z^4,$$

where $\alpha, \beta \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$ and $\beta \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.
- Case (ii): $\alpha \neq 0$ and $\beta = 0$. In this case, $f = w(wz + \alpha z^2)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = wz + \alpha z^2 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible quasi-smooth curve. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\} \subset \operatorname{Sing}(X)$, and S is quasi-smooth at p_z since $z^5 t \in F$ and $S = H_x$. Finally, we have $\operatorname{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.
- Case (iii): $\alpha = 0$ and $\beta \neq 0$. In this case, $f = z(w^2 + \beta t z^3)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + \beta t z^3 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} \supset \Delta \cap \operatorname{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\} \subset \operatorname{Sing}(X)$, and S is quasi-smooth at p_t since $t^3 y \in F$ and $S = H_x$. Finally, we have $\operatorname{Sing}_\Gamma(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.
- Case (iv): $\alpha = \beta = 0$. In this case, $f = w^2 z$ and we can write

$$F = f_1(z, t)x + f_2(z, t)y + w^2 z + g(x, y, z, t, w),$$

  where $f_1, f_2 \in^m bC[z, t]$ and $g \in \mathbb{C}[x, y, z, t, w]$ are homogeneous polynomials such that $g \in (x, y) \cap (x, y, w)^2$. By Lemma 4.16, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \operatorname{Sm}(X)$.

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4.4.b.15. The family $\mathcal{F}_{37}$

We have $w^2 \in F$, and we can write $f = w^2 + \lambda wz^3 + \alpha t^3 z^2 + \mu z^6$, where $\alpha, \lambda, \mu \in \mathbb{C}$. Replacing w, we may assume $\mu = 0$. Then, by the quasi-smoothness of X at p_z, we have $\lambda \neq 0$. Rescaling z, we can write

$$f = w^2 + wz^3 + \alpha t^3 z^2.$$

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = w(w + z^3)$ and $T|_S = \Gamma + \Delta$, where

  $$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = w + z^3 = 0)$$

  are both quasi-lines. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\}$, and S is quasi-smooth at p_t since $t^4 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.16. The family $\mathcal{F}_{38}$

We have $z^6 \in F$, and we can write

$$f = \alpha wt^2 + \beta t^3 z + z^6,$$

where $\alpha, \beta \in \mathbb{C}$. Note that we have $(\alpha, \beta) \neq (0, 0)$ by the quasi-smoothness of X.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_w \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap X_{\mathrm{sm}}$.
- Case (ii): $\alpha = 0$. Note that $\beta \neq 0$. In this case, $f = z(\beta t^3 + z^5)$ and $T|_S = \Gamma + \Delta$, where

  $$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = \beta t^3 + z^5 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_w\} = \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\} \subset \mathrm{Sing}(X)$, and S is quasi-smooth at p_w since $w^2 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.17. The family $\mathcal{F}_{39}$

We have $w^3 \in F$ and $wz^3 \in F$ by the quasi-smoothness of X. Rescaling w and z, we can write

$$f = w^3 + wz^3 + \alpha t^2 z^2,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = z(w^2 + z^3)$ and we have $T|_S = \Gamma + \Delta$, where

  $$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + z^3 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} = \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\}$, and S is quasi-smooth at p_t since $t^3 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

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4.4.b.18. The family $\mathcal{F}_{40}$

We have $w^2 t \in F$ and $t^3 z \in F$ by the quasi-smoothness of X at p_w and p_t. Rescaling w and z, we can write

$$f = w^2 t + \alpha w z^3 + t^3 z,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = t(w^2 + t^2 z)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = w^2 + t^2 z = 0).$$

We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_z\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\}$, and Sbis quasi-smooth at p_z since $z^4 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.19. The family $\mathcal{F}_{42}$

We have $w^2 \in F$, and we can write $f = w^2 + \lambda w t^2 + \mu t^4 + \alpha t z^5$, where $\alpha, \lambda, \mu \in \mathbb{C}$. Replacing w, we may assume $\mu = 0$. Then, by the quasi-smoothness of X at p_t, we have $\lambda \neq 0$. Rescaling t, we can write

$$f = w^2 + w t^2 + \alpha t z^5.$$

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = w(w + t^2)$ and $T|_S = \Gamma + \Delta$, where

$$\Gamma = (x = y = w = 0), \quad \Delta = (x = y = w + t^2 = 0).$$

We see that Γ and Δ are both quasi-lines. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\} \subset \mathrm{Sing}(X)$, and S is quasi-smooth at p_z since $z^6 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_t\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.20. The family $\mathcal{F}_{49}$

We have $w^3 \in F$, and we can write

$$f = w^3 + \alpha t z^3,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = w^3$ and, after replacing x, y suitably, the defining polynomial F of X can be written as

$$F = w^3 + t^3 y - z^3 x + g_{21}(x, y, z, t, w),$$

where $g_{21} \in \mathbb{C}[x, y, z, t, w]$ is a homogeneous polynomial of degree 21 such that $g \in (x, y) \cap (x, y, w)^2$. By Lemma 4.17, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

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4.4.b.21. The family $\mathcal{F}_{50}$

We have $w^2 \in F$, and we can write

$$f = w^2 + \alpha t z^5,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. We have $S \cdot T = 2\Gamma$, where $\Gamma = (x = y = w = 0)$. By Lemma 4.18, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.22. The family $\mathcal{F}_{52}$

We have $w^2 \in F$, and we can write

$$f = w^2 + \alpha t^2 z^3,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\{\mathsf{p}_z, \mathsf{p}_t\} \subset \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. We have $S \cdot T = 2\Gamma$, where $\Gamma = (x = y = w = 0)$. By Lemma 4.18, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.23. The family $\mathcal{F}_{58}$

We have $w^2 z \in F$ by the quasi-smoothness of X at p_w. Also, we have $z^6 \in F$. Rescaling w and z, we can write

$$f = w^2 z + \alpha w t^2 + z^6,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = z(w^2 + z^5)$ and $T|_S = \Gamma + \Delta$, where

  $$\Gamma = (x = y = z = 0), \quad \Delta = (x = y = w^2 + z^5 = 0).$$

  We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_t\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_t\}$, and S is quasi-smooth at p_t since $t^3 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_t, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.24. The family $\mathcal{F}_{60}$

We have $w^2 t \in F$ by the quasi-smoothness of X at p_w. Also, we have $t^4 \in F$. Rescaling w and t, we can write

$$f = w^2 t + \alpha w z^3 + t^4,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and smooth. By Lemma 4.13, we have $\alpha_{\mathsf{p}}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. In this case, $f = t(w^2 + t^3)$ and $T|_S = \Gamma + \Delta$, where

  $$\Gamma = (x = y = t = 0), \quad \Delta = (x = y = w^2 + t^3 = 0).$$

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We see that Γ is a quasi-line and Δ is an irreducible curve which is quasi-smooth along $\Delta \setminus \{\mathsf{p}_z\} \supset \Delta \cap \mathrm{Sm}(X)$. We have $\Gamma \cap \Delta = \{\mathsf{p}_z\}$, and S is quasi-smooth at p_z since $z^4 y \in F$ and $S = H_x$. Finally, we have $\mathrm{Sing}_\Gamma(X) = \{\mathsf{p}_z, \mathsf{p}_w\}$. Thus, by Lemma 4.14, we have $\alpha_\mathsf{p}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.25. The family $\mathcal{F}_{63}$

We have $w^2 \in F$, and we can write

$$f = w^2 + \alpha t z^6,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_\mathsf{p}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. We have $S \cdot T = 2\Gamma$, where $\Gamma = (x = y = w = 0)$. By Lemma 4.18, we have $\alpha_\mathsf{p}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.4.b.26. The family $\mathcal{F}_{64}$

We have $w^2 \in F$, and we can write

$$f = w^2 + \alpha t z^4,$$

where $\alpha \in \mathbb{C}$.

- Case (i): $\alpha \neq 0$. In this case, $S \cdot T = L_{xy}$ is irreducible and is smooth outside $\mathsf{p}_t \in \mathrm{Sing}(X)$. By Lemma 4.13, we have $\alpha_\mathsf{p}(X) = 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.
- Case (ii): $\alpha = 0$. We have $S \cdot T = 2\Gamma$, where $\Gamma = (x = y = w = 0)$. By Lemma 4.18, we have $\alpha_\mathsf{p}(X) \geq 1/2$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

4.5. *Smooth points on H_x for families with $1 < a_1 = a_2$*

The aim of this section is to prove the following.

Proposition 4.19. *Let $X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$, $a_1 \leq \cdots \leq a_4$, be a member of a family $\mathcal{F}_\mathsf{i}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ such that $1 < a_1 = a_2$. Then*

$$\alpha_\mathsf{p}(X) \geq \frac{1}{2}$$

for any smooth point $\mathsf{p} \in X$ contained in H_x.

Note that a family $\mathcal{F}_\mathsf{i}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ satisfies the assumption of Proposition 4.19 if and only if

$$\mathsf{i} \in \{18, 22, 28\}.$$

4.5.a. The family $\mathcal{F}_{18}$

This subsection is devoted to the proof of Proposition 4.19 for the family $\mathcal{F}_{18}$. Let $X = X_{12} \subset \mathbb{P}(1, 2, 2, 3, 5)$ be a member of $\mathcal{F}_{18}$.

By the quasi-smoothness of X, We have $t^4 \in F$ and we may assume $\mathrm{coeff}_F(t^4) = 1$ by rescaling t. We have $(x = y = z = 0) \cap X = \{\mathsf{p}_w\} \subset \mathrm{Sing}(X)$. Hence, we may assume $\mathsf{p} \in H_y$ and $\mathsf{p} \notin H_z$ after possibly replacing y and z, and we can write $\mathsf{p} = (0\!:\!0\!:\!1\!:\!\lambda\!:\!\mu)$ for some $\lambda, \mu \in \mathbb{C}$. We can write

$$F(0, 0, z, t, w) = \alpha w^2 z + \beta w t z^2 + t^4 + \gamma t^2 z^3 + \delta z^6,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We will derive a contradiction by assuming $\alpha_\mathsf{p}(X) < 1/2$. By the assumption, there exists an irreducible $\mathbb{Q}$-divisor $D \in |A|_\mathbb{Q}$ such that $\mathrm{lct}_\mathsf{p}(X; D) < 1/2$.

Suppose $\lambda \neq 0$. Then, by replacing w by $\lambda w - \mu z t$, we may assume $\mathsf{p} = (0:0:1:\lambda:0)$. Let S be a general member of the pencil $|\mathcal{I}_{\mathsf{p}}(2A)|$ so that $\frac{1}{2}S \neq D$. We can take a $\mathbb{Q}$-divisor $T \in |5A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ since $\{x, y, w\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = 2 \cdot 5 \cdot (A^3) = 2,$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1/2$. This is impossible, and we have $\lambda = 0$.

By rescaling w, we may assume $\mathsf{p} = (0:0:1:0:1)$. Suppose $\alpha \neq 0$. Then we have $\delta = -\alpha$ since $F(\mathsf{p}) = 0$. In this case, H_x is smooth at p since $(\partial F/\partial z)(\mathsf{p}) = -5\alpha \neq 0$. In particular, $H_x \neq D$. We can take a $\mathbb{Q}$-divisor $T \in |3A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$ since $\{x, y, t\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_x \cdot T)_{\mathsf{p}} \leq (D \cdot H_x \cdot T) = 3(A^3) = \frac{3}{5},$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 5/3$. This is impossible, and we have $\alpha = 0$. Note that $\delta = 0$ since $F(\mathsf{p}) = 0$, and we have

$$F(0, 0, z, t, w) = t(\beta w z^2 + t^3 + \gamma t z^3).$$

We claim $\mathrm{mult}_{\mathsf{p}}(H_x) \leq 2$. We set $\zeta := \mathrm{coeff}_F(w^2 y)$ and $\eta := \mathrm{coeff}_F(z^5 y)$. By the quasi-smoothness of X, we see $\zeta, \eta \neq 0$ since $w^2 y, z^6 \notin F$. We have

$$\frac{\partial F}{\partial y}(\mathsf{p}) = \zeta + \eta, \quad \frac{\partial F}{\partial t}(\mathsf{p}) = \beta.$$

If either $\zeta + \eta \neq 0$ or $\beta \neq 0$, then we have $\mathrm{mult}_{\mathsf{p}}(H_x) = 1$. If $\zeta + \eta = \beta = 0$, then we have $\mathrm{mult}_{\mathsf{p}}(H_x) = 2$ since the term $\zeta y(w^2 - z^5)$ appears in F. Thus, the claim is proved.

By the claim, we have $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$ and in particular $D \neq H_x$. We can take a $\mathbb{Q}$-divisor $T \in |10A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of $D \cdot H_x$ since $\{x, y, t, w^2 - z^5\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_x \cdot T)_{\mathsf{p}} \leq (D \cdot H_x \cdot T) = 10(A^3) = 2,$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1/2$. This is a contradiction and the proof is completed.

4.5.b. The family $\mathcal{F}_{22}$

This subsection is devoted to the proof of Proposition 4.19 for the family $\mathcal{F}_{22}$. Let $X = X_{14} \subset \mathbb{P}(1, 2, 2, 3, 7)$ be a member of $\mathcal{F}_{22}$.

By the quasi-smoothness of X, we have $w^2 \in F$ and we may assume $\mathrm{coeff}_F(w^2) = 1$ by rescaling w. We see that $(x = y = z = 0) \cap X = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$. Hence, we may assume $\mathsf{p} = (0:0:1:\lambda:\mu)$ for some $\lambda, \mu \in \mathbb{C}$ after possibly replacing y and z. We can write

$$F(0, 0, z, t, w) = w^2 + \alpha w t z^2 + \beta t^4 z + \gamma t^2 z^4 + \delta z^7,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. We will derive a contradiction by assuming $\alpha_{\mathsf{p}}(X) < 1/2$. By the assumption, there exists an irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ such that $\mathrm{lct}_{\mathsf{p}}(X; D) < 1/2$. Let S be a general member of the pencil $|\mathcal{I}_{\mathsf{p}}(2A)|$ so that $\frac{1}{2}S \neq D$.

Suppose $\lambda = 0$. In this case, we can take a $\mathbb{Q}$-divisor $T \in |3A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ since $\{x, y, t\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = 2 \cdot 3 \cdot (A^3) = 1,$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$. This is impossible, and we have $\lambda \neq 0$.

Replacing w by $\lambda w - \mu t z^2$, we may assume $\mu = 0$, that is, $\mathsf{p} = (0:0:1:\lambda:0)$. We see that the set $\{x, y, t^2 - \lambda^2 z^3\}$ isolates p. It follows that we can take a $\mathbb{Q}$-divisor $T \in |6A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ which does not contain any component of $D \cdot S$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = 2 \cdot 6 \cdot (A^3) = 2,$$

which implies $\alpha_{\mathsf{p}}(X) \geq 1/2$. This is a contradiction, and the proof is completed.

4.5.c. The family $\mathcal{F}_{28}$

This subsection is devoted to the proof of Proposition 4.19 for the family $\mathcal{F}_{28}$. Let $X = X_{15} \subset \mathbb{P}(1,3,3,4,5)$ be a member of $\mathcal{F}_{28}$.

By the quasi-smoothness of X, we have $w^3 \in F$ and we may assume $\mathrm{coeff}_F(w^3) = 1$ by rescaling w. We see that $(x = y = z = 0) \cap X = \{\mathsf{p}_t\} \subset \mathrm{Sing}(X)$. Hence, we may assume $\mathsf{p} = (0:0:1:\lambda:\mu)$ for some $\lambda, \mu \in \mathbb{C}$ after possibly replacing y and z. We can write

$$F(0,0,z,t,w) = w^3 + \alpha w t z^2 + \beta t^3 z + \gamma z^5,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. We will derive a contradiction by assuming $\alpha_{\mathsf{p}}(X) < 1/2$. By the assumption, there exists an irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ such that $\mathrm{lct}_{\mathsf{p}}(X; D) < 1/2$. Let S be a general member of the pencil $|\mathcal{I}_{\mathsf{p}}(3A)|$ so that $\mathrm{Supp}(S) \neq \mathrm{Supp}(D)$.

Suppose $\lambda \neq 0$ and $\mu \neq 0$. In this case, the set $\{x, y, \mu t^2 - \lambda^2 w z\}$ isolates p, and we can take a $\mathbb{Q}$-divisor $T \in |8A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = 3 \cdot 8 \cdot (A^3) = 2,$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1/2$. This is impossible, and we have either $\lambda = 0$ or $\mu = 0$.

Suppose $\lambda = 0$. In this case, we can take a $\mathbb{Q}$-divisor $T \in |4A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ since $\{x, y, t\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = 3 \cdot 4 \cdot (A^3) = 1,$$

which implies $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$. This is impossible. We have $\lambda \neq 0$ and $\mu = 0$. In this case, we may assume $\lambda = 1$ by rescaling t, that is, we may assume $\mathsf{p} = (0:0:1:1:0)$.

We claim $\mathrm{mult}_{\mathsf{p}}(H_x) \leq 2$. We set $\zeta := \mathrm{coeff}_F(t^3 y)$ and $\eta := \mathrm{coeff}_F(z^4 y)$. We have $\beta + \gamma = 0$ since $F(\mathsf{p}) = 0$. Then

$$\frac{\partial F}{\partial z}(\mathsf{p}) = \beta + 5\gamma = 4\gamma, \quad \frac{\partial F}{\partial y}(\mathsf{p}) = \zeta + \eta.$$

If either $\gamma \neq 0$ or $\zeta + \eta \neq 0$, then we have $\mathrm{mult}_{\mathsf{p}}(H_x) = 1$. It remains to consider the case where $\gamma = \zeta + \eta = 0$. Note that we have $\beta = 0$ since $\beta + \gamma = 0$. By the quasi-smoothness of X at p_t, we have $\zeta \neq 0$. Then we see that $\mathrm{mult}_{\mathsf{p}}(H_x) = 2$ since the term $\zeta y(t^3 - z^4)$ appears in F. Thus, the claim is proved.

By the claim, we have $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$ and in particular $D \neq H_x$. We can take a $\mathbb{Q}$-divisor $T \in |12A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$ since $\{x, y, w, t^3 - z^4\}$ isolates p. Then we have

$$\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_x \cdot T)_{\mathsf{p}} \leq (D \cdot H_x \cdot T) = 12(A^3) = 1,$$

which implies $\alpha_{\mathsf{p}}(X) \geq 1$. This is a contradiction and the proof is completed.

5. Singular points

The aim of this section is to prove the following.

Theorem 5.1. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}$$

for any singular point $\mathsf{p} \in X$.

Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$. Then the inequality $\alpha_{\mathsf{p}}(X) \geq 1/2$ will follow from Propositions 5.2, 5.3, 5.5 for singular points $\mathsf{p} \in X$ which are not BI centers; from Proposition 5.6 for EI centers; and from Propositions 5.15, 5.16 and 5.18 for QI centers. This will complete the proof of Theorem 5.1.

5.1. Non-BI centers

Throughout the present section, let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$.

5.1.a. Computation by $\overline{\mathrm{NE}}(Y)$

Proposition 5.2. *Let $\mathsf{p} \in X$ be a singular point with subscript $\heartsuit$ in the fifth column of Table 7, and let $\varphi \colon Y \to X$ be the Kawamata blowup at p. Then $(-K_Y)^2 \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$ and $\tilde{D} \sim -K_Y$ for the proper transform of a general member $D \in |A|$. In particular, we have*

$$\alpha_{\mathsf{p}}(X) \geq 1.$$

Proof. Let $r > 1$ be the index of the quotient singular point $\mathsf{p} \in X$. For every instance, we have either $a_1 = 1$ or $d - 1$ is not divisible by r. This means that we can take x as a part of local orbifold coordinates of X at p, and hence $\tilde{D} \sim -K_Y$ for a general $D \in |A|$. The point p is excluded as a maximal center by either Lemma 3.2.2 or 3.2.4 of [CP17].

We set $S := \tilde{D} \sim -K_Y$, where $D \in |A|$ is a general member. If p is excluded by [CP17, Lemma 3.2.2], then it follows from its proof that $(-K_Y)^2 = (-K_Y) \cdot S \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$. If p is excluded by [CP17, Lemma 3.2.4], then there exists a nef divisor T on Y such that $(T \cdot S \cdot -K_Y) \leq 0$, which implies $(-K_Y)^2 = (-K_Y) \cdot S \notin \operatorname{Int} \overline{\mathrm{NE}}(Y)$. The latter assertion follows from Lemma 3.30. □

5.1.b. Computation by L_{xy}

Proposition 5.3. *Let $\mathsf{p} \in X$ be a singular point with the subscript $\diamondsuit$ or $\diamondsuit'$ in the fifth column of Table 7, and let $q = q_{\mathsf{p}}$ be the quotient morphism of $\mathsf{p} \in X$. We denote by r the index of the cyclic quotient singularity $\mathsf{p} \in X$. Let $S \in |A|$ and $T \in |a_1 A|$ be general members. Then the following assertions hold.*

1. *The pair (X, S) is log canonical at p.*
2. *The intersection $S \cap T$ is irreducible, and we have $q^*S \cdot q^*T = \check{\Gamma}$, where $\check{\Gamma}$ is an irreducible and reduced curve such that*

$$0 < \operatorname{mult}_{\check{\mathsf{p}}}(\check{\Gamma}) \leq a_1.$$

3. *We have*

$$r a_1 (A^3) \leq \begin{cases} 1, & \text{if the subscript of } \mathsf{p} \text{ is } \diamondsuit, \\ \frac{3}{2}, & \text{if the subscript of } \mathsf{p} \text{ is } \diamondsuit'. \end{cases}$$

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In particular,

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} 1, & \text{if the subscript of } \mathsf{p} \text{ is } \diamond, \\ \frac{3}{2}, & \text{if the subscript of } \mathsf{p} \text{ is } \diamond'. \end{cases}$$

Proof. Let $\mathsf{p} \in X$ be as in the statement. The assertion (3) can be checked individually, and it remains to consider (1) and (2).

It is straightforward to check that X is a member of a family $\mathcal{F}_{\mathsf{i}}$ which is listed in one of the Tables 1 and 2. It follows that $S \cap T = L_{xy}$ is irreducible. It is easy to check that we may assume $\mathsf{p} = \mathsf{p}_v$ for some $v \in \{z, t, w\}$ after replacing coordinates. Let $\rho = \rho_v \colon \breve{U}_v \to U_v$ be the orbifold chart. We set $\breve{\Gamma} = (\breve{x} = \breve{y} = 0) \subset U_v$. We see that $\breve{\Gamma}$ is an irreducible and reduced curve since so is L_{xy}, and that $\rho^* S \cdot \rho^* T = \breve{\Gamma}$. Note that q can be identified with ρ over a suitable analytic neighborhood of $\mathsf{p} \in U_v$, and hence it is enough to prove the inequality $\mathrm{mult}_{\breve{\mathsf{p}}}(\breve{\Gamma}) \leq a_1$ for the proof of (2).

If X is listed in Table 1, then $\breve{\Gamma}$ is irreducible and smooth by Lemma 4.9. In this case, S is quasi-smooth at p and thus both (1) and (2) are clearly satisfied.

Suppose that X is listed in Table 2. Then X is a member of a family $\mathcal{F}_{\mathsf{i}}$, where

$$\mathsf{i} \in \{44, 47, 61, 62, 65, 69, 77, 79, 83, 85\}.$$

If p is not the unique singular point of L_{xy} which is described in Table 2, then (1) and (2) follow immediately. Suppose that p is the unique singular point of L_{xy}. Then we have $\mathsf{i} \in \{44, 61, 83\}$ and $\mathsf{p} = \mathsf{p}_t$. By the equation given in Table 2, we compute $\mathrm{mult}_{\breve{\mathsf{p}}}(\breve{\Gamma}) = 2$. This shows (2) since $a_1 \geq 2$. We see that $r = a_3$ does not divide $d - 1$, which implies that S is quasi-smooth at p and hence (1) follows. Therefore, (1), (2) and (3) are verified and the assertion on $\alpha_{\mathsf{p}}(X)$ follows from Lemma 3.17. $\square$

5.1.c. Computation by isolating class

Proposition 5.4. *Let $\mathsf{p} \in X$ be a singular point with subscript $\clubsuit$ in the fifth column of Table 7 which is also listed in Table 3. Then the set of coordinates given in the fifth column of Table 3 isolates p. In particular, we have*

$$\alpha_{\mathsf{p}}(X) \geq \min\{1, c\} \geq \frac{1}{2},$$

where c is the number given in the seventh column of Table 3.

Proof. Let $\mathcal{C}$ be the set of homogeneous coordinates given in the fifth column of Table 3. It is straightforward to check that

$$\bigcap_{v \in \mathcal{C}} (v = 0) \cap X$$

is a finite set of points including p, which shows that $\mathcal{C}$ isolates p.

Let c be the number listed in the seventh column of Table 3, and assume that $\alpha_{\mathsf{p}}(X) < \min\{1, c\}$. Then there exists an irreducible $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} A$ such that (X, cD) is not log canonical at p. In particular, we have $\mathrm{omult}_{\mathsf{p}}(D) > 1/c$. If H_x (resp. $|nA|$ for some $n > 0$) is given in the fourth column of Table 3, then we set $S := H_x$ (resp. we define S to be a general member of $|nA|$). We set $n = 1$ if $S = H_x$ so that $S \sim nA$ in any case. Let r be the index of the cyclic quotient singularity $\mathsf{p} \in X$. We claim that $\mathrm{Supp}(D)$ is not contained in S. This is clear when $S \in |nA|$ is a general member. Suppose that $S = H_x$. Then we see that $d - 1$ is not divisible by r, which implies that $S = H_x$ is quasi-smooth at p. Hence, (X, S) is log canonical at p and we have $D \neq H_x$ as desired. By the claim, $D \cdot S$ is an effective 1-cycle on X. Let e be the integer given in the sixth column of Table 3. Note that $e = \max\{\deg v \mid v \in \mathcal{C}\}$ and

$$rne_{\max}(A^3) = \frac{1}{c}.$$

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Table 3. *Isolating set.*

No.	Pt.	Type	S	Isol. set	$e_{\max}$	c
10	p_t	$\frac{1}{3}(1,1,2)$	$\lvert A\rvert$	$\{x,y,z\}$	1	1/2
23	p_{yt}	$\frac{1}{2}(1,1,1)$	H_x	$\{x,z,w\}$	5	6/7
23	p_z	$\frac{1}{3}(1,1,2)$	H_x	$\{x,y,t,w\}$	5	4/7
23	p_t	$\frac{1}{4}(1,1,3)$	H_x	$\{x,y,z\}$	3	5/7
29	p_{zw}	$\frac{1}{2}(1,1,1)$	$\lvert A\rvert$	$\{x,y,t\}$	5	1/2
29	p_t	$\frac{1}{5}(1,2,3)$	$\lvert A\rvert$	$\{x,y,z\}$	2	1/2
31	p_{zw}	$\frac{1}{2}(1,1,1)$	$\lvert A\rvert$	$\{x,y,t\}$	5	3/4
33	p_z	$\frac{1}{3}(1,1,2)$	H_x	$\{x,y,t,w\}$	7	10/17
37	p_{zw}	$\frac{1}{3}(1,1,2)$	H_x	$\{x,y,t\}$	4	1
39	p_{yw}	$\frac{1}{3}(1,1,2)$	H_x	$\{x,z,t\}$	5	1
39	p_z	$\frac{1}{4}(1,1,3)$	H_x	$\{x,y,t\}$	5	1
40	p_z	$\frac{1}{4}(1,1,3)$	H_x	$\{x,y,t,w\}$	7	15/19
40	p_t	$\frac{1}{5}(1,2,3)$	H_x	$\{x,y,z,w\}$	4	1
50	p_t	$\frac{1}{7}(1,3,4)$	$\lvert A\rvert$	$\{x,y,z\}$	3	1/2
61	p_y	$\frac{1}{4}(1,1,3)$	H_x	$\{x,z,t,w\}$	9	7/5
63	p_t	$\frac{1}{8}(1,3,5)$	H_x	$\{x,y,z\}$	3	1
64	p_z	$\frac{1}{5}(1,2,3)$	$\lvert 2A\rvert$	$\{x,y,t\}$	6	1/2
66	p_y	$\frac{1}{5}(1,1,4)$	H_x	$\{x,z,t\}$	7	1
68	p_y	$\frac{1}{3}(1,1,2)$	$\lvert 4A\rvert$	$\{x,z,t\}$	7	1/2
80	p_y	$\frac{1}{3}(1,1,2)$	$\lvert 4A\rvert$	$\{x,z,t\}$	10	1/2
93	p_y	$\frac{1}{7}(1,3,4)$	$\lvert 8A\rvert$	$\{x,z,t\}$	10	1/2
95	p_y	$\frac{1}{5}(1,2,3)$	$\lvert 6A\rvert$	$\{x,z,t\}$	22	1/2

There exists an irreducible $\mathbb{Q}$-divisor $T \sim_{\mathbb{Q}} eA$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of $D \cdot S$ since $\mathcal{C}$ isolates p. It follows that

$$\frac{1}{c} < \operatorname{omult}_{\mathsf{p}}(D) \leq (q_{\mathsf{p}}^* D \cdot q_{\mathsf{p}}^* S \cdot q_{\mathsf{p}}^* T)_{\check{\mathsf{p}}} \leq r(D \cdot S \cdot T) = rne(A^3) = \frac{1}{c},$$

where $q = q_{\mathsf{p}}$ is the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ is the preimage of p via q. This is a contradiction and the inequality $\alpha_{\mathsf{p}}(X) \geq \min\{1, c\}$ is proved. □

5.1.d. Remaining non-BI centers

Proposition 5.5. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$, and let $\mathsf{p} \in X$ be a singular point with subscript ♠ in the fifth column of Table 7 which is also listed below.*

- $\mathsf{i} = 12$ *and singular points of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 13$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 24$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 27$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 32$ *and the singular point of type* $\frac{1}{3}(1,1,2)$.
- $\mathsf{i} = 33$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 40$ *and the singular point of type* $\frac{1}{3}(1,1,2)$.
- $\mathsf{i} = 47$ *and the singular point of type* $\frac{1}{5}(1,2,3)$.
- $\mathsf{i} = 48$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 49$ *and the singular point of type* $\frac{1}{5}(1,2,3)$.
- $\mathsf{i} = 62$ *and the singular point of type* $\frac{1}{5}(1,2,3)$.
- $\mathsf{i} = 65$ *and the singular point of type* $\frac{1}{2}(1,1,1)$.
- $\mathsf{i} = 67$ *and the singular point of type* $\frac{1}{9}(1,4,5)$.
- $\mathsf{i} = 82$ *and the singular point of type* $\frac{1}{5}(1,2,3)$.
- $\mathsf{i} = 84$ *and the singular point of type* $\frac{1}{7}(1,2,5)$.

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Then we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}.$$

The rest of this subsection is to prove Proposition 5.5 which will be separately for each family.

5.1.d.1. The family $\mathcal{F}_{12}$, points of type $\frac{1}{2}(1,1,1)$

Let $X = X_{10} \subset \mathbb{P}(1,1,2,3,4)$ be a member of $\mathcal{F}_{12}$ and p a singular point of type $\frac{1}{2}(1,1,1)$. We may assume $\mathsf{p} = \mathsf{p}_z$ after replacing w. Then we have $z^3 w \in F$ by the quasi-smoothness of X at p. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{2 \cdot 1 \cdot 3 \cdot (A^3)} = \frac{4}{5},$$

and the proof is completed in this case.

5.1.d.2. The family $\mathcal{F}_{13}$, the singular point of type $\frac{1}{2}(1,1,1)$

Let $X = X_{11} \subset \mathbb{P}(1,1,2,3,5)$ be a member of $\mathcal{F}_{13}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{2}(1,1,1)$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{2 \cdot 1 \cdot 3 \cdot (A^3)} = \frac{10}{11}, & \text{if } z^3 w \in F, \\ \frac{2}{2 \cdot 1 \cdot 5 \cdot (A^3)} = \frac{6}{11}, & \text{if } z^3 w \notin F \text{ and } z^4 t \in F. \end{cases}$$

It remains to consider the case where $z^3 w, z^4 t \notin F$. Then, by choosing x, y suitably, we can write

$$F = z^5 x + z^4 f_3 + z^3 f_5 + z^2 f_7 + z f_9 + f_{11},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_3$ and $w \notin f_5$.

We claim $w^2 y \in F$. Assume $w^2 y \notin F$. Then, by the quasi-smoothness of X at p_w, we have $w^2 x \in F$ and we may assume $\mathrm{coeff}_F(w^2 x) = -1$. We can write $F = (z^5 - w^2)x + f'$, where $f' = z^4 f_3 + z^3 f_5 + z^2 f_7 + z f_9 + f_{11} + w^2 x$. It is straightforward to check that $f' \in (x,y,t)^2$ and thus X is not quasi-smooth at the point $(0\!:\!0\!:\!1\!:\!0\!:\!1) \in X$, which is a contradiction. Thus, $w^2 y \in F$.

We see that $\bar{F} := F(0,y,1,t,w) \in (y,t,w)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^2 y \in \bar{F}$ and $w^3 \notin \bar{F}$. Thus, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ by Lemma 3.28.

5.1.d.3. The family $\mathcal{F}_{24}$, the point of type $\frac{1}{2}(1,1,1)$

Let $X = X_{15} \subset \mathbb{P}(1,1,2,5,7)$ be a member of $\mathcal{F}_{24}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{2}(1,1,1)$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{2 \cdot 1 \cdot 5 \cdot (A^3)} = \frac{14}{15}, & \text{if } z^4 w \in F, \\ \frac{2}{2 \cdot 1 \cdot 7 \cdot (A^3)} = \frac{2}{3}, & \text{if } z^4 w \notin F \text{ and } z^5 t \in F. \end{cases}$$

Suppose $z^4 w, z^5 t \notin F$. Then we can write

$$F = z^7 x + z^6 f_3 + z^5 f_5 + z^4 f_7 + z^3 f_9 + z^2 f_{11} + z f_{13} + f_{15},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_5$ and $w \notin f_7$.

We claim $w^2 y \in F$. Assume to the contrary $w^2 y \notin F$. Then we can write $F = (z^7 + g)x + h$, where $g \in \mathbb{C}[x,y,z,t,w]$ and $h \in \mathbb{C}[y,z,t,w]$ are quasi-homogeneous polynomials such that $h \in (y,t)^2$. But

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then X is not quasi-smooth along the nonempty subset

$$(x = y = t = z^7 + g = 0) \subset X.$$

This is a contradiction, and the claim is proved.

We see that $\bar{F} := F(0, y, 1, t, w) \in (y, t, w)^3$, $w^2 y \in \bar{F}$ and $w^3 \notin \bar{F}$. In particular, $\bar{F}$ cannot be a cube of a linear form and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$ by Lemma 3.28.

5.1.d.4. The family $\mathcal{F}_{27}$, the point of type $\frac{1}{2}(1, 1, 1)$

Let $X = X_{15} \subset \mathbb{P}(1, 2, 3, 5, 5)$ be a member of $\mathcal{F}_{27}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{2}(1, 1, 1)$. If either $y^5 w \in F$ or $y^5 t \in F$, then we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{2 \cdot 3 \cdot 5 \cdot (A^3)} = \frac{2}{3}$$

by Lemma 3.29.

Suppose $y^5 w, y^5 t \notin F$ and $y^6 z \in F$. Then we can write

$$F = y^6 z + y^5 f_5 + y^4 f_7 + y^3 f_9 + y^2 f_{11} + y f_{13} + f_{15},$$

where $f_i = f_i(x, z, t, w)$ is a quasi-homogeneous polynomial of degree i with $w, t \notin f_5$. We see that $\operatorname{omult}_{\mathsf{p}}(H_z) = 3$ and thus $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{3} H_z) \geq 1$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{3} H_z$. Then we can take a $\mathbb{Q}$-divisor $T \in |5A|_{\mathbb{Q}}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of $D \cdot H_x$ since $\{x, z, t, w\}$ isolates p. Since $\operatorname{omult}_{\mathsf{p}}(H_z) = 3$, we have

$$3 \operatorname{omult}_{\mathsf{p}}(D) \leq 3(q^* D \cdot q^* H_z \cdot q^* T)_{\check{\mathsf{p}}} \leq 2(D \cdot H_z \cdot T) = 3,$$

where $q = q_{\mathsf{p}}$ is the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ is the preimage of p via q. This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 1$ and thus $\alpha_{\mathsf{p}}(X) \geq 1$.

Suppose $y^5 w, y^5 t, y^6 z \notin F$. Then we can write

$$F = y^7 x + y^6 f_3 + y^5 f_5 + y^4 f_7 + y^3 f_9 + y^2 f_{11} + y f_{13} + f_{15},$$

where $f_i = f_i(x, z, t, w)$ is a quasi-homogeneous polynomial of degree i with $z \notin f_3$ and $w, t \notin f_5$. We see that $\bar{F} = F(0, 1, z, t, w) \in (z, t, w)^3$ and the cubic part of $\bar{F}$ cannot be cube of a linear form since $F(0, 1, 0, t, w) = F(0, 0, 0, t, w)$ is a product of three linearly independent linear forms in t, w by the quasi-smoothness of X. By Lemma 3.28, we have $\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$.

5.1.d.5. The family $\mathcal{F}_{32}$, the point of type $\frac{1}{3}(1, 1, 2)$

Let $X = X_{16} \subset \mathbb{P}(1, 2, 3, 4, 7)$ be a member of $\mathcal{F}_{32}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{3}(1, 1, 2)$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{3 \cdot 2 \cdot 4 \cdot (A^3)} = \frac{7}{8}, & \text{if } z^3 w \in F, \\ \frac{2}{3 \cdot 2 \cdot 7 \cdot (A^3)} = \frac{1}{2}, & \text{if } z^3 w \notin F \text{ and } z^4 t \in F. \end{cases}$$

Suppose $z^3 w, z^4 t \notin F$. Then $z^5 x \in F$ by the quasi-smoothness of X at p and we can write

$$F = z^5 x + z^4 f_4 + z^3 f_7 + z^2 f_{10} + z f_{13} + f_{16},$$

where $f_i = f_i(x, y, t, w)$ is a quasi-homogeneous polynomial of degree i. We have $w^2 y \in F$ by the quasi-smoothness of X at p_w. It follows that either $\bar{F} = F(0, y, 1, t, w) \in (y, t, w)^2 \setminus (y, t, w)^3$ or $\bar{F} \in (y, t, w)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^3 \notin \bar{F}$. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

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5.1.d.6. The family $\mathcal{F}_{33}$, the point of type $\frac{1}{2}(1,1,1)$

Let $X = X_{17} \subset \mathbb{P}(1,2,3,5,7)$ be a member of $\mathcal{F}_{33}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{2}(1,1,1)$.

Suppose that at least one of $y^5 w$, $y^6 t$ and $y^7 z$ appear in F with nonzero coefficient. In this case, H_x is quasi-smooth at p and we have $\mathrm{lct}_{\mathsf{p}}(X; H_x) = 1$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_x. We can take a $\mathbb{Q}$-divisor $T \in |7A|_{\mathbb{Q}}$ such that $\mathrm{omult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of of the effective 1-cycle $D \cdot H_x$ since the set $\{x, z, t, w\}$ isolates p. It follows that

$$\mathrm{omult}_{\mathsf{p}}(D) \leq (q_{\mathsf{p}}^* D \cdot q_{\mathsf{p}}^* H_x \cdot q_{\mathsf{p}}^* T)_{\check{\mathsf{p}}} \leq 2(D \cdot H_x \cdot T) = \frac{17}{15}.$$

Thus, $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 15/17$ and we have $\alpha_{\mathsf{p}}(X) \geq 15/17$.

Suppose $y^5 w, y^6 t, y^7 z \notin F$. Then $y^8 x \in F$ by the quasi-smoothness of X at p and we can write

$$F = y^8 x + y^7 f_3 + y^6 f_5 + y^5 f_7 + y^4 f_9 + y^3 f_{11} + y^2 f_{13} + y f_{15} + f_{17},$$

where $f_i = f_i(x,z,t,w)$ is a quasi-homogeneous polynomial of degree i. We see that $\bar{F} := F(0,1,z,t,w) \in (z,t,w)^3$, $w^3 \notin \bar{F}$ and $w^2 z \in \bar{F}$. It follows that $\bar{F} \in (z,t,w)^3$, and it cannot be a cube of a linear form. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

5.1.d.7. The family $\mathcal{F}_{40}$, the point of type $\frac{1}{3}(1,1,2)$

Let $X = X_{19} \subset \mathbb{P}(1,3,4,5,7)$ be a member of $\mathcal{F}_{40}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{3}(1,1,2)$.

Suppose that either $y^4 w \in F$ or $y^5 z \in F$. In this case, $\mathrm{lct}_{\mathsf{p}}(X; H_x) = 1$ since H_x is quasi-smooth at p. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_x. We can take a $\mathbb{Q}$-divisor $T \in |7A|_{\mathbb{Q}}$ such that $\mathrm{omult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of of the effective 1-cycle $D \cdot H_x$ since the set $\{x, z, t, w\}$ isolates p. It follows that

$$\mathrm{omult}_{\mathsf{p}}(D) \leq (q_{\mathsf{p}}^* D \cdot q_{\mathsf{p}}^* H_x \cdot q_{\mathsf{p}}^* T)_{\check{\mathsf{p}}} \leq 3(D \cdot H_x \cdot T) = \frac{19}{20}.$$

Thus, $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 20/19$ and we have $\alpha_{\mathsf{p}}(X) \geq 1$.

Suppose $y^4 w, y^5 z \notin F$. Then $y^6 x \in F$ by the quasi-smoothness of X at p and we can write

$$F = y^6 x + y^5 f_4 + y^4 f_7 + y^3 f_{10} + y^2 f_{13} + y f_{16} + f_{19},$$

where $f_i = f_i(x,z,t,w)$ is a quasi-homogeneous polynomial of degree i. We set $\bar{F} := F(0,1,z,t,w) \in (z,t,w)^3$. It is easy to see that $w^3 \notin \bar{F}$ and $w^2 z \in \bar{F}$. It follows that either $\bar{F} \in (z,t,w)^2$ or $\bar{F} \in (z,t,w)^3$, and it cannot be a cube of a linear form. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

5.1.d.8. The family $\mathcal{F}_{47}$, the point of type $\frac{1}{5}(1,2,3)$

Let $X = X_{21} \subset \mathbb{P}(1,1,5,7,8)$ be a member of $\mathcal{F}_{47}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{5}(1,2,3)$. We can write

$$F = z^4 x + z^3 f_6 + z^2 f_{11} + z f_{16} + f_{21},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i. By the quasi-smoothness of X at p_w, we have $w^2 \in f_{16}$, which implies $\bar{F} = F(0,y,1,t,w) \in (y,t,w)^2 \setminus (y,t,w)^3$. Thus, by Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

5.1.d.9. The family $\mathcal{F}_{48}$, the point of type $\frac{1}{2}(1,1,1)$

Let $X = X_{21} \subset \mathbb{P}(1,2,3,7,9)$ be a member of $\mathcal{F}_{48}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{2}(1,1,1)$. By Lemma 3.29, we have

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$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{2\cdot 3\cdot 7\cdot (A^3)} = \frac{6}{7}, & \text{if } y^6 w \in F, \\ \frac{2}{2\cdot 3\cdot 9\cdot (A^3)} = \frac{2}{3}, & \text{if } y^6 w \notin F \text{ and } y^7 t \in F. \end{cases}$$

Suppose $y^6 w, y^7 t \notin F$ and $y^9 z \in F$. We can write

$$F = y^9 z + y^8 f_5 + y^7 f_7 + y^6 f_9 + \cdots + f_{21},$$

where $f_i = f_i(x, z, t, w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_7$ and $w \notin f_9$. We have $\operatorname{omult}_{\mathsf{p}}(H_x) = 1$ and thus $\operatorname{lct}_{\mathsf{p}}(X; H_x) = 1$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. We can take a $\mathbb{Q}$-divisor $T \in |9A|_{\mathbb{Q}}$ with such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of $D \cdot H_x$ since $\{x, z, t, w\}$ isolates p. Then we have

$$\operatorname{omult}_{\mathsf{p}}(D) \leq (q^* D \cdot q^* H_x \cdot q^* T)_{\check{\mathsf{p}}} \leq 2(D \cdot H_x \cdot T) = 2 \cdot 1 \cdot 9 \cdot (A^3) = 1,$$

where $q = q_{\mathsf{p}}$ is the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ is the preimage of p via q. This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 1$ and thus $\alpha_{\mathsf{p}}(X) \geq 1$.

Suppose $y^6 w, y^7 t, y^9 z \notin F$. Then $y^{10} x \in F$ and we can write

$$y^{10} x + y^9 f_3 + y^8 f_5 + y^7 f_7 + y^6 f_9 + \cdots + f_{21},$$

where $f_i = f_i(x, z, t, w)$ is a quasi-homogeneous polynomial of degree i with $z \notin f_3$, $t \notin f_7$ and $w \notin f_9$. We see that $\bar{F} := F(0, 1, z, t, w) \in (z, t, w)^3$, and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^2 z \in \bar{F}$ and $w^3 \notin \bar{F}$. Thus, by Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

5.1.d.10. The family $\mathcal{F}_{49}$, the point of type $\frac{1}{5}(1,2,3)$

Let $X = X_{21} \subset \mathbb{P}(1, 3, 5, 6, 7)$ be a member of $\mathcal{F}_{49}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{5}(1, 2, 3)$. If $z^3 t \in F$, then

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{5 \cdot 3 \cdot 7 \cdot (A^3)} = \frac{4}{7}$$

by Lemma 3.29.

Suppose $z^3 t \notin F$. Then $z^4 x \in F$ and we can write

$$F = z^4 x + z^3 f_6 + z^2 f_{11} + z f_{16} + f_{21},$$

where $f_i = f_i(x, y, t, w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_6$. We have $w^3, t^3 y \in F$, and we may assume $\operatorname{coeff}_F(w^3) = \operatorname{coeff}_F(t^3 y) = 1$.

We claim $\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. If $y^2 \in f_6$, then $\operatorname{omult}_{\mathsf{p}}(H_x) = 2$ and thus $\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. We assume $y^2 \notin f_6$. Then we can write

$$F(0, y, z, t, w) = z(\alpha w t y + \beta w y^3) + w^3 + y(t^3 + \gamma t^2 y^2 + \delta t y^4 + \varepsilon y^6),$$

where $\alpha, \beta, \gamma, \delta, \varepsilon \in \mathbb{C}$. We set $\bar{F} := F(0, y, 1, t, w) \in \mathbb{C}[y, t, w]$.

- Suppose $\alpha \neq 0$. Then $\bar{F} \in (y, t, w)^3$ and its cubic part is $\alpha w t y + w^3$. By Lemma 3.28, we have $\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$ in this case.
- Suppose $\alpha = 0$ and $\beta \neq 0$. Then the lowest weight part of $\bar{F}$ with respect to $\operatorname{wt}(y, t, w) = (6, 7, 9)$ is $\beta w y^3 + w^3 + t^3 y$. By Lemma 3.27, we have

$$\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq \min\left\{\frac{22}{27}, \operatorname{lct}(\tilde{\mathbb{P}}, \operatorname{Diff}; \mathcal{D})\right\},$$

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where
- $\tilde{\mathbb{P}} = \mathbb{P}(2,7,9)_{\tilde{y},\tilde{t},\tilde{w}} = \mathbb{P}(6,7,9)^{\mathrm{wf}}$,
- $\mathrm{Diff} = \frac{2}{3}H_{\tilde{t}}$ with $H_{\tilde{t}} = (\tilde{t} = 0) \subset \tilde{\mathbb{P}}$, and
- $\mathcal{D}$ is the prime divisor $(\beta\tilde{w}\tilde{y}^3 + \tilde{w}^3 + \tilde{t}\tilde{y} = 0)$ on $\tilde{\mathbb{P}}$.

We see that $\mathcal{D}$ is quasi-smooth and it intersects $H_{\tilde{t}}$ transversally. It follows that $\mathrm{lct}(\tilde{\mathbb{P}}, \mathrm{Diff}; \mathcal{D}) = 1$ and thus $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 22/27$.

- Suppose $\alpha = \beta = 0$. Then the lowest weight part of $\bar{F}$ with respect to $\mathrm{wt}(y,z,t) = (3,6,7)$ is $w^3 + t^3y + \gamma t^2y^3 + \delta t y^5 + \varepsilon y^7$. By Lemma 3.27, we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq \min\left\{\frac{16}{21}, \mathrm{lct}(\tilde{\mathbb{P}}, \mathrm{Diff}; \mathcal{D})\right\},$$

where
- $\tilde{\mathbb{P}} = \mathbb{P}(1,2,9)_{\tilde{y},\tilde{t},\tilde{w}} = \mathbb{P}(3,6,7)^{\mathrm{wf}}$,
- $\mathrm{Diff} = \frac{2}{3}H_{\tilde{w}}$ with $H_{\tilde{w}} = (\tilde{w} = 0) \subset \tilde{\mathbb{P}}$, and
- $\mathcal{D}$ is the prime divisor $(\tilde{w} + \tilde{t}^3\tilde{y} + \gamma\tilde{t}^2\tilde{y}^3 + \delta\tilde{t}\tilde{y}^5 + \varepsilon\tilde{y}^7 = 0)$ on $\tilde{\mathbb{P}}$.

We see that $\mathcal{D}$ is quasi-smooth. The solutions of the equation $\tilde{t}^3\tilde{y} + \gamma\tilde{t}^2\tilde{y}^3 + \delta\tilde{t}\tilde{y}^5 + \varepsilon\tilde{y}^7 = 0$ corresponds to the three points of type $\frac{1}{3}(1,1,2)$ on X. In particular, the equation has three distinct solutions. It follows that $\mathcal{D}$ intersects $H_{\tilde{w}}$ transversally, and we have $\mathrm{lct}(\tilde{\mathbb{P}}, \mathrm{Diff}; \mathcal{D}) = 1$. Thus, $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 16/21$.

Thus, the claim is proved.

The point p is not a maximal center, and the pair (X, H_x) is not canonical by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \geq \min\{1, \mathrm{lct}_{\mathsf{p}}(X; H_x)\} \geq \frac{1}{2}$$

by Lemma 3.5.

5.1.d.11. The family $\mathcal{F}_{62}$, the point of $\frac{1}{5}(1,2,3)$

Let $X = X_{26} \subset \mathbb{P}(1,1,5,7,13)$ be a member of $\mathcal{F}_{62}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{5}(1,2,3)$. Replacing x and y, we can write

$$F = z^5x + z^4f_6 + z^3f_{11} + z^2f_{16} + zf_{21} + f_{26},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i. We have $\mathrm{omult}_{\mathsf{p}}(H_x) = 2$ since $w^2 \in F$. Hence, $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. The point p is not a maximal center, and the pair (X, H_x) is not canonical at p by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \geq \min\{1, \mathrm{lct}_{\mathsf{p}}(X; H_x)\} \geq \frac{1}{2}$$

by Lemma 3.5.

5.1.d.12. The family $\mathcal{F}_{65}$, the point of type $\frac{1}{2}(1,1,1)$

Let $X = X_{27} \subset \mathbb{P}(1,2,5,9,11)$ be a member of $\mathcal{F}_{65}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{2}(1,1,1)$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{2\cdot 5\cdot 9\cdot (A^3)} = \frac{22}{27}, & \text{if } y^8w \in F, \\ \frac{2}{2\cdot 5\cdot 11\cdot (A^3)} = \frac{2}{3}, & \text{if } y^8w \notin F \text{ and } y^9t \in F. \end{cases}$$

Suppose $y^8w, y^9t \notin F$ and $y^{11}z \in F$. Then we can write

$$F = y^{11}z + y^{10}f_7 + \cdots + yf_{25} + f_{27},$$

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where $f_i = f_i(x,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_9$ and $w \notin f_{11}$. We have $\operatorname{omult}_{\mathsf{p}}(H_x) = 1$ and thus $\operatorname{lct}_{\mathsf{p}}(X; H_x) = 1$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. We see that $\{x,z,t,w\}$ isolates p, hence we can take a $\mathbb{Q}$-divisor $T \in |11A|_{\mathbb{Q}}$ such that $\operatorname{omult}_{\mathsf{p}}(T) \ge 1$ and $\operatorname{Supp}(T)$ does not contain any component of $D \cdot H_x$. Then we have

$$\operatorname{omult}_{\mathsf{p}}(D) \le (q_{\mathsf{p}}^* D \cdot q_{\mathsf{p}}^* H_x \cdot q_{\mathsf{p}}^* T)_{\check{\mathsf{p}}} \le 2(D \cdot H_x \cdot T) = 2 \cdot 1 \cdot 11 \cdot (A^3) = \frac{3}{5},$$

where q_{p} is the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ is the preimage of p via q_{p}. This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \ge 1$ and thus $\alpha_{\mathsf{p}}(X) \ge 1$.

Suppose that $y^8 w, y^9 t, y^{11} z \notin F$. Then $y^{13} x \in F$ and we can write

$$F = y^{13} x + y^{12} f_3 + \cdots + y f_{25} + f_{27},$$

where $f_i = f_i(x,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $z \notin f_5$, $t \notin f_9$ and $w \notin f_{11}$. We see that $\bar{F} := F(0,1,z,t,w) \in (z,t,w)^3$, and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^2 z \in \bar{F}$ and $w^3 \notin \bar{F}$. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \ge 1/2$.

5.1.d.13. The family $\mathcal{F}_{67}$, the point of $\frac{1}{9}(1,4,5)$

Let $X = X_{28} \subset \mathbb{P}(1,1,4,9,14)$ be a member of $\mathcal{F}_{67}$ and $\mathsf{p} = \mathsf{p}_t$ the singular point of type $\frac{1}{9}(1,4,5)$. Replacing x and y, we can write

$$F = t^3 x + t^2 f_{10} + t f_{19} + f_{28},$$

where $f_i = f_i(x,y,z,w)$ is a quasi-homogeneous polynomial of degree i. We have $\operatorname{omult}_{\mathsf{p}}(H_x) = 2$ since $w^2 \in F$. Hence, $\operatorname{lct}_{\mathsf{p}}(X; H_x) \ge 1/2$ by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \ge \min\{1, \operatorname{lct}_{\mathsf{p}}(X; H_x)\} \ge \frac{1}{2}$$

by Lemma 3.5.

5.1.d.14. The family $\mathcal{F}_{82}$, the point of $\frac{1}{5}(1,2,3)$

Let $X = X_{36} \subset \mathbb{P}(1,1,5,12,18)$ be a member of $\mathcal{F}_{82}$ and $\mathsf{p} = \mathsf{p}_z$ the singular point of type $\frac{1}{5}(1,2,3)$. Replacing x and y, we can write

$$F = z^7 x + z^6 f_6 + z^5 f_{11} + \cdots + f_{36},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i. We have $\operatorname{omult}_{\mathsf{p}}(H_x) = 2$ since $w^2 \in F$. Hence, $\operatorname{lct}_{\mathsf{p}}(X; H_x) \ge 1/2$ by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \ge \min\{1, \operatorname{lct}_{\mathsf{p}}(X; H_x)\} \ge \frac{1}{2}$$

by Lemma 3.5.

5.1.d.15. The family $\mathcal{F}_{84}$, the point of type $\frac{1}{7}(1,2,5)$

Let $X = X_{36} \subset \mathbb{P}(1,7,8,9,12)$ be a member of $\mathcal{F}_{84}$ and $\mathsf{p} = \mathsf{p}_y$ the singular point of type $\frac{1}{7}(1,2,5)$. By the quasi-smoothness of X, either $y^4 z \in F$ or $y^5 x \in F$. Moreover, we have $w^3, t^4, z^3 w \in F$ and we assume $\operatorname{coeff}_F(w^3) = \operatorname{coeff}_F(t^4) = \operatorname{coeff}_F(z^3 w) = 1$ by rescaling z, t, w.

Suppose $y^4 z \in F$. Let $\rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}}$ the orbifold chart of X containing p. Then we have $\rho_{\mathsf{p}}^* H_x \cdot \rho_{\mathsf{p}}^* H_z = \check{\Gamma}$, where

$$\check{\Gamma} = (\check{x} = \check{z} = \check{w}^3 + \check{t}^4 = 0) \subset \check{U}_{\mathsf{p}}$$

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is an irreducible and reduced curve with $\mathrm{mult}_{\check{\mathsf{p}}}(\check{\Gamma}) = 3$. We see that (X, H_x) is log canonical at p since H_x is quasi-smooth at p. Thus, by Lemma 3.17, we have $\alpha_{\mathsf{p}}(X) \geq 1$.

Suppose $y^4 z \notin F$. Then $y^5 x \in F$ and we can write

$$F = y^5 x + y^4 f_8 + y^3 f_{15} + y^2 f_{22} + y f_{29} + f_{36},$$

where $f_i = f_i(x, z, t, w)$ is a quasi-homogeneous polynomial of degree i with $z \notin f_8$. By setting $\alpha := \mathrm{coeff}_F(ywtz)$, we have

$$\bar{F} := F(0, 1, z, t, w) = \alpha wtz + w^3 + wz^3 + t^4.$$

- If $\alpha \neq 0$, then $\bar{F} \in (z, t, w)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form. Hence, $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$ by Lemma 3.28.
- If $\alpha = 0$, then the lowest weight part of $\bar{F}$ with respect to $\mathrm{wt}(z, t, w) = (8, 9, 12)$ is $\bar{F} = w^3 + wz^3 + t^4$. By Lemma 3.27, we have

  $$\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq \min\left\{\frac{29}{36}, \mathrm{lct}(\tilde{\mathbb{P}}, \mathrm{Diff}; \mathcal{D})\right\},$$

  where
  - $\tilde{\mathbb{P}} = \mathbb{P}(2, 3, 1)_{\tilde{z}, \tilde{t}, \tilde{w}} = \mathbb{P}(8, 9, 12)^{\mathrm{wf}}$,
  - $\mathrm{Diff} = \frac{2}{3} H_{\tilde{z}} + \frac{3}{4} H_{\tilde{t}}$ with $H_{\tilde{z}} = (\tilde{z} = 0) \subset \tilde{\mathbb{P}}$, $H_{\tilde{t}} = (\tilde{t} = 0) \subset \tilde{\mathbb{P}}$, and
  - $\mathcal{D}$ is the prime divisor $(\tilde{w}^3 + \tilde{w}\tilde{z} + \tilde{t} = 0)$ on $\tilde{\mathbb{P}}$.

  We see that $\mathcal{D}$ is quasi-smooth, $\mathcal{D} \cap H_{\tilde{z}} \cap H_{\tilde{t}} = \emptyset$ and any two of $\mathcal{D}, H_{\tilde{z}}, H_{\tilde{t}}$ intersect transversally. It follows that $\mathrm{lct}(\tilde{\mathbb{P}}, \mathrm{Diff}; \mathcal{D}) = 1$ and thus $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 29/36$.

Note that $\mathsf{p} \in X$ is not a maximal center and the pair (X, H_x) is not canonical at p by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \geq \min\{1, \mathrm{lct}_{\mathsf{p}}(X; H_x)\} \geq \frac{1}{2}$$

by Lemma 3.5.

5.2. *EI centers*

Proposition 5.6. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$ and $\mathsf{p} \in X$ an EI center. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}.$$

Proof. We have $\alpha_{\mathsf{p}}(X) \geq 1$ by Proposition 5.3 for a member X of $\mathcal{F}_{\mathsf{i}}$ and $\mathsf{p} \in X$, where

- $\mathsf{i} = 36$ and p is of type $\frac{1}{4}(1, 1, 3)$.
- $\mathsf{i} = 44$ and p is of type $\frac{1}{6}(1, 1, 5)$.
- $\mathsf{i} = 61$ and p is of type $\frac{1}{7}(1, 2, 5)$.
- $\mathsf{i} = 76$ and p is of type $\frac{1}{8}(1, 3, 5)$.

We have $\alpha_{\mathsf{p}}(X) \geq 1/2$ by Proposition 5.4 for a member X of $\mathcal{F}_{\mathsf{i}}$ and $\mathsf{p} \in X$, where

- $\mathsf{i} = 23$ and p is of type $\frac{1}{4}(1, 1, 3)$.
- $\mathsf{i} = 40$ and p is of type $\frac{1}{5}(1, 2, 3)$.

It remains to consider members of families $\mathcal{F}_7$ and $\mathcal{F}_{20}$, and singular points of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 1, 2)$, respectively.

Let $X = X_8 \subset \mathbb{P}(1,1,2,2,3)$ be a member of $\mathcal{F}_7$ and p a singular point of type $\frac{1}{2}(1,1,1)$. Replacing homogeneous coordinates, we may assume $\mathsf{p} = \mathsf{p}_t$ and we can write

$$F = t^3 z + t^2 f_4 + t f_6 + f_8,$$

where $f_i = f_i(x,y,z,w)$ is a quasi-homogeneous polynomial of degree i. Hence, by Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{2 \cdot 1 \cdot 3 \cdot (A^3)} = \frac{1}{2}.$$

Let $X = X_{13} \subset \mathbb{P}(1,1,3,4,5)$ be a member of $\mathcal{F}_{20}$ and $\mathsf{p} = \mathsf{p}_z$ be the singular point of type $\frac{1}{3}(1,1,2)$. If $z^3 t \in F$, then we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{3 \cdot 1 \cdot 5 \cdot (A^3)} = \frac{8}{13}$$

by Lemma 3.29. Suppose $z^3 t \notin F$. Then we can write that

$$F = z^4 x + z^3 f_4 + z^2 f_7 + z f_{10} + f_{13},$$

where $f_i = f_i(x,y,t,w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_4$. We have $\operatorname{omult}_{\mathsf{p}}(H_x) = 2$ since $w^2 z \in F$ by the quasi-smoothness of X at p_w. This shows $\operatorname{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. The point p is not a maximal singularity, and the pair (X, H_x) is not canonical at p by Lemma 3.6. Thus,

$$\alpha_{\mathsf{p}}(X) \geq \min\{1, \operatorname{lct}_{\mathsf{p}}(X; H_x)\} \geq \frac{1}{2}$$

by Lemma 3.5. This completes the proof. □

5.3. *Equations for QI centers*

Let

$$X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)_{x_0, \dots, x_4} =: \mathbb{P}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$. We set $a_0 = 1$ and let $F = F(x_0, \dots, x_4)$ be the defining polynomial of X.

Definition 5.7. Let $\mathsf{p} \in X$ be a QI center, and let j, k be such that $j \neq k$, $d = 2a_k + a_j$ and the index of $\mathsf{p} \in X$ coincides with a_k. Then we can choose coordinates so that $\mathsf{p} = \mathsf{p}_{x_k}$. We say that p is *an exceptional QI center* if $x_k^2 x_l \notin F$ for any $l \in \{0, \dots, 4\}$.

Lemma 5.8. *Let $\mathsf{p} \in X$ be a nonexceptional QI center. Then we can choose homogeneous coordinates $x_{i_1}, x_{i_2}, x_{i_3}, x_j, x_k$ of $\mathbb{P}$, where $\{i_1, i_2, i_3, j, k\} = \{0,1,2,3,4\}$, such that $a_{i_1}, a_{i_2}, a_{i_3} < a_k$, $\mathsf{p} = \mathsf{p}_{x_k}$ and*

$$F = x_k^2 x_j + x_k f(x_{i_1}, x_{i_2}, x_{i_3}, x_j) + g(x_{i_1}, x_{i_2}, x_{i_3}, x_j) \tag{5.1}$$

for some quasi-homogeneous polynomials $f, g \in \mathbb{C}[x_{i_1}, x_{i_2}, x_{i_3}, x_j]$ of degree $d - a_k$, d, respectively.

Proof. Basically, this follows by looking at Table 7. See also [CPR00, Theorem 4.9]. □

Let $\mathsf{p} \in X$ be a nonexceptional QI center, and we choose and fix homogeneous coordinates $x_{i_1}, x_{i_2}, x_{i_3}, x_j, x_k$ of $\mathbb{P}$ as in Lemma 5.8.

Definition 5.9. We say that p is a *degenerate QI center* if $f(x_{i_1}, x_{i_2}, x_{i_3}, 0) = 0$ as a polynomial, otherwise we call p a *nondegenerate QI center*.

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Remark 5.10. It is proved in [CP17, Section 4.1] that a QI center $\mathsf{p} \in X$ is a maximal center if and only if it is nondegenerate.

Lemma 5.11. *Let* p *be a degenerate QI center. Then we can choose homogeneous coordinates* $x_{i_1}, x_{i_2}, x_{i_3}, x_j, x_k$ *of* $\mathbb{P}$ *such that* $a_{i_1}, a_{i_2}, a_{i_3} < a_k$, $\mathsf{p} = \mathsf{p}_{x_k}$ *and*

$$F = x_k^2 x_j + g(x_{i_1}, x_{i_2}, x_{i_3}, x_j) \tag{5.2}$$

for some quasi-homogeneous polynomial $g \in \mathbb{C}[x_{i_1}, x_{i_2}, x_{i_3}, x_j]$ *of degree d. Moreover, the hypersurface*

$$(g(x_{i_1}, x_{i_2}, x_{i_3}, 0) = 0) \subset \mathbb{P}(a_{i_1}, a_{i_2}, a_{i_3})_{x_{i_1}, x_{i_2}, x_{i_3}}$$

is quasi-smooth.

Proof. We have $f = x_j f'$ for some $f' \in \mathbb{C}[x_{i_1}, x_{i_2}, x_{i_3}, x_j]$ since p is degenerate. Filtering off terms divisible by x_j in equation (5.1), we have

$$F = x_j(x_k^2 + x_k f') + g.$$

We can eliminate the term $x_k x_j f'$ by replacing $x_k \mapsto x_k - f'/2$. This shows the first assertion.

We choose and fix homogeneous coordinates so that F is of the form (5.2). We set $\bar{g} = g(x_{i_1}, x_{i_2}, x_{i_3}, 0)$. Then we can write $g = \bar{g} + x_j h$, where $h = h(x_{i_1}, x_{i_2}, x_{i_3}, x_j)$. Suppose to the contrary that $(\bar{g} = 0) \subset \mathbb{P}(a_{i_1}, a_{i_2}, a_{i_3})$ is not quasi-smooth at a point $(\alpha_1 : \alpha_2 : \alpha_3)$. We choose and fix $\beta \in \mathbb{C}$ such that $\beta^2 + h(\alpha_1, \alpha_2, \alpha_3, 0) = 0$ and set

$$\mathsf{q} := (\alpha_1 : \alpha_2 : \alpha_3 : 0 : \beta) \in \mathbb{P}(a_{i_1}, a_{i_2}, a_{i_3}, a_j, a_k) = \mathbb{P}.$$

It is easy to see that $(\partial F/\partial v)(\mathsf{q}) = 0$ for any $v \in \{x_{i_1}, x_{i_2}, x_{i_3}, x_j, x_k\}$. This is impossible since X is quasi-smooth. Therefore, $(\bar{g} = 0) \subset \mathbb{P}(a_{i_1}, a_{i_2}, a_{i_3})$ is quasi-smooth. □

Lemma 5.12. *Let X be a member of a family* $\mathcal{F}_{\mathsf{i}}$ *with* $\mathsf{i} \in \mathsf{I} \setminus \{2, 8\}$. *Suppose that X has a QI center. Then one of the following holds.*

1. *X has a unique QI center. In this case, by a choice of homogeneous coordinates, we have*

$$X = X_{2r+c} \subset \mathbb{P}(1, a, b, c, r)_{x,s,u,v,w},$$

where a is coprime to b, $a < b$, $a + b = r$, $c < r$, *and the unique QI center is the point* $\mathsf{p} = \mathsf{p}_w$, *which is of type* $\frac{1}{r}(1, a, b)$.

2. *X has exactly three distinct QI centers. In this case, by a choice of homogeneous coordinates, we have*

$$X = X_{3r} \subset \mathbb{P}(1, a, b, r, r)_{x,y,z,t,w},$$

where a is coprime to b, $a \le b$ *and* $a + b = r$. *The three QI centers are the three points in* $(x = y = z = 0) \cap X$, *and they are all of type* $\frac{1}{r}(1, a, b)$.

3. *X has exactly two distinct QI centers, and their singularity types are equal. In this case, by a choice of homogeneous coordinates, we have*

$$X = X_{4r} \subset \mathbb{P}(1, a, b, r, 2r)_{x,y,z,t,w},$$

where a is coprime to b, $a \le b$ *and* $a+b = r$. *The QI centers are the two points in* $(x = y = z = 0) \cap X$, *and they are both of type* $\frac{1}{r}(1, a, b)$.

4. *X has exactly two distinct QI centers, and their singularity types are distinct. In this case, by a choice of homogeneous coordinates, we have*

$$X = X_{4a+3b} \subset \mathbb{P}(1, a, b, r_1, r_2)_{x,u,v,t,w},$$

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where a is coprime to b, $a+b=r_1$ and $2a+b=r_2$. The QI centers are p_t and p_w which are of types $\frac{1}{r_1}(1,a,b)$ and $\frac{1}{r_2}(1,a,a+b)$, respectively.

Proof. Let

$$X = X_d \subset \mathbb{P}(1,a_1,a_2,a_3,a_4)_{x_0,x_1,x_2,x_3,x_4}$$

be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \{2,8\}$. We set $a_0 = 1$ and assume $a_1 \le \cdots \le a_4$. We assume that X has at least one QI center.

Suppose $d = 3a_4$. Let $\mathsf{p} \in X$ be a QI center. Then, after replacing homogeneous coordinates, we may assume $\mathsf{p} = \mathsf{p}_{x_i}$ and $x_i^2 x_j \in F$ for some $i \in \{0,1,2,3,4\}$ and $j \in \{0,1,2,3,4\} \setminus \{i\}$. In particular, we have $d = 3a_4 = 2a_i + a_j$, which is possible if and only if $a_i = a_j = a_4$. Thus, we have $a_3 = a_4$ and we may assume $i=4$, $j=3$. We see that $\mathsf{p} \in X$ is of type $\frac{1}{a_4}(1,a_1,a_2)$ and $\mathsf{p} \in X$ is terminal. It follows that $a_1 + a_2 = a_4$ and a_l is coprime to a_4 for $l = 1,2$. By setting $a := a_1$, $b := a_2$ and $r := a_3 = a_4$, this case corresponds to (2). In the following, we assume $d < 3a_4$.

Suppose $d = 2a_4$. We have $a_4 = a_1 + a_2 + a_3$ since $d = a_1 + a_2 + a_3 + a_4 = 2a_4$. Let $\mathsf{p} \in X$ be a QI center. Then we may assume $\mathsf{p} = \mathsf{p}_{x_i}$ and $x_i^2 x_j \in F$ for some $i \in \{0,1,2,3\}$ and $j \in \{0,1,2,3,4\} \setminus \{i\}$. In particular, we have $d = 2a_i + a_j$, and hence

$$2a_i + a_j = a_1 + a_2 + a_3 + a_4 = 2(a_1 + a_2 + a_3),$$

which is only possible when $j=4$ and $i=3$. Hence, $i=3$, $j=4$, and we have $a_4 = 2a_3$ since $d = 2a_3 + a_4 = 2a_4$. We see that $\mathsf{p} \in X$ is of type $\frac{1}{a_3}(1,a_1,a_2)$ and $\mathsf{p} \in X$ is terminal. It follows that $a_3 = a_1 + a_2$ and a_l is coprime to a_3 for $l = 1,2$. By setting $a := a_1$, $b := a_2$, $r := a_3$, this case corresponds to (3).

Suppose $d = 2a_4 + a_3$. We have $a_4 = a_1 + a_2$ since $d = a_1 + a_2 + a_3 + a_4$. We see that $\mathsf{p}_4 \in X$ is of type $\frac{1}{a_4}(1,a_1,a_2)$, and it is a QI center. It follows that a_4 is coprime to a_l for $l = 1,2$ since $\mathsf{p}_4 \in X$ is a terminal singularity. If X admits a QI center other than p_4, then we have $d = 2a_i + a_j$, where $i \in \{1,2,3\}$ and $j \in \{0,1,2,3,4\} \setminus \{i\}$ which is impossible. Thus, $\mathsf{p}_4 \in X$ is a unique QI center, and we are in case (1) by setting $a := a_1$, $b := a_2$, $c := a_3$ and $r := a_4$. Note that we have $a < b$ because otherwise we have $a_1 = a_2 = 1$ and $a_4 = 2$ and X belongs to a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{2,8\}$ which is impossible.

Suppose $d = 2a_4 + a_2$. Then $a_4 = a_1 + a_3$. We see that $\mathsf{p}_4 \in X$ is of type $\frac{1}{a_4}(1,a_1,a_3)$, and it is a QI center. If p_4 is a unique QI center, then we are in case (1) by setting $a := a_1$, $b := a_3$, $c := a_2$ and $r := a_4$. We assume that X admits a QI center $\mathsf{p} \in X$ other than p_4. We may assume $\mathsf{p} = \mathsf{p}_i$ after replacing homogeneous coordinates, and we have $d = 2a_i + a_j$ for some $i \in \{1,2,3\}$ and $j \in \{0,1,2,3,4\} \setminus \{i\}$. Then we have $a_j = a_4$ and $a_i = a_3$. Thus, $i = 3$, $j = 4$ and we have $a_3 = a_1 + a_2$. The singularity of $\mathsf{p} = \mathsf{p}_i \in X$ is of type $\frac{1}{a_3}(1,a_1,a_2)$ and it is terminal. It follows that a_1 is coprime to a_2. Thus, we are in case (4) by setting $a := a_1$, $b := a_2$, $r_1 := a_3$ and $r_2 = a_4$.

Suppose $d = 2a_4 + a_1$. Then, by interchanging the role of a_1 and a_2 in the previous arguments, we conclude that this case corresponds to either (1) or (4). This completes the proof. □

Lemma 5.13. *Let*

$$X = X_{2r+c} \subset \mathbb{P}(1,a,b,c,r)_{x,s,u,v,w}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \{2,8\}$ with a unique QI center, where a is coprime to b, $a < b$, $r = a+b$ and $c < r$. Then the following assertions hold.

1. *If $c = 1$, then X belongs to a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{24, 46\}$.*
2. *If $2r + c$ is not divisible by b, then $b = a+1$, $c = a+2$, $r = 2a+1$ and $a \in \{2,3,4\}$.*

Proof. This follows from Table 7. □

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Lemma 5.14. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$ and let $\mathsf{p} \in X$ be an exceptional QI center. Then we are in Case (4) of Lemma 5.12 and $\mathsf{p} = \mathsf{p}_t$. Moreover, we can write*

$$F = t^3 u + t^2 f_{2a+b} + t f_{3a+2b} + f_{4a+3b}, \tag{5.3}$$

where $f_i \in \mathbb{C}[x, u, v, w]$ is a quasi-homogeneous polynomial of degree i with $w \notin f_{2a+b}$.

Proof. We are in (1), (2), (3) and (4) of Lemma 5.12. Suppose that we are in (1). Then $\mathsf{p} = \mathsf{p}_w$. Since $\mathsf{p} \in X$ is exceptional and X is quasi-smooth at p, we have $w^m q \in F$ for some $m \geq 3$ and a homogeneous coordinate $q \in \{x, s, u, v\}$. This implies

$$2r + c = d = mr + \deg q \geq 3r + 1,$$

which is impossible since $c < r$. By similar arguments we can show that (2) and (3) are both impossible.

It follows that we are in Case (4). In this case either $\mathsf{p} = \mathsf{p}_t$ or $\mathsf{p} = \mathsf{p}_w$. The latter is impossible since $d = 4a + 3b < 3r_2$. Hence, $\mathsf{p} = \mathsf{p}_t$. We have $t^m q \in F$ for some integer $m \geq 3$ and a homogeneous coordinate $q \in \{x, u, v, w\}$. It is easy to see that this is possible if and only if $m = 3$ and $\deg q = a$. Possibly replacing coordinates we may assume $q = u$. Then it is straightforward to see that F can be written as equation (5.3). □

5.4. QI centers: exceptional case

The aim of this section is to prove the following.

Proposition 5.15. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \mathsf{I}_1$, and let $\mathsf{p} \in X$ be an exceptional QI center. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}.$$

Let X be a member of $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \{2, 8\}$ which admits an exceptional QI center, Then, by Lemma 5.14 and Table 7, we have

$$\mathsf{i} \in \{12, 13, 20, 25, 31, 33, 38, 58\}.$$

The rest of this section is devoted to the proof of Proposition 5.15 which will be done by division into cases. By Lemma 5.14, we can choose coordinates x, u, v, t, w of $\mathbb{P} = \mathbb{P}(1, a, b, r_1, r_2)$ as in Case (4) of Lemma 5.12 with $\mathsf{p} = \mathsf{p}_t$ and the defining polynomial F is as in equation (5.3).

5.4.a. Case: $a \geq 2$ and $4a \leq 3b$

This case corresponds to families $\mathcal{F}_{33}$ and $\mathcal{F}_{58}$. We have $w^2 v \in f_{4a+3b}$ since X is quasi-smooth at p_w. Moreover, we see that no quadratic monomial in variables x, v, w appear in $f_{2a+b}, f_{3a+2b}, f_{4a+3b}$. This implies $\operatorname{omult}_{\mathsf{p}}(H_u) = 3$, and we have

$$\alpha_{\mathsf{p}}\left(X; \frac{1}{a} H_u\right) \geq \frac{a}{3} \geq \frac{2}{3}.$$

Let $D \in |A|_{\mathbb{Q}}$ be an effective $\mathbb{Q}$-divisor other than $\frac{1}{a} H_u$. We can take a $\mathbb{Q}$-divisor $T \in |r_2 A|_{\mathbb{Q}}$ such that $\operatorname{omult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_u$ since $\{x, u, v, w\}$ isolates p. Let $q = q_{\mathsf{p}}$ be the quotient morphism of $\mathsf{p} \in X$, and let $\check{\mathsf{p}}$ be the preimage of p via q. Then we have

$$3 \operatorname{omult}_{\mathsf{p}}(D) \leq (q^* D \cdot q^* H_u \cdot q^* T)_{\check{\mathsf{p}}} \leq r_1 (D \cdot H_u \cdot T) = \frac{4a + 3b}{b} \leq 6$$

since $4a \leq 3b$. Thus, $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 1/2$ and the inequality $\alpha_{\mathsf{p}}(X) \geq 1/2$ is proved.

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5.4.b. Case: $a = 1$

This case corresponds to families $\mathcal{F}_{12}$, $\mathcal{F}_{20}$ and $\mathcal{F}_{31}$. We have either $\bar{F} = F(x, 0, v, 1, w) \in (x, v, w)^2 \setminus (x, v, w)^3$ or $\bar{F} \in (x, v, w) \in (x, v, w)^3$, and its cubic part is not a cube of a linear form since $w^2 v \in F$ and $w^3 \notin F$. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ since $a = 1$.

5.4.c. Case: X is a member of the family $\mathcal{F}_{13}$

Let

$$X = X_{11} \subset \mathbb{P}(1, 1, 2, 3, 5)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_{13}$ and $\mathsf{p} \in X$ an exceptional QI center. Then we have

$$F = t^3 z + t^2 f_5 + t f_8 + f_{11},$$

where $f_i \in \mathbb{C}[x, y, z, w]$ is a quasi-homogeneous polynomial of degree i with $w \notin f_5$, and $\mathsf{p} = \mathsf{p}_t$. Let $S, T \in |A|$ be general members. We have

$$F(0, 0, z, t, w) = t^3 z + \alpha t z^4 + \beta w z^3 = z(t^3 + \alpha t z^3 + \beta w z^2),$$

where $\alpha, \beta \in \mathbb{C}$. We set $\Gamma = (x = y = z = 0)$, which is a quasi-line of degree $1/15$. If $\beta \neq 0$, then we set

$$\Delta = (x = y = t^3 + \alpha t z^3 + \beta w z^2 = 0),$$

which is clearly an irreducible and reduced curve of degree $3/10$ and does not pass through p. Moreover, we have

$$T|_S = \Gamma + \Delta.$$

Claim 9. *If $\beta \neq 0$, then the intersection matrix $M(\Gamma, \Delta)$ satisfies the condition $(\star)$.*

Proof of Claim 9. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\}$, and it is easy to see that S is quasi-smooth at p_w since $S \in |A|$ is general. By Lemma 3.9, S is quasi-smooth along Γ and we have $\mathrm{Sing}_\Gamma(S) = \{\mathsf{p}_t, \mathsf{p}_w\}$, where $\mathsf{p}_t, \mathsf{p}_w \in S$ are of types $\frac{1}{3}(1, 2)$, $\frac{1}{5}(2, 3)$, respectively. By Remark 3.10, we have

$$(\Gamma^2)_S = -2 + \frac{2}{3} + \frac{4}{5} = -\frac{8}{15}.$$

By taking intersection numbers of $T|_S = \Gamma + \Delta$ and Γ, Δ, we have

$$(\Gamma \cdot \Delta)_S = \frac{3}{5}, \quad (\Delta^2)_S = -\frac{3}{10}.$$

Thus, $M(\Gamma, \Delta)$ satisfies the condition $(\star)$. □

Suppose $\beta = 0$ and $\alpha \neq 0$. We set

$$\Xi = (x = y = t = 0), \quad \Theta = (x = y = t^2 + \alpha z^3 = 0),$$

which are irreducible and reduced curves of degrees $1/10$, $1/5$, respectively, which do not pass through p. We have

$$T|_S = \Gamma + \Xi + \Theta.$$

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Claim 10. *If $\beta = 0$ and $\alpha \neq 0$, then the intersection matrix $M(\Gamma, \Xi, \Theta)$ satisfies the condition* $(\star)$.

Proof of Claim 10. We have $(\Gamma^2)_S = -8/15$ by the proof of Claim 9. We see that $\Xi \cap (\Gamma \cup \Theta) = \{\mathsf{p}_w\}$ and S is quasi-smooth at p_w. By Lemma 3.9, S is quasi-smooth along Ξ and we have $\operatorname{Sing}_{\Xi}(S) = \{\mathsf{p}_z, \mathsf{p}_w\}$, where $\mathsf{p}_z, \mathsf{p}_w \in S$ are of types $\frac{1}{2}(1,1)$, $\frac{1}{5}(2,3)$, respectively. By Remark 3.10, we have

$$(\Xi^2)_S = -2 + \frac{1}{2} + \frac{4}{5} = -\frac{7}{10}.$$

We compute the intersection number $(\Gamma \cdot \Xi)_S$. We have $\Gamma \cap \Xi = \{\mathsf{p}_w\}$, and the germ $\mathsf{p}_w \in S$ is analytically isomorphic to $\bar{o} \in \mathbb{A}^2_{z,t}/\boldsymbol{\mu}_5$, where the $\boldsymbol{\mu}_5$-action on $\mathbb{A}^2_{z,t}$ is given by

$$(z, t) \mapsto (\zeta^2 z, \zeta^3 t),$$

and $\bar{o}$ is the image of the origin $o \in \mathbb{A}^2_{z,t}$. Under the isomorphism, Γ and Ξ corresponds to the quotient of $(z = 0)$ and $(t = 0)$. It follows that

$$(\Gamma \cdot \Xi)_S = (\Gamma \cdot \Xi)_{\mathsf{p}_w} = \frac{1}{5}.$$

Then, by taking intersection numbers of $T|_S = \Gamma + \Xi + \Theta$ and Γ, Ξ, Θ, we have

$$(\Gamma \cdot \Theta)_S = \frac{2}{5}, \quad (\Xi \cdot \Theta)_S = \frac{3}{5}, \quad (\Theta^2)_S = -\frac{4}{5}.$$

Thus, $M(\Gamma, \Xi, \Theta)$ satisfies the condition $(\star)$. □

Suppose $\beta = \alpha = 0$. Then

$$T|_S = \Gamma + 3\Xi,$$

where $\Xi = (x = y = t = 0)$.

Claim 11. *If $\beta = \alpha = 0$, then the intersection matrix $M(\Gamma, \Xi)$ satisfies the condition* $(\star)$.

Proof of Claim 11. We have $(\Gamma^2)_S = -8/15$ by the proof of Claim 9. By taking intersection numbers of $T|_S = \Gamma + 3\Xi$ and Γ, Ξ, we have

$$(\Gamma \cdot \Xi)_S = \frac{1}{5}, \quad (\Xi^2)_S = -\frac{1}{30}.$$

Thus, $M(\Gamma, \Xi)$ satisfies the condition $(\star)$. □

By Claims 9, 10, 11 and Lemma 3.21, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{1}{3(A^3) + 1 - 3\deg\Gamma}\right\} = \frac{10}{19}.$$

5.4.d. Case: X is a member of the family $\mathcal{F}_{25}$

Let

$$X = X_{15} \subset \mathbb{P}(1,1,3,4,7)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_{25}$, and let p be an exceptional QI center. Then we have

$$F = t^3 z + t^2 f_7 + t f_{11} + f_{15},$$

where $f_i \in \mathbb{C}[x, y, z, w]$ is a quasi-homogeneous polynomial of degree i with $w \notin f_7$, and we have $\mathsf{p} = \mathsf{p}_t$. By the quasi-smoothness we have $z^5 \in f_{15}$ and we may assume $\operatorname{coeff}_{f_{15}}(z^5) = 1$. Then we have

$$F(0, 0, z, t, w) = t^3 z + z^5 = z(t^3 + z^4).$$

Let $S, T \in |A|$ be general members. Then we have

$$T|_S = \Gamma + \Delta,$$

where $\Gamma = (x = y = z = 0)$ is a quasi-line of degree $1/28$ and $\Delta = (x = y = t^3 + z^4)$ is an irreducible and reduced curve of degree $1/7$ that does not pass through p.

Claim 12. *The intersection matrix $M(\Gamma, \Delta)$ satisfies the condition* $(\star)$.

Proof of Claim 12. We have $\Gamma \cap \Delta = \{\mathsf{p}_w\}$ and S is quasi-smooth at p_w. Hence, S is quasi-smooth along Γ by Lemma 3.9, and we have $\operatorname{Sing}_\Gamma(S) = \{\mathsf{p}_t, \mathsf{p}_w\}$, where $\mathsf{p}_t, \mathsf{p}_w \in S$ are of types $\frac{1}{4}(1, 3)$, $\frac{1}{7}(3, 4)$, respectively. By Remark 3.10, we have

$$(\Gamma^2)_S = -2 + \frac{3}{4} + \frac{6}{7} = -\frac{11}{28}.$$

By taking intersection numbers of $T|_S = \Gamma + \Delta$ and Γ, Δ, we have

$$(\Gamma \cdot \Delta)_S = \frac{3}{7}, \quad (\Delta^2)_S = -\frac{2}{7}.$$

It follows that $M(\Gamma, \Delta)$ satisfies the condition $(\star)$. $\square$

By Claim 12 and Lemma 3.21, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{1}{4(A^3) + 1 - 4\deg\Gamma}\right\} = \frac{7}{11}.$$

5.4.e. Case: X is a member of the family $\mathcal{F}_{38}$

Let

$$X = X_{18} \subset \mathbb{P}(1, 2, 3, 5, 8)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_{38}$ and $\mathsf{p} \in X$ an exceptional QI center. Then we have

$$F = t^3 z + t^2 f_8 + t f_{13} + f_{18},$$

where $f_i \in \mathbb{C}[x, y, z, w]$ is a quasi-homogeneous polynomial of degree i with $w \notin f_8$, and $\mathsf{p} = \mathsf{p}_t$. By the quasi-smoothness of X, we have $z^6 \in f_{18}$ and we may assume $\operatorname{coeff}_{f_{18}}(z^6) = 1$. Then we have

$$F(0, 0, z, t, w) = t^3 z + z^6 = z(t^3 + z^5).$$

We set $S = H_x$ and $T = H_y$. We have

$$T|_S = \Gamma + \Delta,$$

where $\Gamma = (x = y = z = 0)$ is a quasi-line of degree $1/40$ and $\Delta = (x = y = t^3 + z^5 = 0)$ is an irreducible and reduced curve of degree $1/8$ that does not pass through p.

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Claim 13. *The intersection matrix $M(\Gamma, \Delta)$ satisfies the condition $(\star)$.*

Proof of Claim 13. By similar arguments as in the proof of Claim 12, we have

$$(\Gamma^2)_S = -2 + \frac{4}{5} + \frac{7}{8} = -\frac{13}{40},$$

and

$$(\Gamma \cdot \Delta)_S = \frac{3}{8}, \quad (\Delta^2)_S = -\frac{1}{8}.$$

Thus, $M(\Gamma, \Delta)$ satisfies the condition $(\star)$. □

By Claim 13 and Lemma 3.21, we have

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \frac{1}{5 \cdot 2 \cdot (A^3) + \frac{1}{2} - 5 \deg \Gamma}\right\} = \frac{8}{9}.$$

This completes the proof of Proposition 5.15.

5.5. *QI centers: degenerate case*

The aim of this section is to prove the following, which gives the exact value of $\alpha_{\mathsf{p}}(X)$ for a degenerate QI center $\mathsf{p} \in X$.

Let

$$X = X_d \subset \mathbb{P}(a_0, a_1, \dots, a_4)_{x_0, x_1, x_2, x_3, x_4}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$, where $1 = a_0 \leq a_1 \leq \cdots \leq a_4$, and let $\mathsf{p} \in X$ be a degenerate QI center. We choose homogeneous coordinates as in Lemma 5.11.

Proposition 5.16. *Let the notation as above, and let $\mathsf{p} = \mathsf{p}_{x_k} \in X$ be a degenerate QI center. Then*

$$\alpha_{\mathsf{p}}(X) = \begin{cases} \frac{a_k+1}{2a_k+1}, & \text{if } a_j = 1, \\ 1, & \text{otherwise}. \end{cases}$$

In particular, we have $\alpha_{\mathsf{p}}(X) > \frac{1}{2}$.

Proof. Let $\varphi \colon Y \to X$ be the Kawamata blowup at p with exceptional divisor E. Note that we can choose $x_{i_1}, x_{i_2}, x_{i_3}$ as a system of orbifold coordinates at p and φ is the weighted blowup with weight $\mathrm{wt}(x_{i_1}, x_{i_2}, x_{i_3}) = \frac{1}{a_k}(a_{i_1}, a_{i_2}, a_{i_3})$. Filtering off terms divisible by x_j in equation (5.2), we have

$$x_j(x_k^2 + \cdots) = g(x_{i_1}, x_{i_2}, x_{i_3}, 0) =: \bar{g}.$$

Since the polynomial $x_k^2 + \cdots$ does not vanish at p, the vanishing order of x_j along E coincides with that of $\bar{g}$, which is clearly d/a_k. Hence, we have

$$K_Y + \frac{1}{a_j}\tilde{H}_{x_j} + \frac{2}{a_j}E = \varphi^*\left(K_X + \frac{1}{a_j}H_{x_j}\right), \tag{5.4}$$

where $\tilde{H}_{x_j}$ is the proper transform of H_{x_j} on Y. In particular, $(X, \frac{1}{a_j}H_{x_j})$ is not canonical at p. By Lemma 3.5, we have $\alpha_{\mathsf{p}}(X) \geq 1$ if $(X, \frac{1}{a_j}H_{x_j})$ is log canonical at p, and otherwise $\alpha_{\mathsf{p}}(X) = \mathrm{lct}_{\mathsf{p}}(X; \frac{1}{a_j}H_{x_j})$.

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Suppose $a_j > 1$. The pair $(E, \frac{1}{a_j}\tilde{H}_{x_j}|_E)$ is log canonical since $\tilde{H}_{x_j}|_E$ is isomorphic to

$$(g(x_{i_1}, x_{i_2}, x_{i_3}, 0) = 0) \subset \mathbb{P}(a_{i_1}, a_{i_2}, a_{i_3}) \cong E,$$

which is quasi-smooth by Lemma 5.11. By the inversion of adjunction, the pair $(Y, \frac{1}{a_j}\tilde{H}_{x_j} + E)$ is log canonical along E and so is the pair $(Y, \frac{1}{a_j}\tilde{H}_{x_j} + \frac{2}{a_j}E)$ since $2/a_j \le 1$. By equation (5.4), the pair $(X, \frac{1}{a_j}H_{x_j})$ is log canonical at p. Thus, $\alpha_{\mathsf{p}}(X) \ge 1$. The existence of the prime divisor $H_{x_0} \in A$ passing through p shows $\alpha_{\mathsf{p}}(X) \le 1$, and we conclude $\alpha_{\mathsf{p}}(X) = 1$ in this case.

Suppose $a_j = 1$. We set

$$\theta = \frac{a_k + 1}{2a_k + 1}$$

and prove $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{a_j}H_{x_j}) = \theta$. For a rational number $c \ge 0$, it is easy to see that the discrepancy of the pair $(X, \frac{c}{a_j}H_{x_j})$ along E is

$$\frac{1}{a_k} - \frac{cd}{a_j a_k}$$

and it is at least -1 if and only if $c \le \theta$. This shows $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{a_j}H_{x_j}) \le \theta$. Moreover, since

$$K_Y + \frac{\theta}{a_j}\tilde{H}_{x_j} + E = \varphi^*\left(K_X + \frac{\theta}{a_j}H_{x_j}\right)$$

and the pair $(Y, \frac{\theta}{a_j}\tilde{H}_{x_j} + E)$ is log canonical along E, the pair $(X, \frac{\theta}{a_j}H_{x_j})$ is log canonical at p. This shows $\alpha_{\mathsf{p}}(X) = \theta$, and the proof is completed. □

Example 5.17. Let $X = X_{21} \subset \mathbb{P}(1, 1, 3, 7, 10)$ be a member of the family $\mathcal{F}_{46}$ and $\mathsf{p} = \mathsf{p}_w$ the $\frac{1}{10}(1, 3, 7)$ point, which is the center of a quadratic involution. Assume that p is degenerate, which is equivalent to X being birationally superrigid. Then, by Proposition 5.16, we have

$$\alpha(X) \le \alpha_{\mathsf{p}}(X) = \frac{11}{21}.$$

5.6. *QI centers: nondegenerate case*

The aim of this section is to prove the following.

Proposition 5.18. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I} \setminus \{2, 5, 8\}$ and $\mathsf{p} \in X$ be a nondegenerate QI center. Then*

$$\alpha_{\mathsf{p}}(X) \ge \frac{1}{2}.$$

The rest of this section is entirely devoted to the proof of Proposition 5.18, which will be done by dividing into several cases.

5.6.a. Case: X has a unique QI center

By Lemma 5.12, we can choose homogeneous coordinates so that

$$X = X_{2r+c} \subset \mathbb{P}(1, a, b, c, r)_{x,s,u,v,w},$$

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where a is corprime to b, $a < b$, $r = a + b$ and $c < r$. Let $\mathsf{p} \in X$ be the QI center. Then $\mathsf{p} = \mathsf{p}_w$ and the defining polynomial F of X can be written as

$$F = w^2 v + w f_{r+c} + f_{2r+c},$$

where $f_{r+c} = f_{r+c}(x, s, u)$ and $f_{2r+c} = f_{2r+c}(x, s, u, v)$ are quasi-homogeneous polynomials of the indicated degree. We will show that $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{c} H_v) \geq 1/2$.

Claim 14. *Suppose that $c \geq 2$ and $2r + c$ is divisible by b. Then*

$$\mathrm{lct}_{\mathsf{p}}\left(X; \frac{1}{c} H_v\right) \geq \frac{1}{2}.$$

Proof of Claim 14. We first show that $\mathsf{p}_u \notin X$ unless X belongs to the family $\mathcal{F}_7$. Suppose $\mathsf{p}_u \in X$. By the quasi-smoothness of X at p_u, we have $d = mb + e$, where $m \in \mathbb{Z}_{>0}$ and $e \in \{1, a, c, r\}$. Since d is divisible by b, we see that e is divisible by b. This is possible only when $e = c$ since $r = a + b$ and a are both coprime to b. Thus, we can write $c = kb$ for some $k \in \mathbb{Z}_{>0}$. Take any point $\mathsf{q} \in (x = s = w = 0) \cap X$. The singularity $\mathsf{q} \in X$ is of type $\frac{1}{b}(1, a, r) = \frac{1}{b}(1, a, a)$. It follows that $a = 1$ and $b = 2$ since $\mathsf{q} \in X$ is terminal. We have $r = a + b = 3$ and $c = 2$ since $c = kb = 2k < r = 3$. Thus, $X = X_8 \subset \mathbb{P}(1, 1, 2, 2, 3)$ and this belongs to $\mathcal{F}_7$.

We first consider the case where $\mathsf{p}_u \notin X$. This means that $u^m \in F$ for some $m \in \mathbb{Z}_{>0}$. We have $\mathrm{omult}_{\mathsf{p}}(H_v) \leq m = (2r + c)/b$ and hence

$$\mathrm{lct}_{\mathsf{p}}\left(X; \frac{1}{c} H_v\right) \geq \frac{bc}{2r + c}.$$

It remains to prove the inequality $bc/(2r + c) \geq 1/2$, which is equivalent to $(2b - 1)c \geq 2r$. We have

$$(2b - 1)c \geq 2(2b - 1) = 2(b - 1) + 2b \geq 2a + 2b = 2r$$

since $c \geq 2$ and $b > a$. This shows $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{c} H_v) \geq 1/2$.

We next consider the case where $\mathsf{p}_u \in X$. In this case, X belongs to the family $\mathcal{F}_7$ and $X = X_8 \subset \mathbb{P}(1, 1, 2, 2, 3)_{x,s,u,v,w}$ with defining polynomial

$$F = w^2 v + w f_5(x, s, u) + f_8(x, s, u, v).$$

We have $u^4 \notin F$ since $\mathsf{p}_u \in X$. We show that $f_5(x, s, u)$ contains a monomial involving u. Suppose to the contrary that $f_5 = f_5(x, s)$ is a polynomial in variables x and s. We can write $f_8 = u^3 g_2 + u^2 g_4 + u g_6 + g_8 + v h_6$, where $g_i = g_i(x, s)$ and $h_6 = h_6(x, s, u, v)$ are quasi-homogeneous polynomials of indicated degree. Then we have $F = v(w^2 + h_6) + g$, where $g = w f_5 + u^3 g_2 + u^2 g_4 + u g_6 + g_8 \in (x, s)^2$, and we see that X is not quasi-smooth at any point in the nonempty set

$$(t = w^2 + h_6 = x = s = 0) \subset \mathbb{P}(1, 1, 2, 2, 3).$$

This is a contradiction. It follows that there is a monomial involving u which appears in f_5 with nonzero coefficient. This implies $\mathrm{omult}_{\mathsf{p}}(H_v) \leq 4$, and we have $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{2} H_v) \geq 1/2$ as desired. □

Claim 15. *Suppose that $c \geq 2$ and $2r + c$ is not divisible by b. Then*

$$\mathrm{lct}_{\mathsf{p}}\left(X; \frac{1}{c} H_v\right) \geq \frac{c}{5} \geq \frac{4}{5}.$$

Proof of Claim 15. By Lemma 5.13, we have

$$X = X_{5a+4} \subset \mathbb{P}(1, a, a + 1, a + 2, 2a + 1)_{x,s,u,v,w}$$

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with $a \in \{2, 3, 4\}$. Moreover, $\mathsf{p} = \mathsf{p}_w$ and the defining polynomial of X can be written as

$$F = w^2 v + w f_{3a+3}(x, s, u) + f_{5a+4}(x, s, u, v).$$

By the quasi-smoothness of X at p_z, we have either $u^3 \in f_{3a+3}$ or $u^4 s \in f_{5a+4}$. This implies $\operatorname{omult}_{\mathsf{p}}(H_v) \leq 5$, and we have

$$\operatorname{lct}_{\mathsf{p}}\left(X; \frac{1}{c} H_v\right) \geq \frac{c}{5} = \frac{a+2}{5} \geq \frac{4}{5}.$$

This proves the claim. □

It remains to consider the case where $c = 1$. By Lemma 5.13, X belongs to a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{24, 46\}$.

Claim 16. *Suppose X is a member of the family $\mathcal{F}_{24}$. Then* $\operatorname{lct}_{\mathsf{p}}(X; H_v) \geq 1/2$.

Proof of Claim 16. We have

$$X = X_{15} \subset \mathbb{P}(1, 2, 5, 1, 7)_{x,s,u,v,w}$$

and $\mathsf{p} = \mathsf{p}_w$ is of type $\frac{1}{7}(1, 2, 5)$. We can write

$$F = w^2 v + w f_8(x, s, u) + f_{15}(x, s, u, v),$$

where $f_8 = f_8(x, s, u) \neq 0$ and $f_{15} = f_{15}(x, s, u, v)$ are quasi-homogeneous polynomials of degrees 8 and 15, respectively. We have $u^3 \in f_{15}$, and we may assume $\operatorname{coeff}_{f_{15}}(u^3) = 1$. We set $\bar{F} := F(x, s, u, 0, 1)$. For a given $\underline{c} = (c_1, c_2, c_3) \in (\mathbb{Z}_{>0})^3$, we denote by $G_{\underline{c}} \in \mathbb{C}[x, s, u]$ the lowest weight part of $\bar{F}$ with respect to the weight $\operatorname{wt}(x, s, u) = \underline{c}$ and let

$$\mathcal{D}_{\underline{c}} = \mathcal{D}^{\mathrm{wf}}_{G_{\underline{c}}}$$

be the effective $\mathbb{Q}$-divisor on $\mathbb{P}(\underline{c})^{\mathrm{wf}}$ associated to $G_{\underline{c}}$. By Lemma 3.27, we have

$$\operatorname{lct}_{\mathsf{p}}(X; H_v) \geq \min\left\{\frac{c_1 + c_2 + c_3}{\deg G_{\underline{c}}},\ \operatorname{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}})\right\}, \tag{5.5}$$

where $\deg G_{\underline{c}}$ is the degree with respect to the weight $\operatorname{wt}(x, s, u) = \underline{c}$.

Suppose $usx \in f_8$. In this case, we may assume $\operatorname{coeff}_{f_8}(usx) = 1$ and we have $G_{\underline{c}} = usx + u^3$ for $\underline{c} = (1, 1, 1)$. In this case, $\mathbb{P}(\underline{c})^{\mathrm{wf}} = \mathbb{P}^2$, $\mathrm{Diff} = 0$ and $\mathcal{D}_{\underline{c}}$ is the sum of a line and a conic intersection at two distinct points. It is straightforward to check

$$\operatorname{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) = \operatorname{lct}(\mathbb{P}^2; \mathcal{D}_{\underline{c}}) = 1,$$

and we have $\operatorname{lct}_{\mathsf{p}}(X; H_v) \geq 1$ in this case.

Suppose that $usx \notin f_8$ and $s^4 \in f_8$. In this case, we may assume $\operatorname{coeff}_{f_8}(s^4) = 1$ and $\operatorname{coeff}_{f_{15}}(us^5) = 0$ by replacing s and w. Hence, we have

$$F(0, s, u, 0, 1) = s^4 + u^3,$$

and thus $\operatorname{lct}_{\mathsf{p}}(X; H_v) \geq \operatorname{lct}_{\mathsf{p}}(H_x, H_v|_{H_x}) = 7/12$, where the equality follows from [Kol97, 8.21 Proposition] (or by Lemma 3.27 with $\operatorname{wt}(s, u) = (3, 4)$).

Suppose $ux^3 \in f_8$. In this case, we may assume $\operatorname{coeff}_{f_8}(usx) = 1$ by rescaling x. We consider a weight $\underline{c} = (2, e, 3)$, where e is a sufficiently large integer which is coprime to 6. Then $G_{\underline{c}} = u(x^3 + u^2)$,

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$\mathbb{P}(\underline{c})^{\mathrm{wf}} = \mathbb{P}(2, e, 3)$, Diff $= 0$ and $\mathcal{D}_{\underline{c}}$ is the union of two quasi-smooth curves $(u = 0)$ and $(x^3 + u^2 = 0)$. We have

$$\mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) = \mathrm{lct}(\mathbb{P}(2, e, 3); \mathcal{D}_{\underline{c}}) = \mathrm{lct}_{(0:1:0)}(\mathbb{P}(2, e, 3); \mathcal{D}_{\underline{c}}) = \frac{5}{9},$$

and thus $\mathrm{lct}_{\mathsf{p}}(X; H_v) \geq 5/9$ in this case.

In the following, we assume $usx, s^4, ux^3 \notin f_8$. We can write

$$\bar{F} = (\alpha_1 s^3 x^2 + \alpha_2 s^2 x^4 + \alpha_3 s x^6 + \alpha_4 x^8) + (u^3 + \beta u s^5 + \gamma s^7 x + g_{15}),$$

where $\alpha_1, \dots, \alpha_4, \beta, \gamma \in \mathbb{C}$ and $g_{15} = g_{15}(x, s, u) \not\ni u^3$ is a quasi-homogeneous polynomial of degree 15 which is contained in the ideal $(x, u)^2 \subset \mathbb{C}[x, s, u]$. Note that at least one of $\alpha, \beta, \gamma, \delta$ and ε is nonzero since $f_8(x, s, u) \neq 0$. Note also that $\lambda u s^5$ and $\mu s^7 x$ are the only terms in $\bar{F}$ which is not contained in $(x, u)^2$. It follows from the quasi-smoothness of X that $(\lambda, \mu) \neq (0, 0)$.

Suppose $\beta \neq 0$. Replacing u, we may assume $\beta = 1$ and $\gamma = 0$. There exists $j \in \{1, 2, 3, 4\}$ such that $\alpha_j \neq 0$ since $f_8 \neq 0$ as a polynomial, and thus we set $i = \min\{ j \mid \alpha_j \neq 0 \} \in \{1, 2, 3, 4\}$. We may assume $\alpha_i = 1$ by rescaling x. We set $\underline{c} = (2i + 7, 4i, 10i)$. We have $G_{\underline{c}} = s^{4-i} x^{2i} + u^3 + u s^5$ for $1 \leq i \leq 4$. Moreover, we see that

$$\mathbb{P}(\underline{c})^{\mathrm{wf}} = \begin{cases} \mathbb{P}(2i + 7, 2, 5)_{\tilde{x}, \tilde{s}, \tilde{u}}, & \text{if } 1 \leq i \leq 3, \\ \mathbb{P}(3, 2, 1)_{\tilde{x}, \tilde{s}, \tilde{u}}, & \text{if } i = 4, \end{cases}$$

and

$$(\mathrm{Diff}, \mathcal{D}_{\underline{c}}) = \begin{cases} (\frac{2i-1}{2i} H_{\tilde{x}}, D_i), & \text{if } 1 \leq i \leq 3, \\ (\frac{7}{8} H_{\tilde{x}} + \frac{4}{5} H_{\tilde{s}}, D'), & \text{if } i = 4, \end{cases}$$

where

$$D_i = (\tilde{s}^{4-i} \tilde{x} + \tilde{u}^3 + \tilde{u} \tilde{s}^5 = 0), \quad D' = (\tilde{x} + \tilde{u}^3 + \tilde{u} \tilde{s} = 0)$$

are prime divisors on $\mathbb{P}(\underline{c})^{\mathrm{wf}}$. We first consider the case where $1 \leq i \leq 3$. We see that $H_{\tilde{x}}$ is quasi-smooth, and D_i is quasi-smooth outside $\{\mathsf{q}\}$, where $\mathsf{q} = (1\!:\!0\!:\!0) \in \mathbb{P}(\underline{c})^{\mathrm{wf}}$. Moreover, they intersect at two points $(0\!:\!1\!:\!0)$ and $(0\!:\!-1\!:\!1)$ transversally. It follows that $\mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) = \min\{1, \ \mathrm{lct}_{\mathsf{q}}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathcal{D}_{\underline{c}})\}$ since $H_{\tilde{x}}$ does not pass through q. If $i = 3$, then D_3 is also quasi-smooth at q, which implies $\mathrm{lct}(\mathbb{P}(\underline{c}), \mathrm{Diff}; \mathcal{D}_{\underline{c}}) = 1$. If $i = 1, 2$, then we have

$$\begin{aligned} \mathrm{lct}_{\mathsf{q}}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathcal{D}_{\underline{c}}) &= \mathrm{lct}_{(0,0)}(\mathbb{A}^2_{\tilde{s}, \tilde{u}}, (\tilde{s}^{4-i} + \tilde{u}^3 + \tilde{u}\tilde{s}^5 = 0)) \\ &= \mathrm{lct}_{(0,0)}(\mathbb{A}^2_{\tilde{s}, \tilde{u}}, (\tilde{s}^{4-i} + \tilde{u}^3 = 0)) \\ &= \frac{3 + (4 - i)}{3(4 - i)} = \frac{7 - i}{3(4 - i)} \end{aligned}$$

since $(\tilde{s}^{4-i} + \tilde{u}^3 + \tilde{u}\tilde{s}^5 = 0)$ is analytically equivalent to $(\tilde{s}^{4-i} + \tilde{u}^3 = 0)$. Thus, by Lemma 3.27, we have

$$\begin{aligned} \mathrm{lct}_{\mathsf{p}}(X; H_v) &\geq \min\left\{ \frac{(2i + 7) + 4i + 10i}{30i}, \ \mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) \right\} \\ &= \begin{cases} \frac{2}{3}, & \text{if } i = 1, \\ \frac{13}{20}, & \text{if } i = 2, \\ \frac{11}{18}, & \text{if } i = 3. \end{cases} \end{aligned}$$

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Suppose $\beta = 0$. In this case, we have $\gamma \ne 0$. We set $i = \min\{ j \mid \alpha_j \ne 0 \} \in \{1, 2, 3, 4\}$. We may assume $\gamma = \alpha_i = 1$ by rescaling x and s appropriately. We set

$$\underline{c} = \begin{cases} (3(i+3), 3(2i-1), 15i-4), & \text{if } 1 \le i \le 3, \\ (3, 3, 8), & \text{if } i = 4. \end{cases}$$

We have $G_{\underline{c}} = s^{4-i}x^{2i} + u^3 + s^7 x$ for $1 \le i \le 4$. Moreover, we see that

$$\mathbb{P}(\underline{c})^{\mathrm{wf}} = \begin{cases} \mathbb{P}(i+3, 2i-1, 15i-4)_{\tilde{x},\tilde{s},\tilde{u}}, & \text{if } 1 \le i \le 3, \\ \mathbb{P}(1, 1, 8)_{\tilde{x},\tilde{s},\tilde{u}}, & \text{if } i = 4, \end{cases}$$

and

$$\mathrm{Diff} = \frac{2}{3} H_{\tilde{u}}, \quad \mathcal{D}_{\underline{c}} = (\tilde{s}^{4-i}\tilde{x}^{2i} + \tilde{u} + \tilde{s}^7 \tilde{x} = 0).$$

We see that $H_{\tilde{u}}$ and $\mathcal{D}_{\underline{c}}$ are both quasi-smooth. If $i = 3, 4$, then $H_{\tilde{u}}$ and $\mathcal{D}_{\underline{c}}$ intersect transversally and thus we have $\mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) = 1$. Suppose $i = 1, 2$. Then $H_{\tilde{u}}$ and $\mathcal{D}_{\underline{c}}$ intersect transversally except at $\mathsf{p}_{\tilde{x}} = (1\!:\!0\!:\!0) \in \mathbb{P}(\underline{c})^{\mathrm{wf}}$, and we have

$$\begin{aligned} \mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) &= \mathrm{lct}_{\mathsf{p}_{\tilde{x}}}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) \\ &= \mathrm{lct}(\mathbb{A}^2_{\tilde{s},\tilde{u}}, \tfrac{2}{3}(\tilde{u} = 0); (\tilde{s}^{4-i} + \tilde{u} + \tilde{s}^7 = 0)) \\ &= \mathrm{lct}(\mathbb{A}^2_{\tilde{s},\tilde{u}}, \tfrac{2}{3}(\tilde{u} = 0); (\tilde{s}^{4-i} + \tilde{u} = 0)) \\ &= \begin{cases} \frac{2}{3}, & \text{if } i = 1, \\ 1, & \text{if } i = 2. \end{cases} \end{aligned}$$

Thus, by Lemma 3.27, we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_v) \ge \begin{cases} \frac{2}{3}, & \text{if } i = 1, \\ \frac{25}{39}, & \text{if } i = 2, \\ \frac{74}{123}, & \text{if } i = 3, \\ \frac{7}{12}, & \text{if } i = 4. \end{cases}$$

This proves the claim. □

Claim 17. *Suppose X is a member of the family $\mathcal{F}_{46}$. Then* $\mathrm{lct}_{\mathsf{p}}(X; H_v) \ge 1/2$.

Proof of Claim 17. We have

$$X = X_{21} \subset \mathbb{P}(1, 3, 7, 1, 10)_{x,s,u,v,w}$$

and $\mathsf{p} = \mathsf{p}_w$ is of type $\frac{1}{10}(1, 3, 7)$. We can write

$$F = w^2 v + w f_{11}(x, s, u) + f_{21}(x, s, u, v),$$

where $f_{11} = f_{11}(x, s, u) \ne 0$ and $f_{21} = f_{21}(x, s, u, v)$ are quasi-homogeneous polynomials of degree 11 and 21, respectively. We have $u^3, s^7 \in F$ by the quasi-smoothness of X, and we may assume $\mathrm{coeff}_F(u^3) = \mathrm{coeff}_F(s^7) = 1$. We set $\bar{F} = F(x, s, u, 0, 1)$, which can be written as

$$\bar{F} = (\alpha usx + \beta s^3 x^2 + \gamma ux^4 + \delta s^2 x^5 + \varepsilon sx^8 + \zeta x^{11}) + (u^3 + s^7),$$

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Table 4. *Family $\mathcal{F}_{46}$: Weights and LCT.*

Case	$\mathbb{P}(\underline{c})^{\mathrm{wf}}$	Diff	$G^{\mathrm{wf}}_{\underline{c}}$	η	θ
(i)	$\mathbb{P}(1,1,1)$	0	$\tilde{u}(\tilde{s}\tilde{x}+\tilde{u}^2)$	1	1
(ii)	$\mathbb{P}(2,1,7)$	$\frac{2}{3}H_{\tilde{u}}$	$\tilde{s}^3\tilde{x}^2+\tilde{u}+\tilde{s}^7$	2/3	2/3
(iii)	$\mathbb{P}(1,3,1)$	$\frac{1}{2}H_{\tilde{x}}+\frac{6}{7}H_{\tilde{s}}$	$\tilde{u}\tilde{x}^2+\tilde{u}^3+\tilde{s}$	1	9/14
(iv)	$\mathbb{P}(1,1,7)$	$\frac{2}{3}H_{\tilde{u}}$	$\tilde{s}^2\tilde{x}^5+\tilde{u}+\tilde{s}^7$	5/6	13/21
(v)	$\mathbb{P}(3,1,7)$	$\frac{3}{4}H_{\tilde{x}}+\frac{2}{3}H_{\tilde{u}}$	$\tilde{s}\tilde{x}^2+\tilde{u}+\tilde{s}^7$	1	7/12
(vi)	$\mathbb{P}(1,1,1)$	$\frac{10}{11}H_{\tilde{x}}+\frac{6}{7}H_{\tilde{s}}+\frac{2}{3}H_{\tilde{u}}$	$\tilde{x}+\tilde{u}+\tilde{s}$	1	131/231

where $\alpha, \beta, \ldots, \zeta \in \mathbb{C}$. We introduce various 3-tuples ($\underline{c} = (c_1, c_2, c_3)$ of positive integers according to the following division into cases. We denote by $G_{\underline{c}}$ the lowest weight part of $\bar{F}$ with respect to $\mathrm{wt}(x, s, u) = \underline{c}$.

(i) $\alpha \neq 0$. In this case, we may assume $\alpha = 1$. We set $\underline{c} = (1, 1, 1)$. Then we have $G_{\underline{c}} = usx + u^3$.
(ii) $\alpha = 0$ and $\beta \neq 0$. In this case, we may assume $\beta = 1$. We set $\underline{c} = (6, 3, 7)$. Then we have $G_{\underline{c}} = s^3x^2 + u^3 + s^7$.
(iii) $\alpha = \beta = 0$ and $\gamma \neq 0$. In this case, we may assume $\gamma = 1$. We set $\underline{c} = (7, 6, 14)$. Then we have $G_{\underline{c}} = ux^4 + u^3 + s^7$.
(iv) $\alpha = \beta = \gamma = 0$ and $\delta \neq 0$. In this case, we may assume $\delta = 1$. We set $\underline{c} = (3, 3, 7)$. In this case, we have $G_{\underline{c}} = s^2x^5 + u^3 + s^7$.
(v) $\alpha = \beta = \gamma = \delta = 0$ and $\varepsilon \neq 0$. In this case, we may assume $\varepsilon = 1$. We set $\underline{c} = (9, 12, 28)$. In this case, we have $G_{\underline{c}} = sx^8 + u^3 + s^7$.
(vi) $\alpha = \beta = \gamma = \delta = \varepsilon = 0$. In this case, we may assume $\delta = 1$. We set $\underline{c} = (21, 33, 77)$. In this case, we have $x^{11} + u^3 + s^7$.

The descriptions of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$, Diff and $G^{\mathrm{wf}}_{\underline{c}}$ are given in Table 4, where we choose $\tilde{x}, \tilde{s}, \tilde{u}$ as homogeneous coordinates of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$.

We set $\mathcal{D}_{\underline{c}} = \mathcal{D}^{\mathrm{wf}}_{G_{\underline{c}}}$. We explain the computation of $\eta := \mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}})$ whose value is given in the fifth column of Table 4. The computation $\eta = 1$ is straightforward when we are in case (i) since $\mathcal{D}_{\underline{c}}$ is the union of of a line and a conic on $\mathbb{P}^2$ intersecting at two points. In the other cases, $\mathcal{D}_{\underline{c}}$ is the divisor defined by $G^{\mathrm{wf}}_{\underline{c}} = 0$ which is a quasi-line in $\mathbb{P}(\underline{c})^{\mathrm{wf}}$. If we are in one of the cases (iii), (v) and (vi), then any two of the components of Diff $+\mathcal{D}_{\underline{c}}$ intersect transversally, which implies $\eta = 1$. If we are in case (ii) or (iv), then $H_{\tilde{u}}$ and $\mathcal{D}_{\underline{c}}$ intersect transversally except at $\mathsf{q} = (1\!:\!0\!:\!0) \in \mathbb{P}(\underline{c})^{\mathrm{wf}}$. We set $e = 3, 2$ if we are in case (ii), (iv), respectively. Then we have

$$\begin{aligned}\mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) &= \mathrm{lct}_{\mathsf{q}}(\mathbb{P}(\underline{c})^{\mathrm{wf}}, \mathrm{Diff}; \mathcal{D}_{\underline{c}}) \\ &= \mathrm{lct}_{(0,0)}(\mathbb{A}^2_{\tilde{s},\tilde{u}}, \tfrac{2}{3}(\tilde{u} = 0); (\tilde{s}^e + \tilde{u} + \tilde{s}^7 = 0)) \\ &= \mathrm{lct}_{(0,0)}(\mathbb{A}^2_{\tilde{s},\tilde{u}}, \tfrac{2}{3}(\tilde{u} = 0); (\tilde{s}^e + \tilde{u} = 0)).\end{aligned}$$

This completes the explanations of the computations of η. We set

$$\theta = \left\{ \frac{c_1 + c_2 + c_3}{\deg_{\underline{c}}(G^{\mathrm{wf}}_{\underline{c}})}, \eta \right\}$$

which is described in the sixth column of Table 4. By Lemma 3.27, we have $\mathrm{lct}_{\mathsf{p}}(X; H_v) \geq \theta \geq 1/2$ and the claim is proved. $\square$

By Claims 14, 15, 16 and 17, we have $\mathrm{lct}_{\mathsf{p}}(X; \frac{1}{c}H_v) \geq 1/2$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{c}H_v$. We set $\lambda = (r + c)/(2r + c)$, and we will show that $\mathrm{lct}_{\mathsf{p}}(X; D) \geq \lambda$. Suppose not, that is, $(X, \lambda D)$ is not log canonical at p. Let $\varphi\colon Y \to X$ be the Kawamata blowup of $\mathsf{p} \in X$. Then, for the proper transforms $\tilde{H}_t$ and $\tilde{D}$ of H_t and D, respectively, we have

$$\tilde{H}_t \sim c\varphi^* A - \frac{r+c}{r} E,$$
$$\tilde{D} \sim_{\mathbb{Q}} \varphi^* A - \frac{e}{r} E,$$

where $e \in \mathbb{Q}_{\geq 0}$. By [Kaw96], the discrepancy of the pair $(X, \lambda D)$ along E is negative, and thus we have $e > 1/\lambda$. By [CPR00, Theorem 4.9], $-K_Y \sim_{\mathbb{Q}} \varphi^* A - \frac{1}{r} E$ is nef (more precisely, $-mK_Y$ defines the flopping contraction for a sufficiently divisible $m > 0$). Hence, $(-K_Y \cdot \tilde{H}_t \cdot \tilde{D}) \geq 0$ and we have

$$\begin{aligned} 0 \leq (-K_Y \cdot \tilde{H}_t \cdot \tilde{D}) &= c(A^3) - \frac{e(r+c)}{r^3}(E^3) \\ &= \frac{2r+c}{abr} - \frac{e(r+c)}{abr} < \frac{2r+c}{abr} - \frac{r+c}{\lambda abr} = 0. \end{aligned}$$

This is a contradiction. Therefore, $\operatorname{lct}_{\mathsf{p}}(X; D) \geq \lambda$ and thus

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{ \operatorname{lct}_{\mathsf{p}}\left(X; \frac{1}{c} H_v\right), \frac{r+c}{2r+c} \right\} \geq \frac{1}{2}.$$

This completes the proof of Proposition 5.18 when X has a unique QI center.

5.6.b. Case: X has exactly three distinct QI centers

By Lemma 5.12, we can choose homogeneous coordinates so that

$$X = X_{3r} \subset \mathbb{P}(1, a, b, r, r)_{x,y,z,t,w},$$

where a is coprime to b, $a \leq b$ and $a + b = r$. Let $\mathsf{p} \in X$ be a QI center. Then we may assume $\mathsf{p} = \mathsf{p}_w$ by replacing t and w suitably. Then the defining polynomial F of X can be written as

$$F = w^2 t + w f_{2r} + f_{3r},$$

where $f_{2r}(x, y, z)$ and $f_{3r}(x, y, z, t)$ are quasi-homogeneous polynomials of degrees $2r$ and $3r$, respectively. We have $(A^3) = 3r/abr^2 = 3/abr$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{rab(A^3)} = \frac{2}{3},$$

and Proposition 5.18 is proved when X has exactly three distinct QI centers.

5.6.c. Case: X has exactly two distinct QI centers and their singularity types are equal

By Lemma 5.12, we can choose homogeneous coordinates so that

$$X = X_{4r} \subset \mathbb{P}(1, a, b, r, 2r)_{x,y,z,t,w},$$

where a is coprime to b, $a \leq b$ and $a + b = r$. Let $\mathsf{p} \in X$ be a QI center. We may assume $\mathsf{p} = \mathsf{p}_t$ by replacing w suitably. Then the defining polynomial F of X can be written as

$$F = t^2 w + t f_{3r} + f_{4r},$$

where $f_{3r}(x, y, z)$ and $f_{4r}(x, y, z, w)$ are quasi-homogeneous polynomials of degrees $3r$ and $4r$, respectively. Note that $(A^3) = 4r/2abr^2 = 2/abr$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{rab(A^3)} = 1,$$

and Proposition 5.18 is proved in this case.

5.6.d. Case: *X* has exactly two distinct QI centers and their singularity types are distinct

By Lemma 5.12, we have

$$X = X_{4a+3b} \subset \mathbb{P}(1, a, b, r_1, r_2)_{x,u,v,t,w},$$

where a is coprime to b, $r_1 = a + b$ and $r_2 = 2a + b$.

We first consider the QI center $\mathsf{p} = \mathsf{p}_t \in X$ of type $\frac{1}{r_1}(1, a, b)$. The defining polynomial F of X can be written as

$$F = t^2 w + t f_{3a+2b} + f_{4a+3b},$$

where $f_{3a+2b} = f_{3a+2b}(x, u, v)$ and $f_{4a+3b} = f_{4a+3b}(x, u, v, w)$ are quasi-homogeneous polynomials of the indicated degrees. Note that $(A^3) = (4a + 3b)/abr_1r_2$. By Lemma 5.12, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{r_1 ab(A^3)} = \frac{2r_2}{4a + 3b} = \frac{4a + 2b}{4a + 3b} > \frac{2}{3}.$$

We next consider the QI center $\mathsf{p} = \mathsf{p}_w \in X$ of type $\frac{1}{r_2}(1, a, a + b)$. Then the defining polynomial F of X can be written as

$$F = w^2 v + w f_{2a+2b} + f_{4a+3b},$$

where $f_{2a+2b} = f_{2a+2b}(x, u, t)$ and $f_{4a+3b} = f_{4a+3b}(x, u, v, t)$ are quasi-homogeneous polynomials of the indicated degree.

Suppose $t^2 w \in F$, that is, $t^2 \in f_{2a+2b}$. Then $\operatorname{omult}_{\mathsf{p}}(H_v) = 2$ and we have $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{b} H_v) \geq b/2 \geq 1/2$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{b} H_v$. We see that the set $\{x, u, v\}$ isolates p since $t^2 w \in F$. In particular, a general member $T \in |aA|$ does not contain any component of the effective 1-cycle $D \cdot H_z$. Then we have

$$\begin{aligned} 2 \operatorname{omult}_{\mathsf{p}}(D) &\leq (\rho_{\mathsf{p}}^* D \cdot \rho_{\mathsf{p}}^* H_v \cdot \rho_{\mathsf{p}}^* T)_{\check{\mathsf{p}}} \leq r_2 (D \cdot H_v \cdot T) \\ &= r_2 ba(A^3) = \frac{4a + 3b}{a + b}. \end{aligned}$$

This implies

$$\operatorname{lct}_{\mathsf{p}}(X; D) \geq \frac{2a + 2b}{4a + 3b} > \frac{1}{2}.$$

Therefore, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$.

In the following, we consider the case where $t^2 w \notin F$.

Claim 18. *If* $b \geq 2$, *then* $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{b} H_v) \geq 1/2$.

Proof of Claim 18. By the quasi-smoothness of X at p_t, we have $t^3 u \in f_{4a+3b}$ since $t^2 w \notin F$ by assumption. Hence, we have $\operatorname{omult}_{\mathsf{p}}(H_v) \leq 4$ and this shows $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{b} H_v) \geq 1/2$. □

If $b = 1$, then X is a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{13, 25\}$.

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Table 5. *Family $\mathcal{F}_{13}$: weights and LCT.*

Case	$\mathbb{P}(\underline{c})^{\mathrm{wf}}$	Diff	$G_{\underline{c}}^{\mathrm{wf}}$	η	θ
(i)	$\mathbb{P}(e,3,2)$	0	$\tilde{u}(\tilde{u}^2+\tilde{t}^3)$	$5/9$	$5/9$
(ii)	$\mathbb{P}(7,2,3)$	$\frac{1}{2}H_{\tilde{x}}$	$\tilde{u}(\tilde{u}\tilde{x}+\tilde{t}^3+\lambda\tilde{t}\tilde{u}^3)$	$2/3$	$2/3$
(iii)	$\mathbb{P}(4,1,3)$	$\frac{2}{3}H_{\tilde{x}}+\frac{1}{2}H_{\tilde{t}}$	$\tilde{t}^{1/2}(\tilde{x}+\tilde{t}\tilde{u}+\lambda\tilde{u}^4)$	$\geq 5/9$	$\geq 5/9$
(iv)	$\mathbb{P}(3,2,1)$	$\frac{3}{4}H_{\tilde{x}}+\frac{2}{3}H_{\tilde{u}}$	$\tilde{u}^{1/3}(\tilde{x}+\tilde{t}^3+\lambda\tilde{t}\tilde{u})$	$\geq 7/12$	$\geq 7/12$
(v)	$\mathbb{P}(11,2,3)$	$\frac{5}{6}H_{\tilde{x}}$	$\tilde{x}+\tilde{t}^3\tilde{u}+\lambda\tilde{t}\tilde{u}^4$	$\geq 1/2$	$\geq 1/2$

Claim 19. *If X is a member of the family $\mathcal{F}_{13}$, then* $\operatorname{lct}_{\mathsf{p}}(X;H_v)\geq 1/2$.

Proof of Claim 19. We have

$$F=w^2v+wf_6(x,u,v,t)+f_{11}(x,u,v,t).$$

Note that $f_6(x,u,0,t)\neq 0$ as a polynomial since $\mathsf{p}\in X$ is nondegenerate. We set $\bar{F}=F(x,u,0,t,1)\in\mathbb{C}[x,u,t]$. We have $t^3u\in F$, and we may assume $\operatorname{coeff}_F(t^3u)=1$. If $tux\in f_6$, then the cubic part of $\bar{F}$ is not a cube of a linear form, and thus we have $\operatorname{lct}_{\mathsf{p}}(X;H_v)\geq 1/2$ by Lemma 3.28. In the following, we assume $tux\notin f_6$. Then we can write

$$\bar{F}=(\alpha u^3+\beta u^2x^2+\gamma tx^3+\delta ux^4+\varepsilon x^6)+(t^3u+\lambda tu^4+xg_{10}),$$

where $\alpha,\beta,\dots,\varepsilon,\lambda\in\mathbb{C}$ and $g_{10}=g_{10}(x,u,t)$ is a quasi-homogeneous polynomial of degree 10. We introduce 3-tuples $\underline{c}=(c_1,c_2,c_3)$ of positive integers according to the following division into cases. We denote by $G_{\underline{c}}$ the lowest weight part of $\bar{F}$ with respect to $\operatorname{wt}(x,u,t)=\underline{c}$.

(i) $\alpha\neq 0$. In this case, we may assume $\alpha=1$. We choose and fix a sufficiently large integer e which is coprime to 2 and 3, and we set $\underline{c}=(e,3,2)$. Then we have $G_{\underline{c}}=u^3+t^3u$.

(ii) $\alpha=0$ and $\beta\neq 0$. In this case, we may assume $\beta=1$. We set $\underline{c}=(7,4,6)$. Then we have $G_{\underline{c}}=u^2x^2+t^3u+\lambda tu^4$.

(iii) $\alpha=\beta=0$ and $\gamma\neq 0$. In this case, we may assume $\gamma=1$. We set $\underline{c}=(8,6,9)$. Then $G_{\underline{c}}=tx^3+t^3u+\lambda tu^4$.

(iv) $\alpha=\beta=\gamma=0$ and $\delta\neq 0$. In this case, we may assume $\delta=1$. We set $\underline{c}=(9,8,12)$. Then $G_{\underline{c}}=ux^4+t^3u+\lambda tu^4$.

(v) $\alpha=\beta=\gamma=\delta=0$. In this case, we may assume $\varepsilon=1$. We set $\underline{c}=(11,12,18)$. Then we have $x^6+t^3u+\lambda tu^4$.

The descriptions of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$, Diff and $G_{\underline{c}}^{\mathrm{wf}}$ are given in Table 5, where we choose $\tilde{x},\tilde{u},\tilde{t}$ as homogeneous coordinates of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$.

We set $\mathcal{D}_{\underline{c}}=\mathcal{D}^{\mathrm{wf}}_{G_{\underline{c}}}$. We explain the computation of $\eta:=\operatorname{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}},\mathrm{Diff};\mathcal{D}_{\underline{c}})$ whose value (or lower bound) is given in the fifth column of Table 5. □

Claim 20. *If X is a member of the family $\mathcal{F}_{25}$, then* $\operatorname{lct}_{\mathsf{p}}(X;H_v)\geq 1/2$.

Proof of Claim 20. We have

$$F=w^2v+wf_8(x,u,v,t)+f_{15}(x,u,v,t).$$

Note that $f_8(x,u,0,t)\neq 0$ as a polynomial since $\mathsf{p}\in X$ is nondegenerate. We set $\bar{F}=F(x,u,0,t,1)\in\mathbb{C}[x,u,t]$. We have $t^3u,u^5\in F$, and we may assume $\operatorname{coeff}_F(t^3u)=\operatorname{coeff}_F(u^5)=1$. If $tux\in f_8$, then the cubic part of $\bar{F}$ is not a cube of a linear form and thus we have $\operatorname{lct}_{\mathsf{p}}(X;H_v)\geq 1/2$ by Lemma 3.28. In the following, we assume $tux\notin f_8$. Then we can write

$$\bar{F}=(\alpha u^2x^2+\beta tx^4+\gamma ux^5+\delta x^8)+(t^3u+u^5+xg_{14}),$$

Table 6. *Family $\mathcal{F}_{25}$: Weights and LCT.*

Case	$\mathbb{P}(\underline{c})^{\mathrm{wf}}$	Diff	$G_{\underline{c}}^{\mathrm{wf}}$	η	θ
(i)	$\mathbb{P}(3,1,4)$	$\frac{1}{2}H_{\tilde{x}}+\frac{2}{3}H_{\tilde{t}}$	$\tilde{u}(\tilde{u}\tilde{x}+\tilde{t}+\tilde{u}^4)$	1	23/30
(ii)	$\mathbb{P}(11,3,4)$	$\frac{3}{4}H_{\tilde{x}}$	$\tilde{t}\tilde{x}+\tilde{t}^3\tilde{u}+\tilde{u}^5$	1	13/20
(iii)	$\mathbb{P}(1,1,1)$	$\frac{4}{5}H_{\tilde{x}}+\frac{3}{4}H_{\tilde{u}}+\frac{2}{3}H_{\tilde{t}}$	$\tilde{u}^{1/4}(\tilde{x}+\tilde{t}+\tilde{u})$	1	47/75
(iv)	$\mathbb{P}(5,1,4)$	$\frac{7}{8}H_{\tilde{x}}+\frac{3}{4}H_{\tilde{t}}$	$\tilde{x}+\tilde{t}\tilde{u}+\tilde{u}^5$	1	71/120

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ and $g_{14}=g_{14}(x,u,t)$ is a quasi-homogeneous polynomial of degree 14. We introduce 3-tuples $\underline{c}=(c_1,c_2,c_3)$ of positive integers according to the following division into cases. We denote by $G_{\underline{c}}$ the lowest weight part of $\bar{F}$ with respect to $\mathrm{wt}(x,u,t)=\underline{c}$.

(i) $\alpha\neq 0$. In this case, we may assume $\alpha=1$. We set $\underline{c}=(9,6,8)$. Then we have $G_{\underline{c}}=u^2x^2+t^3u+u^5$.
(ii) $\alpha=0$ and $\beta\neq 0$. In this case, we may assume $\beta=1$. We set $\underline{c}=(11,12,16)$. Then we have $G_{\underline{c}}=tx^4+t^3u+u^5$.
(iii) $\alpha=\beta=0$ and $\gamma\neq 0$. In this case, we may assume $\gamma=1$. We set $\underline{c}=(12,15,20)$. Then $G_{\underline{c}}=ux^5+t^3u+u^5$.
(iv) $\alpha=\beta=\gamma=0$ and $\delta\neq 0$. In this case, we may assume $\delta=1$. We set $\underline{c}=(15,24,32)$. Then $G_{\underline{c}}=x^8+t^3u+u^5$.

The descriptions of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$, Diff and $G_{\underline{c}}^{\mathrm{wf}}$ are given in Table 6, where we choose $\tilde{x},\tilde{u},\tilde{t}$ as homogeneous coordinates of $\mathbb{P}(\underline{c})^{\mathrm{wf}}$.

We set $\mathcal{D}_{\underline{c}}=\mathcal{D}^{\mathrm{wf}}_{G_{\underline{c}}}$. We explain the computation of $\eta:=\mathrm{lct}(\mathbb{P}(\underline{c})^{\mathrm{wf}},\mathrm{Diff};\mathcal{D}_{\underline{c}})$ whose value is given in the fifth column of Table 6. Suppose that we are in case (ii) or (iv). Then $\mathcal{D}_{\underline{c}}$ is a prime divisor which is quasi-smooth and intersects any component of Diff transversally. This shows $\eta=1$. Suppose that we are in case (i). Then $\mathcal{D}_{\underline{c}}=H_{\tilde{u}}+\Gamma$, where $\Gamma=(\tilde{u}\tilde{x}+\tilde{t}+\tilde{u}^4=0)$ is a quasi-line. We see that any two of $H_{\tilde{x}},H_{\tilde{u}},H_{\tilde{t}},\Gamma$ intersect transversally, and thus $\eta=1$. Suppose that we are in case (iii). Then $\mathcal{D}_{\underline{c}}=\frac{1}{4}H_{\tilde{u}}+\Gamma$, where $\Gamma=(\tilde{x}+\tilde{t}\tilde{u}+\tilde{u}^5=0)$ is a quasi-line. We see that any two of $H_{\tilde{x}},H_{\tilde{u}},H_{\tilde{t}}$ and Γ intersect transversally, and thus $\eta=1$.

We set

$$\theta:=\min\left\{\frac{c_1+c_2+c_3}{\mathrm{wt}_{\underline{c}}(\bar{F})},\eta\right\},$$

which is listed in the sixth column of Table 6. By Lemma 3.27, we have $\mathrm{lct}_{\mathsf{p}}(X;H_v)\geq\theta\geq 1/2$ and the claim is proved. □

By Claims 18, 19 and 20, we have $\mathrm{lct}_{\mathsf{p}}(X;\frac{1}{b}H_v)\geq 1/2$. Suppose $\alpha_{\mathsf{p}}(X)<1/2$. Then there exists an irreducible $\mathbb{Q}$-divisor $D\in|A|_{\mathbb{Q}}$ other than $\frac{1}{b}H_v$ such that $(X,\frac{1}{2}D)$ is not log canonical at p. Let $\varphi\colon Y\to X$ be the Kawamata blowup at p with exceptional divisor E. We set $\lambda=\mathrm{ord}_E(D)$. Since the pair $(X,\frac{1}{2}D)$ is not canonical at p, the discrepancy of $(X,\frac{1}{2}D)$ along E is negative, which implies

$$\lambda>\frac{2}{r_2}.$$

By [CPR00, Theorem 4.9], the divisor $-K_Y\sim_{\mathbb{Q}}\varphi^*A-\frac{1}{r_2}E$ is nef. We see that $\tilde{D}\cdot\tilde{H}_v$ is an effective 1-cycle on Y, where $\tilde{D}$ and $\tilde{H}_v$ are proper transforms of D and H_v, respectively. It follows that

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$$0 \le (-K_Y \cdot \tilde{D} \cdot \tilde{H}_v) = b(A^3) - \frac{(2a+2b)\lambda}{r_2^2}(E^3)$$
$$= \frac{(4a+3b) - (2a+2b)r_2\lambda}{ar_1r_2} < -\frac{b}{ar_1r_2} < 0.$$

This is a contradiction, and we have $\alpha_{\mathsf{p}}(X) \ge 1/2$. Therefore, the proof of Proposition 5.18 is completed.

6. Families $\mathcal{F}_2$, $\mathcal{F}_4$, $\mathcal{F}_5$, $\mathcal{F}_6$, $\mathcal{F}_8$, $\mathcal{F}_{10}$ and $\mathcal{F}_{14}$

This section is devoted to the proof of the following theorem.

Theorem 6.1. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}_1$. Then*

$$\alpha(X) \ge \frac{1}{2}.$$

6.1. Families $\mathcal{F}_6$, $\mathcal{F}_{10}$ and $\mathcal{F}_{14}$

In this section, we prove Theorem 6.1 for families $\mathcal{F}_6$, $\mathcal{F}_{10}$ and $\mathcal{F}_{14}$ whose member is a weighted hypersurface

$$X = X_{2(a+2)} \subset \mathbb{P}(1,1,1,a,a+2)_{x,y,z,t,w},$$

where $a = 2, 3, 4$, respectively. Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{6, 10, 14\}$.

Let $\mathsf{p} \in X$ be a smooth point. We may assume $\mathsf{p} = \mathsf{p}_x$ by a suitable choice of coordinates. By Lemma 4.3 (see also Remark 4.4), we have

$$\alpha_{\mathsf{p}}(X) \ge \frac{1}{1 \cdot a \cdot (A^3)} = \frac{1}{2}.$$

Let $\mathsf{p} \in X$ be a singular point. If $\mathsf{i} = 14$, then $\mathsf{p} \in X$ is of type $\frac{1}{2}(1,1,1)$ and we have $\alpha_{\mathsf{p}}(X) \ge 1$ by Proposition 5.2. If $\mathsf{i} = 6, 10$, then $\mathsf{p} \in X$ is of type $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, respectively, and in both cases we have $\alpha_{\mathsf{p}}(X) \ge 1/2$ by Proposition 5.4. Thus, the proof of Theorem 6.1 for families $\mathcal{F}_6$, $\mathcal{F}_{10}$ and $\mathcal{F}_{14}$ is completed.

6.2. The family $\mathcal{F}_2$

This section is devoted to the proof Theorem 6.1 for the family $\mathcal{F}_2$. In the following, let

$$X = X_5 \subset \mathbb{P}(1,1,1,1,2)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_2$ with defining polynomial $F = F(x,y,z,t,w)$.

6.2.a. Smooth points

Let $\mathsf{p} \in X$ be a smooth point. In this subsection, we will prove $\alpha_{\mathsf{p}}(X) \ge 1/2$. We may assume $\mathsf{p} = \mathsf{p}_x$ by a choice of coordinates. The proof will be done by division into cases.

6.2.a.1. Case: $x^3w \in F$

In this case, we can write

$$F = x^3w + x^2 f_3 + x f_4 + f_5,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i. We have $\mathrm{mult}_{\mathsf{p}}(H_w) \ge 3$.

Claim 21. $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{2}H_w) \geq 1/2$.

Proof of Claim 21. This is obvious when $\operatorname{mult}_{\mathsf{p}}(H_w) \leq 4$, hence we assume $\operatorname{mult}_{\mathsf{p}}(H_w) \geq 5$. Then we can write

$$F = x^3w + x^2wa_1 + x(\alpha w^2 + wb_2) + w^2c_1 + wd_3 + e_5,$$

where $\alpha \in \mathbb{C}$ and $a_1, b_2, c_1, d_3, e_5 \in \mathbb{C}[y,z,t]$ are quasi-homogeneous polynomials of indicated degrees. We show that $(e_5 = 0) \subset \mathbb{P}(1,1,1)_{y,z,t}$ is smooth. Indeed, if it has a singular point at $(y\!:\!z\!:\!t) = (\lambda\!:\!\mu\!:\!\nu)$, then, by setting $\theta \in \mathbb{C}$ to be a solution of the equation

$$x^3 + x^2a_1(\lambda,\mu,\nu) + xb_2(\lambda,\mu,\nu) + d_3(\lambda,\mu,\nu) = 0,$$

we see that X is not quasi-smooth at the point $(\theta : \lambda : \mu : \nu : 0)$ and this is a contradiction. The lowest weight part of $F(1,y,z,t,0) = e_5$ with respect to $\operatorname{wt}(y,z,t) = (1,1,1)$ is e_5 which defines a smooth hypersurface in $\mathbb{P}^2$. By Lemma 3.27, we have $\operatorname{lct}_{\mathsf{p}}(X, H_w) \geq 3/5$. Thus, $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{2}H_w) \geq 6/5$ in this case and the claim is proved. □

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{2}H_w$. We can take a $\mathbb{Q}$-divisor $T \in |A|_{\mathbb{Q}}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_w$ since $\{y,z,t\}$ isolates p. It follows that

$$3\operatorname{mult}_{\mathsf{p}}(D) \leq (D \cdot H_w \cdot T)_{\mathsf{p}} \leq (D \cdot H_w \cdot T) = 5.$$

This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 3/5$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$.

6.2.a.2. Case: $x^3w \notin F$

By a choice of coordinates, we can write

$$F = x^4t + x^3f_2 + x^2f_3 + xf_4 + f_5,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $w \notin f_2$.

Suppose $w^2 \in f_4$. In this case, $\operatorname{mult}_{\mathsf{p}}(H_t) = 2$ and hence $\operatorname{lct}_{\mathsf{p}}(X; H_t) \geq 1/2$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_t. We can take a $\mathbb{Q}$-divisor $T \in |A|_{\mathbb{Q}}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_t$ since $\{y,z,t\}$ isolates p so that

$$2\operatorname{mult}_{\mathsf{p}}(D) \leq (D \cdot H_t \cdot T)_{\mathsf{p}} \leq (D \cdot H_t \cdot T) = \frac{5}{2}.$$

This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 4/5$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$ in this case.

Suppose $w^2 \notin f_4$. We have $\operatorname{Bs}|\mathcal{I}_{\mathsf{p}}(A)| = \Gamma$, where $\Gamma = (y = z = t = 0) \subset X$ is a quasi-line. We assume $\alpha_{\mathsf{p}}(X) < 1/2$. Then there exists an irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ such that $(X, \frac{1}{2}D)$ is not log canonical at p. Let $S \in |\mathcal{I}_{\mathsf{p}}(A)|$ be a general member so that $S \neq \operatorname{Supp}(D)$. Then S is a normal surface by Lemma 3.7 and it is quasi-smooth along Γ. Moreover, for another general $T \in |\mathcal{I}_{\mathsf{p}}(A)|$, the multiplicity of $T|_S$ along Γ is 1, that is, we can write

$$T|_S = \Gamma + \Delta,$$

where Δ is an effective divisor on S such that $\Gamma \not\subset \operatorname{Supp}(\Delta)$. We see that Γ is a quasi-line, S is quasi-smooth at p_w, Γ passes through the $\frac{1}{2}(1,1)$ point p_w of S and $(K_S \cdot \Gamma) = 0$. It follows that

$$(\Gamma^2)_S = -2 + \frac{1}{2} = -\frac{3}{2},$$

by Remark 3.10. Hence,

$$(\Delta \cdot \Gamma)_S = (T|_S \cdot \Gamma)_S - (\Gamma^2)_S = 2.$$

The divisor $D|_S$ on S is effective, and we write $\frac{1}{2}D|_S = \gamma\Gamma + \Xi$, where $\gamma \geq 0$ and Ξ is an effective divisor on S such that $\Gamma \not\subset \mathrm{Supp}(\Xi)$. Since $\mathrm{Bs}\,|\mathcal{I}_{\mathsf{p}}(A)| = \Gamma$ and S is general, we may assume that $\mathrm{Supp}(\Xi)$ does not contain any component of $\mathrm{Supp}(\Delta)$. In particular, $(\Xi \cdot \Delta)_S \geq 0$. Note also that

$$(D|_S \cdot \Delta)_S = (T|_S \cdot \Delta)_S = ((A^3) - (T \cdot \Gamma)_S) = 2.$$

It follows that

$$2 = (D|_S \cdot \Delta)_S \geq 2\gamma(\Gamma \cdot \Delta)_S = 4\gamma,$$

which implies $\gamma \leq \frac{1}{2}$. We see that $(X, \frac{1}{2}D|_S)$ is not log canonical at p, and hence $(S, \Gamma + \Xi) = (S, \frac{1}{2}D|_S + (1-\gamma)\Gamma)$ is not log canonical at p. By the inversion of adjunction, we have

$$1 \geq \frac{1}{4} + \frac{3}{2}\gamma = ((\frac{1}{2}D|_S - \gamma\Gamma) \cdot \Gamma)_S = (\Delta \cdot \Gamma)_S \geq \mathrm{mult}_{\mathsf{p}}(\Delta|_\Gamma) > 1.$$

This is a contradiction and the inequality $\alpha_{\mathsf{p}}(X) \geq 1/2$ is proved.

6.2.b. The singular point of type $\frac{1}{2}(1,1,1)$

Let $\mathsf{p} = \mathsf{p}_w$ be the singular point of type $\frac{1}{2}(1,1,1)$. Note that the point $\mathsf{p} \in X$ is a QI center.

6.2.b.1. Case: p *is nondegenerate*

By a choice of coordinates, we can write

$$F = w^2 t + w f_3(x,y,z) + g_5(x,y,z,t),$$

where f_3, g_5 are nonzero homogeneous polynomials such that $f_3 \neq 0$ as a polynomial. Let $\varphi \colon Y \to X$ be the Kawamata blowup at p with exceptional divisor E.

Claim 22. $\mathrm{lct}_{\mathsf{p}}(X, H_t) \geq \frac{1}{2}$.

Proof of Claim 22. The lowest weight part of $F(x,y,z,0,1)$ with respect to $\mathrm{wt}(x,y,z) = (1,1,1)$ is f_3. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ unless f_3 is a cube of a linear form. Hence, it remains to prove the claim assuming that f_3 is a cube of a linear form. By a choice of coordinates, we may assume $f_3 = z^3$. Let S be the divisor on X defined by $x - \lambda y = 0$ for a general $\lambda \in \mathbb{C}$. By the quasi-smoothness of X, the polynomial F cannot be contained in the ideal $(z,t) \subset \mathbb{C}[x,y,z,t,w]$. This implies $g_5(x,y,0,0) \neq 0$, and hence $g_5(\lambda y, y, 0, 0) \neq 0$. By eliminating x, the surface S is isomorphic to the hypersurface in $\mathbb{P}(1,1,1,2)_{y,z,t,w}$ defined by

$$G := w^2 t + w z^3 + \alpha y^5 + z a_4 + t b_4 = 0,$$

where $a_4 = a_4(y,z)$, $b_4 = b_4(y,z,t)$ are homogeneous polynomials of degree 4 and $\alpha \neq 0$ is a constant. The lowest weight part of $G(x,z,0,1)$ with respect to $\mathrm{wt}(y,z) = (3,5)$ is $z^3 + \alpha y^5$ which defines a smooth point of $\mathbb{P}(3,5)_{y,z}$. By Lemma 3.27, $\mathrm{lct}_{\mathsf{p}}(S; H_t|_S) \geq 8/15$, and hence $\mathrm{lct}_{\mathsf{p}}(X; H_t) \geq 8/15$. Thus, the claim is proved. □

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X other than H_t. We can take $T \in |A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$, and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_t$ since $\{x,y,z,t\}$ isolates p. Then

$$3\,\mathrm{omult}_{\mathsf{p}}(D) < 2(D \cdot H_t \cdot T) = 5$$

since $\mathrm{omult}_{\mathsf{p}}(H_t) = 3$. This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq \frac{3}{5}$ and thus $\alpha_{\mathsf{p}}(X) \geq \frac{1}{2}$.

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6.2.b.2. Case: p *is degenerate*

In this case, we have $\alpha_{\mathsf{p}}(X) = 3/5$ by Proposition 5.16. Therefore, the proof of Theorem 6.1 for the family $\mathcal{F}_2$ is completed.

6.3. The family $\mathcal{F}_4$

This subsection is devoted to the proof of Theorem 6.1 for the family $\mathcal{F}_4$. In the following, let

$$X = X_6 \subset \mathbb{P}(1,1,1,2,2)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_4$ with defining polynomial $F = F(x,y,z,t,w)$.

6.3.a. Smooth points

Let p be a smooth point of X. We will prove $\alpha_{\mathsf{p}}(X) \ge 1/2$. We may assume $\mathsf{p} = \mathsf{p}_x$ by a choice of coordinates. The proof will be done by division into cases.

6.3.a.1. Case: Either $x^4 w \in F$ *or* $x^4 t \in F$

In this case, we have

$$\alpha_{\mathsf{p}}(X) \ge \frac{2}{1 \cdot 1 \cdot 2 \cdot (A^3)} = \frac{2}{3}$$

by Lemma 3.29.

6.3.a.2. Case: $x^4 w, x^4 t \notin F$

We can write

$$F = x^5 y + x^4 f_2 + x^3 f_3 + x^2 f_4 + x f_5 + f_6,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $t, w \notin f_2$.

We claim $\operatorname{lct}_{\mathsf{p}}(X; H_y) \ge 1/2$. This is obvious when $\operatorname{mult}_{\mathsf{p}}(H_y) \le 2$ and hence we assume $\operatorname{mult}_{\mathsf{p}}(H_y) \ge 3$. Then we can write

$$\bar{F} := F(1,0,z,t,w) = \sum_{i=2}^{6} f_i(0,z,t,w) = \alpha z^3 + \beta t z^2 + \gamma w z^2 + c(t,w) + h,$$

where $c(t,w) = f_6(0,0,z,t)$ and $h = h(y,t,w)$ is in the ideal $(y,t,w)^4$. By the quasi-smoothness of X, c cannot be a cube of a linear form. This implies that the cubic part of $\bar{F}$ is not a cube of a linear form. Thus, $\operatorname{lct}_{\mathsf{p}}(X; H_y) \ge 1/2$ by Lemma 3.28 and the claim is proved.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_y. We can take $T \in |2A|_{\mathbb{Q}}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \ge 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_y$ since $\{y,z,t,w\}$ isolates p. Then we have

$$2 \operatorname{mult}_{\mathsf{p}}(D) \le (D \cdot H_y \cdot T) = 2(A^3) = 3$$

since $\operatorname{mult}_{\mathsf{p}}(H_y) \ge 2$. This implies $\operatorname{lct}_{\mathsf{p}}(X; D) \ge 2/3$ and thus $\alpha_{\mathsf{p}}(X) \ge 1/2$.

6.3.b. Singular points of type $\frac{1}{2}(1,1,1)$

Let p be a singular point of type $\frac{1}{2}(1,1,1)$. Then we have $\alpha_{\mathsf{p}}(X) \ge 1/2$ by Proposition 5.18 (actually we have $\alpha_{\mathsf{p}}(X) \ge 2/3$ by the argument in Section 5.6.b). Therefore, the proof of Theorem 6.1 for the family $\mathcal{F}_4$ is completed.

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6.4. *The family* $\mathcal{F}_5$

This subsection is devoted to the proof Theorem 6.1 for the family $\mathcal{F}_5$. In the following, let

$$X = X_7 \subset \mathbb{P}(1,1,1,2,3)_{x,y,z,t,w}$$

be a member of family $\mathcal{F}_5$ with defining polynomial $F = F(x,y,z,t,w)$.

6.4.a. Smooth points

Let p be a smooth point of X. We will prove $\alpha_{\mathsf{p}}(X) \geq 1/2$. The proof will be done by division into cases.

6.4.a.1. Case: $\mathsf{p} \in U_x \cup U_y \cup U_z$

By a choice of coordinates x, y, z, we may assume $\mathsf{p} = \mathsf{p}_x$. By Lemma 3.29, we have

$$\alpha_{\mathsf{p}}(X) \geq \begin{cases} \frac{2}{1\cdot 1\cdot 2\cdot (A^3)} = \frac{6}{7}, & \text{if } x^4w \in F, \\ \frac{2}{1\cdot 1\cdot 3\cdot (A^3)} = \frac{4}{7}, & \text{if } x^4w \notin F \text{ and } x^5t \in F. \end{cases}$$

It remains to consider the case where $x^4w, x^5t \notin F$. In this case, we can write

$$F = x^6y + x^5f_2 + x^4f_3 + x^3f_4 + x^2f_5 + xf_6 + f_7,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $t \notin f_2$ and $w \notin f_3$.

Claim 23. $\mathrm{lct}_{\mathsf{p}}(X; H_y) \geq 1/2$.

Proof of Claim 23. This is obvious when $\mathrm{mult}_{\mathsf{p}}(H_y) = 2$, and we assume $\mathrm{mult}_{\mathsf{p}}(H_y) \geq 3$. It follows that each monomial appearing in F is contained in $(y) \cup (z,t,w)^3$. A monomial of degree $d \in \{2,3,4,5,6,7\}$ in variables y, z, t, w which is contained in $(y) \cup (z,t,w)^3$ is contained in $(y) \cup (z,t)^2$ except for the monomial w^2z of degree 7. Hence, we can write

$$F = x^6y + yg + h + \alpha w^2 z,$$

where $g = g(x,y,z,t,w) \in \mathbb{C}[x,y,z,t,w]$ and $h = h(x,z,t,w) \in (z,t)^2$. If $\alpha = 0$, then X is not quasi-smooth at any point of the nonempty set

$$(y = x^6 + g = z = t = 0) \subset \mathbb{P}(1,1,1,2,3).$$

Thus, $w^2z \in F$ and we see that $\bar{F} = F(1,0,z,t,w) \in (z,t,w)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^2z \in \bar{F}$ and $w^3 \notin \bar{F}$. By Lemma 3.28, we have $\mathrm{lct}_{\mathsf{p}}(X; H_y) \geq 1/2$, and the claim is proved. □

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_y. We can take a $\mathbb{Q}$-divisor $T \in |3A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_y$ since $\{y, z, t, w\}$ isolates p. Then

$$2\,\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot H_y \cdot T)_{\mathsf{p}} \leq (D \cdot H_y \cdot T) = 3(A^3) = \frac{7}{2}$$

since $\mathrm{mult}_{\mathsf{p}}(H_y) \geq 2$. This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 4/7$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$.

6.4.a.2. Case $\mathsf{p} \notin U_x \cup U_y \cup U_z$

If $wt^2 \in F$, then $X \setminus (U_x \cup U_y \cup U_z)$ consists of singular points. Hence, we have $wt^2 \notin F$ in this case, and p is contained in the quasi-line $\Gamma := (x = y = z = 0) \subset X$. We will show $\alpha_{\mathsf{p}}(X) \geq 1$. Assume to the contrary that $\alpha_{\mathsf{p}}(X) < 1$. Then there exists an irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ such that the pair (X, D)

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is not log canonical at p. Let $S \in |A|$ be a general member and write $D|_S = \gamma\Gamma + \Delta$, where $\gamma \geq 0$ is a rational number and Δ is an effective 1-cycle on S such that $\Gamma \not\subset \mathrm{Supp}(\Delta)$.

Claim 24. $(\Gamma^2)_S = -5/6$ *and* $\gamma \leq 1$.

Proof of Claim 24. We see that S has singular points of type $\frac{1}{2}(1,1)$ and $\frac{1}{3}(1,2)$ at p_t and p_w, respectively, and smooth elsewhere since $S \in |A|$ is general. Since Γ is a quasi-line on S passing through $\mathsf{p}_t, \mathsf{p}_w$ and $K_S = (K_X + S)|_S \sim 0$ by adjunction, we have

$$(\Gamma^2)_S = -2 + \frac{1}{2} + \frac{2}{3} = -\frac{5}{6}.$$

We choose a general member $T \in |A|$ which does not contain any component of Δ. This is possible since $\mathrm{Bs}\,|A| = \Gamma$. We write $T|_S = \Gamma + \Xi$, where Ξ is an effective divisor on S such that $\Gamma \not\subset \mathrm{Supp}(\Xi)$. We have

$$(D|_S \cdot \Xi)_S = (D|_S \cdot (T|_S - \Gamma))_S = \frac{7}{6} - \frac{1}{6} = 1,$$
$$(\Gamma \cdot \Xi)_S = (\Gamma \cdot (T|_S - \Gamma))_S = \frac{1}{6} + \frac{5}{6} = 1.$$

Note that Ξ does not contain any component of Δ by our choice of T, and hence

$$1 = (D|_S \cdot \Xi)_S = ((\gamma\Gamma + \Delta) \cdot \Xi)_S \geq \gamma(\Gamma \cdot \Xi)_S = \gamma,$$

as desired. □

The pair $(S, D|_S) = (S, \gamma\Gamma + \Delta)$ is not log canonical at p. Hence, the pair $(S, \Gamma + \Delta)$ is not log canonical at p since $\gamma \leq 1$. By the inversion of adjunction, we have $\mathrm{mult}_{\mathsf{p}}(\Delta|_\Gamma) > 1$ and thus

$$1 < \mathrm{mult}_{\mathsf{p}}(\Delta|_\Gamma) \leq (\Delta \cdot \Gamma)_S = ((D|_S - \gamma\Gamma) \cdot \Gamma)_S = \frac{1}{6} + \frac{5}{6}\gamma \leq 1.$$

This is a contradiction, and we have $\alpha_{\mathsf{p}}(X) \geq 1$.

6.4.b. The singular point of type $\frac{1}{2}(1,1,1)$

Let $\mathsf{p} = \mathsf{p}_t$ be the singular point of type $\frac{1}{2}(1,1,1)$.

6.4.b.1. Case: $t^2w \in F$

In this case, we have

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{2 \cdot 1 \cdot 1 \cdot (A^3)} = \frac{6}{7}$$

by Lemma 3.29.

6.4.b.2. Case: $t^2w \notin F$

Replacing x, y, z, we can write

$$F = t^3x + t^2f_3 + tf_5 + f_7,$$

where $f_i = f_i(x,y,z,w)$ is a quasi-homogeneous polynomial of degree i with $w \notin f_3$.

Claim 25. *If* $\mathrm{mult}_{\mathsf{p}}(H_x) \geq 3$, *then either* $w^2y \in F$ *or* $w^2z \in F$.

Proof of Claim 25. Suppose $w^2y, w^2z \notin F$. Then $h := F(0, y, z, t, w)$ is contained in the ideal $(y, z)^2 \subset \mathbb{C}[y, z, t, w]$, and we can write $F = xg + h$, where $g = g(x, y, z, t, w)$. We see that X is not quasi-smooth at any point in the nonempty subset

$$(x = y = z = g = 0) \subset \mathbb{P}(1, 1, 1, 2, 3).$$

This is a contradiction, and the claim is proved. □

We set $\bar{F} := F(0, y, z, 1, w)$. By Claim 25, either $\bar{F} \in (y, z, w)^2 \setminus (y, z, w)^3$ or $\bar{F} \in (y, z, w)^3$ and the cubic part of $\bar{F}$ is not a cube of a linear form since $w^3 \notin \bar{F}$. By Lemma 3.28, we have $\alpha_{\mathsf{p}}(X) \geq 1/2$ since $\mathsf{p} \in X$ is not a maximal center.

6.4.c. Singular point of type $\frac{1}{3}(1, 1, 2)$

Let $\mathsf{p} = \mathsf{p}_w$ be the singular point of type $\frac{1}{3}(1, 1, 2)$. We can write

$$F = w^2x + w(\alpha t^2 + ta_2(y, z) + b_4(y, z)) + f_7(x, y, z, t),$$

where $\alpha \in \mathbb{C}$ and $a_2 = a_2(x, y), b_4 = b_4(y, z), f_7 = f_7(x, y, z, t)$ are quasi-homogeneous polynomials of degree 2, 4, 7, respectively. Let $q = q_{\mathsf{p}}$ be the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ be the preimage of p.

6.4.c.1. Case: $\alpha \neq 0$

We have $\mathrm{mult}_{\mathsf{p}}(H_x) = 2$ and $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_x. We can take a $\mathbb{Q}$-divisor $T \in |A|_{\mathbb{Q}}$ such that $\mathrm{mult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$ since $\{x, y, z\}$ isolates p. Then

$$2\,\mathrm{omult}_{\mathsf{p}}(D) \leq (q^*D \cdot q^*H_x \cdot q^*T)_{\check{\mathsf{p}}} \leq 3(D \cdot H_x \cdot T) = \frac{7}{2}.$$

This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 4/7$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$.

6.4.c.2. Case $\alpha = 0$ and $a_2 \neq 0$

The cubic part of $F(0, y, z, t, 1)$ is ta_2, and, by Lemma 3.28, we have $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 1/2$. Let $D \sim_{\mathbb{Q}} A$ be an irreducible $\mathbb{Q}$-divisor on X other than H_x. Then we can take a general $T \in |\mathcal{I}_{\mathsf{p}}(2A)| = |2A|$ which does not contain any component of $D \cap H_x$ since $\mathrm{Bs}\,|2A| = \mathsf{p}$. We see that T is defined by $t - q(x, y, z) = 0$ on X, where $q \in \mathbb{C}[x, y, z]$ is a general quadratic form. Let $\rho = \rho_{\mathsf{p}} \colon \check{U}_{\mathsf{p}} \to U_{\mathsf{p}}$ be the orbifold chart of X containing p and let $\check{\mathsf{p}}$ be the preimage of p. It is then easy to see that the effective 1-cycle $\rho^*H_x \cdot \rho^*T$ on $\check{U}_{\mathsf{p}}$ has multiplicity 4 at $\check{\mathsf{p}}$. Then we have

$$4\,\mathrm{omult}_{\mathsf{p}}(D) \leq (\rho^*D \cdot \rho^*H_x \cdot \rho^*T)_{\check{\mathsf{p}}} \leq 3(D \cdot H_x \cdot T) = 7.$$

This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 4/7$ and thus $\alpha_{\mathsf{p}}(X) \geq 4/7$.

6.4.c.3. Case: $\alpha = a_2 = 0$ and $b_4 \neq 0$

By similar arguments as in the proof of Claim 25, we see that either $t^3y \in f_7$ or $t^3z \in f_7$. We choose z and t so that $b_4(0, z) = z^4$ and $\mathrm{coeff}_{f_7}(t^3z) = 1$. Then we have

$$F(0, 0, z, t, 1) = z^4 + t^3z + \beta t^2z^3 + \gamma tz^5 + \delta z^7,$$

where $\beta, \gamma, \delta \in \mathbb{C}$. The lowest weight part of $F(0,0,z,t,1)$ with respect to $\mathrm{wt}(z,t) = (1,1)$ is $z^4 + t^3 z$ which defines four distinct points of $\mathbb{P}^1_{z,t}$. Hence, we have

$$\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq \mathrm{lct}_{\mathsf{p}}(H_y; H_x|_{H_y}) \geq \frac{1}{2}$$

by Lemma 3.27. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_x. We can take a $\mathbb{Q}$-divisor $T \in |2A|_{\mathbb{Q}}$ such that $\mathrm{omult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_x$ since $\{x, y, z, t\}$ isolates p. Then

$$4\,\mathrm{omult}_{\mathsf{p}}(D) \leq (q^*D \cdot q^*H_x \cdot q^*T)_{\check{\mathsf{p}}} \leq 3(D \cdot H_x \cdot T) = 7$$

since $\mathrm{omult}_{\mathsf{p}}(H_x) = 4$. This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 4/7$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$.

6.4.c.4. Case: $\alpha = a_2 = b_4 = 0$

In this case, the point $\mathsf{p} \in X$ is a degenerate QI center and we have $\alpha_{\mathsf{p}}(X) = 4/7$ by Proposition 5.16.

6.5. *The family $\mathcal{F}_8$*

This subsection is devoted to the proof of Theorem 6.1 for the family $\mathcal{F}_8$. In the following, let

$$X = X_9 \subset \mathbb{P}(1,1,1,3,4)_{x,y,z,t,w}$$

be a member of $\mathcal{F}_8$ with defining polynomial $F = F(x,y,z,t,w)$.

6.5.a. Smooth points

Let $\mathsf{p} \in X$ be a smooth point. We will prove $\alpha_{\mathsf{p}}(X) \geq 1/2$. We may assume $\mathsf{p} = \mathsf{p}_x$. The proof will be done by division into cases.

6.5.a.1. Case: $x^5 w \in F$

We can write

$$F = x^5 w + x^4 f_5 + x^3 f_6 + x^2 f_7 + x f_8 + f_9,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i. We have $\mathrm{mult}_{\mathsf{p}}(H_w) = 3$ since $t^3 \in f_9$. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor on X. Let $S \in |\mathcal{I}_{\mathsf{p}}(A)|$ be a general member so that $\mathrm{Supp}(D) \neq S$. Since $\{y, z, w\}$ isolates p, we can take a $\mathbb{Q}$-divisor $T \in |A|_{\mathbb{Q}}$ such that $\mathrm{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot S$ and $\mathrm{mult}_{\mathsf{p}}(T) \geq 3/4$ (Note that T is one of H_y, H_z and $\frac{1}{4} H_w$). Then we have

$$\frac{3}{4}\,\mathrm{mult}_{\mathsf{p}}(D) \leq (D \cdot S \cdot T)_{\mathsf{p}} \leq (D \cdot S \cdot T) = \frac{3}{4}.$$

This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$ and thus $\alpha_{\mathsf{p}}(X) \geq 1$.

6.5.a.2. Case: $x^5 w \notin F$ and $x^6 t \in F$

We can write

$$F = x^6 t + x^5 f_4 + x^4 f_5 + x^3 f_6 + x^2 f_7 + x f_8 + f_9,$$

where $f_i = f_i(y,z,t,w)$ is a quasi-homogeneous polynomial of degree i with $w \notin f_4$. Let $S, T \in |\mathcal{I}_{\mathsf{p}}(A)|$ be general members. Note that S is smooth at p. The intersection $S \cap T$ is isomorphic to the subscheme in $\mathbb{P}(1_x, 3_t, 4_w)$ defined by the equation $F(x, 0, 0, t, w) = 0$, and we can write

$$F(x, 0, 0, t, w) = x^6 t + \alpha x^3 t^2 + \beta x^2 wt + \gamma x w^2 + t^3,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$.

Claim 26. *If $\gamma \neq 0$, then $S \cdot T = \Gamma$, where Γ is an irreducible and reduced curve of degree $3/4$ that is smooth at* p.

Proof of Claim 26. Suppose $\gamma \neq 0$. Then it is easy to see that the polynomial $F(x, 0, 0, t, w)$ is irreducible. Hence, the curve

$$\Gamma = (y = z = F(x, 0, 0, t, w) = 0) \subset \mathbb{P}(1, 1, 1, 3, 4).$$

is irreducible and reduced. It is also obvious that $\deg \Gamma = 3/4$ and Γ is smooth at p. □

If $\gamma \neq 0$, then we have $\alpha_{\mathsf{p}}(X) \geq 1$ by Claim 26 and Lemma 3.17.

In the following, we consider the case where $\gamma = 0$. We set

$$\Delta = (y = z = t = 0) \subset \mathbb{P}(1, 1, 1, 3, 4),$$

which is a quasi-line of degree $1/4$ passing through p. Note that Δ is smooth at p.

Claim 27. *If $\gamma = 0$ and $\beta \neq 0$, then $T|_S = \Delta + \Xi$, where Ξ is an irreducible and reduced curve which does not pass through p. Moreover, the intersection matrix $M(\Delta, \Xi)$ satisfies the condition $(\star)$.*

Proof of Claim 27. We have

$$F(x, 0, 0, t, w) = t(x^6 + \alpha x^3 t + \beta x^2 w + t^2),$$

and the polynomial $x^6 + \alpha x^3 t + \beta x^2 w + t^2$ is irreducible since $\beta \neq 0$. It follows that $T|_S = \Delta + \Xi$, where

$$\Xi = (y = z = x^6 + \alpha x^3 t + \beta x^2 w + t^2 = 0) \subset \mathbb{P}(1, 1, 1, 3, 4)$$

is an irreducible and reduced curve of degree $1/2$ that does not pass through p. We have $\Delta \cap \Xi = \{\mathsf{p}_w, \mathsf{q}\}$, where $\mathsf{q} = (1 : 0 : 0 : 0 : -1/\beta)$. It is easy to see that S is quasi-smooth at p_w and q, hence S is quasi-smooth along Δ by Lemma 3.9. We have $\mathrm{Sing}_\Gamma(S) = \{\mathsf{p}_w\}$ and $\mathsf{p}_w \in S$ is of type $\frac{1}{4}(1, 3)$. By Remark 3.10, we have

$$(\Delta^2)_S = -2 + \frac{3}{4} = -\frac{5}{4}.$$

By taking intersection number of $T|_S = \Delta + \Xi$ and Δ and then $T|_S$ and Ξ, we have

$$(\Delta \cdot \Xi) = \frac{3}{2}, \quad (\Xi^2)_S = -1.$$

It follows that the intersection matrix $M(\Delta, \Xi)$ satisfies the condition $(\star)$. □

Claim 28. *If $\gamma = \beta = 0$ and $\alpha \neq \pm 2$, then $T|_S = \Delta + \Theta_1 + \Theta_2$, where Θ_1 and Θ_2 are distinct quasi-lines which does not pass through p. Moreover, the intersection matrix $M(\Delta, \Theta_1, \Theta_2)$ satisfies the condition $(\star)$.*

Proof of Claim 28. We have

$$F(x, 0, 0, t, w) = t(x^6 + \alpha x^3 t + t^2) = t(t - \lambda x^3)(t - \lambda^{-1} x^3),$$

where $\lambda \neq 0, 1$ is a complex number such that $\alpha = \lambda + \lambda^{-1}$. Hence, we have

$$T|_S = \Delta + \Theta_1 + \Theta_2,$$

where

$$\Xi_1 = (y = z = t - \lambda x^3 = 0), \quad \Xi_2 = (y = z = t - \lambda^{-1} x^3 = 0)$$

are both quasi-lines of degree $1/4$ that do not pass through p. We have $\Delta \cap (\Theta_1 \cup \Theta_2) = \{\mathsf{p}_w\}$ and S is clearly quasi-smooth at p_w. It follows that S is quasi-smooth along Γ by Lemma 3.9, and $\mathrm{Sing}_\Delta(S) = \{\mathsf{p}_w\}$, where $\mathsf{p}_w \in S$ is of type $\frac{1}{4}(1,3)$. Thus, we have

$$(\Delta^2)_S = -\frac{5}{4}.$$

By similar arguments, we see that S is quasi-smooth along Θ_i and $\mathrm{Sing}_\Theta(S) = \{\mathsf{p}_w\}$ for $i = 1, 2$, and hence

$$(\Theta_i^2)_S = -\frac{5}{4}.$$

By taking intersection number of $T|_S = \Delta + \Theta_1 + \Theta_2$ and $\Delta, \Theta_1, \Theta_2$, we conclude

$$(\Delta \cdot \Theta_1)_S = (\Delta \cdot \Theta_2)_S = (\Theta_1 \cdot \Theta_2)_S = \frac{3}{4}.$$

It is then straightforward to see that $M(\Delta, \Theta_1, \Theta_2)$ satisfies the condition $(\star)$. $\square$

Claim 29. *If $\gamma = \beta = 0$ and $\alpha = \pm 2$, then $T|_S = \Delta + 2\Theta$, where Θ is an irreducible and reduced curve which does not pass through p. Moreover, the intersection matrix $M(\Delta, \Theta)$ satisfies the condition $(\star)$.*

Proof of Claim 29. Without loss of generality, we may assume $\alpha = -2$. We have

$$F(x, 0, 0, t, w) = t(t - x^3)^2,$$

and hence

$$T|_S = \Delta + 2\Theta,$$

where

$$\Theta = (y = z = t - x^3 = 0) \subset \mathbb{P}(1, 1, 1, 3, 4)$$

is a quasi-line of degree $1/4$ that does not pass through p. By the same arguments as in Claim 28, we have

$$(\Delta^2)_S = -\frac{5}{4}.$$

Then, by taking intersection number of $T|_S = \Delta + 2\Theta$ and Δ, Θ, we have

$$(\Delta \cdot \Theta)_S = \frac{3}{4}, \quad (\Theta^2)_S = -\frac{1}{4}.$$

Thus, the matrix $M(\Delta, \Theta)$ satisfies the condition $(\star)$. $\square$

By Claims 27, 28, 29 and Lemma 3.21, we conclude

$$\alpha_{\mathsf{p}}(X) \geq \min\left\{1, \ \frac{1}{(A^3) + 1 - \deg \Delta}\right\} = \frac{2}{3}.$$

6.5.a.3. Case: $x^5 w, x^6 t \notin F$

Replacing y and z, we can write

$$F = x^8 y + x^7 f_2 + x^6 f_3 + x^5 f_4 + x^4 f_5 + x^3 f_6 + x^2 f_7 + x f_8 + f_9,$$

where $f_i = f_i(y, z, t, w)$ is a homogeneous polynomial of degree i with $w \notin f_4$ and $t \notin f_3$. Note that we have $2 \leq \operatorname{mult}_{\mathsf{p}}(H_y) \leq 3$ since $t^3 \in F$.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_y. We can take a $\mathbb{Q}$-divisor $T \in |4A|_{\mathbb{Q}}$ such that $\operatorname{mult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_y$ since $\{y, z, t, w\}$ isolates p. Then

$$2 \operatorname{mult}_{\mathsf{p}}(D) \leq (D \cdot H_y \cdot T)_{\mathsf{p}} \leq (D \cdot H_y \cdot T) = 4(A^3) = 3.$$

This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 2/3$ and thus it remains to show that $\operatorname{lct}(X; H_y) \geq 1/2$.

Suppose that either $\operatorname{mult}_{\mathsf{p}}(H_y) = 2$ or $\operatorname{mult}_{\mathsf{p}}(H_y) = 3$, and the cubic part of $\bar{F} := F(1, 0, z, t, w)$ is a cube of a linear form. Then $\operatorname{lct}_{\mathsf{p}}(X; H_y) \geq 1/2$ by Lemma 3.28, and we are done.

In the following, we assume that $\operatorname{mult}_{\mathsf{p}}(H_y) = 3$ and the cubic part of $\bar{F}$ is a cube of a linear form. Since $t^3 \in \bar{F}$ and $w^3 \notin \bar{F}$, we may assume that the cubic part of $\bar{F}$ is t^3 after replacing t.

We claim $w^2 z \in F$. We see that a monomial other than $w^2 z$ which appears in $\bar{F}$ with nonzero coefficient is contained in the ideal $(z, t)^2 \subset \mathbb{C}[z, t, w]$. We can write $F = yG + F(x, 0, z, t, w)$ for some homogeneous polynomial $G(x, y, z, t, w)$. If $w^2 z \notin F$, then $F(x, 0, z, t, w) \in (z, t)^2$ and X is not quasi-smooth at any point contained in the nonempty set

$$(y = z = t = G = 0) \subset \mathbb{P}(1, 1, 1, 3, 4).$$

This is a contradiction and the claim is proved.

Then we may assume $\operatorname{coeff}_F(w^2 z) = 1$ and, by replacing t and w, we can write

$$\begin{aligned}\bar{F} = \alpha_4 z^4 + \alpha_5 z^5 + (\beta t z^3 + \alpha_6 z^6) + (\gamma w z^3 + \delta t z^4 + \alpha_7 z^7) + \\ + (\varepsilon w z^4 + \zeta t^2 z^2 + \eta t z^5 + \alpha_8 z^8) + (w^2 z + t^3 + \theta t^2 z^3 + \lambda t z^6 + \alpha_9 z^9),\end{aligned}$$

where $\alpha_4, \dots, \alpha_9, \beta, \gamma, \dots, \lambda \in \mathbb{C}$. The lowest weight part of $\bar{F}$ with respect to $\operatorname{wt}(z, t, w) = (6, 8, 9)$ is $G := \alpha_4 z^4 + w^2 z + t^3$. We set $\mathbb{P} = \mathbb{P}(6, 8, 9)$. Then $\mathbb{P}^{\mathrm{wf}} = \mathbb{P}(1, 4, 3)_{\tilde{z}, \tilde{t}, \tilde{w}}$ and, by Lemma 3.27, we have

$$\operatorname{lct}_{\mathsf{p}}(X; H_y) \geq \min\left\{\frac{23}{24}, \ \operatorname{lct}(\mathbb{P}^{\mathrm{wf}}, \operatorname{Diff}; \Gamma)\right\},$$

where

$$\begin{aligned}\operatorname{Diff} &= \frac{2}{3} H^{\mathrm{wf}}_{\tilde{t}} + \frac{1}{2} H^{\mathrm{wf}}_{\tilde{w}}, \\ \Gamma &= \mathcal{D}^{\mathrm{wf}}_G = (\alpha_4 \tilde{z}^4 + \tilde{w} \tilde{z} + \tilde{t} = 0) \subset \mathbb{P}(1, 4, 3),\end{aligned}$$

are ($\mathbb{Q}$-)divisors on $\mathbb{P}^{\mathrm{wf}}$ with $H^{\mathrm{wf}}_{\tilde{t}} = (\tilde{t} = 0)$ and $H^{\mathrm{wf}}_{\tilde{w}} = (\tilde{w} = 0)$. It is easy to see that any pair of curves $H^{\mathrm{wf}}_{\tilde{t}}, H^{\mathrm{wf}}_{\tilde{w}}$ and Γ intersect transversally. If $\alpha_4 \neq 0$, then $H^{\mathrm{wf}}_{\tilde{t}} \cap H^{\mathrm{wf}}_{\tilde{w}} \cap \Gamma = \emptyset$, and thus $\operatorname{lct}(\mathbb{P}^{\mathrm{wf}}, \operatorname{Diff}; \Gamma) = 1$. If $\alpha_4 = 0$, then $H^{\mathrm{wf}}_{\tilde{t}} \cap H^{\mathrm{wf}}_{\tilde{w}} \cap \Gamma = \{\mathsf{p}_{\tilde{z}}\}$. In this case, by consider the the blowup at $\mathsf{p}_{\tilde{z}}$, we can confirm the equality $\operatorname{lct}(\mathbb{P}^{\mathrm{wf}}, \operatorname{Diff}; \Gamma) = 5/6$. Thus, we have $\operatorname{lct}_{\mathsf{p}}(X; H_y) \geq 5/6$, and the proof is completed.

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6.5.b. The singular point of type $\frac{1}{4}(1,1,3)$

Let $\mathsf{p} = \mathsf{p}_w$ be the singular point of type $\frac{1}{4}(1,1,3)$. We can write

$$F = w^2 x + w(ta_2(y,z) + b_5(y,z)) + f_9(x,y,z,t),$$

where $a_2 = a_2(y,z)$, $b_5 = b_5(y,z)$ and $f_9 = f_9(x,y,z,t)$ are homogeneous polynomials of degrees 2, 5 and 9, respectively.

Suppose that $a_2 \neq 0$ as a polynomial. Then $\bar{F} := F(0,y,z,t,1) \in (y,z,t)^3$ and its cubic part $ta_2 + t^3$ is not a cube of a linear form. It follows that $\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq 2/3$ by Lemma 3.28. Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than H_x. Since the set $\{x,y,z\}$ isolates p, we can take a $\mathbb{Q}$-divisor $T \in |A|_{\mathbb{Q}}$ such that $\mathrm{omult}_{\mathsf{p}}(T) \geq 1$ and $\mathrm{Supp}(T)$ does not contain any component of $D \cdot H_x$. We have $\mathrm{omult}_{\mathsf{p}}(H_x) = 3$. It follows that

$$3\,\mathrm{omult}_{\mathsf{p}}(D) \leq (q^*D \cdot q^*H_x \cdot q^*T)_{\check{\mathsf{p}}} \leq 4(D \cdot H_x \cdot T) = 3,$$

where $q = q_{\mathsf{p}}$ is the quotient morphism of $\mathsf{p} \in X$ and $\check{\mathsf{p}}$ is the preimage of p via q. This shows $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$ and thus $\alpha_{\mathsf{p}}(X) \geq 1/2$.

Finally, suppose that $a_2 = 0$ and $b_5 \neq 0$. Replacing y and z, we may assume $z^5 \in b_5$ and $\mathrm{coeff}_{b_5}(z^5) = 1$. We may also assume that $\mathrm{coeff}_{f_9}(t^3) = 1$ by rescaling t. Then we have

$$F(0,0,z,t,1) = z^5 + f_9(0,0,z,t).$$

The lowest weight part with respect to the weight $\mathrm{wt}(z,t) = (3,5)$ is $z^5 + t^3$ and thus

$$\mathrm{lct}_{\mathsf{p}}(X; H_x) \geq \mathrm{lct}_{\mathsf{p}}(H_y; H_x|_{H_y}) = \frac{8}{15}.$$

We have $\mathrm{omult}_{\mathsf{p}}(H_x) = 3$ and the set $\{x,y,z\}$ isolates p. Hence, by the same argument as in the the case $a_2 \neq 0$, we have $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$ for any irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ other than H_x. Thus, $\alpha_{\mathsf{p}}(X) \geq 1/2$.

Suppose that $a_2 = b_5 = 0$. Then we have $\alpha_{\mathsf{p}}(X) = 5/9$ by Proposition 5.16, and the proof is completed.

Remark 6.2. The singular point $\mathsf{p} \in X$ of type $\frac{1}{4}(1,1,3)$ is a QI center. When p is nondegenerate, the above proof shows that $\mathrm{lct}_{\mathsf{p}}(X; D) \geq 1$ for any irreducible $\mathbb{Q}$-divisor $D \in |A|_{\mathbb{Q}}$ other than H_x and $\mathrm{lct}_{\mathsf{p}}(X; H_x) > 1/2$.

7. Further results and discussion on related problems

7.1. Birationally superrigid Fano 3-folds of higher codimensions

We can embed a Fano 3-fold into a weighted projective space by choosing (minimal) generators of the anticanonical graded ring. We consider embedded Fano 3-folds. We have satisfactory results on the classification of Fano 3-folds of low codimensions ([IF00], [CCC11], [ABR02]), and the following are known for their birational (super)rigidity.

- Fano 3-folds of codimension 2 are all weighted complete intersections and they consist of 85 families. Among them, there are exactly 19 families whose members are birationally rigid ([Oka14], [AZ16]).
- Fano 3-folds of codimension 3 consist of 69 families of so-called Pfaffian Fano 3-folds and one family of complete intersections of three quadrics in $\mathbb{P}^6$. Among them, there are exactly three families whose members are birationally rigid ([AO18]).
- Constructions of many families of Fano 3-folds of codimension 4 has been known (see, e.g., [BKR12], [CD18]), but their classification is not completed. There are at least two families of birationally superrigid Fano 3-folds of codimension 4 ([Oka20a]).

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For birationally rigid Fano 3-folds of codimension 2 and 3, K-stability and existence of KE metrics are known under some generality assumptions.

Theorem 7.1 [KOW18]. *Let X be a general quasi-smooth Fano 3-folds of codimension $c \in \{2, 3\}$ which is birationally rigid. We assume that X is a complete intersection of a quadric and cubic in $\mathbb{P}^5$ when $c = 2$. Then $\alpha(X) \geq 1$, X is K-stable and admits a KE metric.*

Theorem 7.2 [Zhu20b, Theorem 1.3]. *Let X be a smooth complete intersection of a quadric and cubic in $\mathbb{P}^5$. Then X is K-stable and admits a KE metric.*

Question 7.3. Can we conclude K-stability for any quasi-smooth Fano 3-fold of codimension 2 and 3 which is birationally (super)rigid? How about for Fano 3-folds of codimension 4 or higher?

7.2. Lower bound of alpha invariants

In the context of Theorem 1.6, the following is a very natural question to ask.

Question 7.4. Is it true that $\alpha(X) \geq 1/2$ (or $\alpha(X) > 1/2$) for any birationally superrigid Fano variety? If yes, can we find a lower bound better than $1/2$?

The following example suggests that the number $1/2$ is optimal (or the lower bound can be even smaller).

Example 7.5. For an integer $a \geq 2$, let X_a be a weighted hypersurfaces of degree $2a + 1$ in $\mathbb{P}(1^{a+2}, a) = \operatorname{Proj} \mathbb{C}[x_1, \ldots, x_{a+2}, y]$, given by the equation

$$y^2 x_1 + f(x_1, \ldots, x_{a+2}) = 0,$$

where f is a general homogeneous polynomial of degree $2a + 1$. Then X_a is a quasi-smooth Fano weighted hypersurface of dimension $a + 1$ and Picard number 1 with the unique singular point p of type

$$\frac{1}{a}(\overbrace{1, \ldots, 1}^{a+1}).$$

The singularity $\mathsf{p} \in X$ is terminal. By the same argument as in the proof of Proposition 5.16, we obtain

$$\alpha(X) \leq \alpha_{\mathsf{p}}(X) = \operatorname{lct}_{\mathsf{p}}(X; H_{x_1}) = \frac{a+1}{2a+1}.$$

When $a = 2$, $X_a = X_2$ is a member of the family $\mathcal{F}_2$ and it is birationally superrigid. We expect that X_a is birationally superrigid, although this is not proved at all when $a \geq 3$. If X_a is birationally superrigid for $a \gg 0$, then it follows that there exists a sequence of birationally superrigid Fano varieties whose alpha invariants are arbitrary close to (or less than) $\frac{1}{2}$.

Question 7.6. Let X_a be as in Example 7.5. Is X_a birationally superrigid for $a \geq 3$?

7.3. Existence of KE metrics

For a quasi-smooth Fano 3-fold weighted hypersurface of index 1 which is strictly birationally rigid, we are unable to conclude the existence of a KE metric as a direct consequence of Theorem 1.8. However, for a Fano variety X of dimension n with only quotient singularities, the implication

$$\alpha(X) > \frac{n}{n+1} \implies \text{existence of a KE metric on } X$$

is proved in [DK01, Section 6]. The aim of this section is to prove the existence of KE metrics on quasi-smooth members of suitable families.

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We set

$$\mathsf{I}'_{\mathrm{KE}} = \{42, 44, 45, 61, 69, 74, 76, 79\} \subset \mathsf{I}_{\mathrm{BR}},$$

and

$$\mathsf{I}_{\mathrm{KE}} = \mathsf{I}_{\mathrm{BSR}} \sqcup \mathsf{I}'_{\mathrm{KE}}.$$

Note that $|\mathsf{I}_{\mathrm{KE}}| = 56$. For a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}$, the mark 'KE' is given in the right-most column of Table 7 if and only if $\mathsf{i} \in \mathsf{I}_{\mathrm{KE}}$.

Theorem 7.7. *For a member X of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}'_{\mathrm{KE}}$, we have*

$$\alpha(X) > \frac{3}{4}.$$

In particular, any member of a family $\mathcal{F}_{\mathsf{i}}$ with I_{KE} admits a KE metric and is K-stable.

Proof. By Corollary 1.9 and the above arguments, it is enough to prove the first assertion. Let

$$X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)_{x,y,z,t,w}$$

be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}'_{\mathrm{KE}}$, where we assume $a_1 \le a_2 \le a_3 \le a_4$. Note that $1 < a_1 < a_2$.

Claim 30. $\alpha_{\mathsf{p}}(X) \ge 1$ *for any* $\mathsf{p} \in U_1 \cap \mathrm{Sm}(X)$.

Proof of Claim 30. Let p be a smooth point of X contained in U_1.

Suppose $\mathsf{i} = 42$. Then $d = 20$ is divisible by $a_4 = 10$ and $a_2 a_3 (A^3) = 1$. By Lemma 4.3, we have $\alpha_{\mathsf{p}}(X) \ge 1$.

Suppose $\mathsf{i} \in \{69, 74, 76, 79\}$. Then $a_2 a_4 (A^3) \le 1$. By Lemma 4.2, we have $\alpha_{\mathsf{p}}(X) \ge 1$ in this case.

Suppose $\mathsf{i} \in \{44, 45, 61\}$. Then $a_3 a_4 (A^3) \le 2$. We may assume $\mathsf{p} = \mathsf{p}_x$. Then we have $\alpha_{\mathsf{p}}(X) \ge 2/a_3 a_4 (A^3) \ge 1$ by Lemma 3.29. This completes the proof. □

Claim 31. $\alpha_{\mathsf{p}}(X) \ge 1$ *for any* $\mathsf{p} \in (H_x \setminus L_{xy}) \cap \mathrm{Sm}(X)$.

Proof of Claim 31. This follows immediately from Proposition 4.8. □

Claim 32. $\alpha_{\mathsf{p}}(X) \ge 43/54 > 3/4$ *for any* $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$.

Proof of Claim 32. Let p be a smooth point of X contained in L_{xy}. Suppose that X is a member of one of the families listed in Tables 1 or 2, that is, X is a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \{44, 45, 61, 69, 74, 76, 79\}$. Then the claim follows immediately from Proposition 4.10.

Suppose $\mathsf{i} = 42$. Then, by the proof of Proposition 4.11 (see Section 4.4.b), either $\alpha_{\mathsf{p}}(X) \ge 1$ for any $\mathsf{p} \in L_{xy} \cap \mathrm{Sm}(X)$ or X satisfies the assumption of Lemma 4.14. In the latter case, we have $\alpha_{\mathsf{p}}(X) \ge 43/54$ by Remark 4.15. This completes the proof. □

By Claims 30, 31 and 32, we have $\alpha_{\mathsf{p}}(X) \ge 3/4$ for any smooth point $\mathsf{p} \in X$. It remains to consider singular points.

Claim 33. $\alpha_{\mathsf{p}}(X) > 3/4$ *for any* $\mathsf{p} \in \mathrm{Sing}(X)$.

Proof of Claim 33. Let $\mathsf{p} \in X$ be a singular point. If the subscript $\heartsuit$ (resp. $\diamondsuit$) is given in Table 7, then $\alpha_{\mathsf{p}}(X) \ge 1$ by Proposition 5.2 (resp. Proposition 5.3). It remains to consider the case where $\mathsf{i} = 42$ and p is of type $\frac{1}{5}(1, 2, 3)$. In this case, we have $\alpha_{\mathsf{p}}(X) \ge 1$ by the proof of Proposition 5.18 (see Section 5.6.c). □

This completes the proof of Theorem 7.7. □

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7.4. Birational rigidity and K-stability

7.4.a. Generalizations of the conjecture

Birational superrigidity is a very strong property. It is natural to relax the assumption of birational superrigidity to birational rigidity in Conjecture 1.1, and we still expect a positive answer to the following.

Conjecture 7.8 [KOW18, Conjecture 1.9]. *A birationally rigid Fano variety is K-stable.*

We explain the situation for smooth Fano 3-folds. There are exactly two families of smooth Fano 3-folds which is strictly birationally rigid: One is the family of complete intersections of a quadric and cubic in $\mathbb{P}^6$ ([IP96]), and another is the family of double covers V of a smooth quadric Q of dimension 3 branched along a smooth surface degree 8 on Q ([Isk80]). Former Fano 3-folds are K-stable and admit KE metrics ([Zhu20b]), and so are the latter Fano 3-folds (this follows from [Der16a] since Q is K-semistable). More evidence is already provided by Theorems 1.2 and 7.1, and we will provide further evidences in the next subsection (see Corollary 7.13).

It may be interesting to consider further generalization of Conjecture 7.8. According to systematic studies of Fano 3-folds of codimension 2 [Oka14; Oka18; Oka20b], existence of many birationally birigid Fano 3-folds are verified. Here, a Fano variety X of Picard number 1 is *birationally birigid* if there exists a Fano variety X' of Picard number 1 which is birational but not isomorphic to X, and up to isomorphism $\{X, X'\}$ is all the Mori fiber space in the birational equivalence class of X. Extending the birigidity, tririgidity and so on notion of solid Fano variety is introduced in [AO18]: A Fano variety of Picard number 1 is *solid* if any Mori fiber space in the birational equivalence class is a Fano variety of Picard number 1. Solid Fano varieties are expected to behave nicely in moduli ([Zhu20a]). Only some evidence is known ([KOW19]) for the following question.

Question 7.9. Is it true that any solid Fano variety is K-stable?

7.4.b. On K-stability for 95 families

For strictly birationally rigid members of the 95 families, we are unable to conclude K-stability by Theorem 1.8, except for those treated in Theorem 7.7. The aim of this subsection is to prove K-stability for all the quasi-smooth members of suitable families indexed by I_{BR}. This will be done by combining the inequality $\alpha \ge 1/2$ obtained by Theorem 1.8 and an additional information on local movable alpha invariants which are introduced below.

Definition 7.10. Let X be a Fano variety of Picard number 1 and $\mathsf{p} \in X$ a point. For a nonempty linear system $\mathcal{M}$ on X, we define $\lambda_{\mathcal{M}} \in \mathbb{Q}_{>0}$ to be the rational number such that $\mathcal{M} \sim_{\mathbb{Q}} -\lambda_{\mathcal{M}} K_X$. For a movable linear system $\mathcal{M}$ on X and a positive rational number μ, we define the *movable log canonical threshold* of $(X, \mu\mathcal{M})$ at p to be the number

$$\mathrm{lct}^{\mathrm{mov}}_{\mathsf{p}}(X; \mu\mathcal{M}) = \sup\{\, c \in \mathbb{Q}_{\ge 0} \mid (X, c\mu\mathcal{M}) \text{ is log canonical at } \mathsf{p}\,\},$$

and then we define the *movable alpha invariant of X at* p as

$$\alpha^{\mathrm{mov}}_{\mathsf{p}}(X) = \inf\{\, \mathrm{lct}^{\mathrm{mov}}_{\mathsf{p}}(X, \lambda_{\mathcal{M}}^{-1}\mathcal{M}) \mid \mathcal{M} \text{ is a movable linear system on } X\}.$$

Proposition 7.11 (cf. [SZ19, Corollary 3.1]). *Let X be a quasi-smooth Fano 3-fold weighted hypersurface of index 1. Assume that, for any maximal center $\mathsf{p} \in X$, we have*

$$\alpha^{\mathrm{mov}}_{\mathsf{p}}(X) \ge 1 \quad \textit{and} \quad (\alpha^{\mathrm{mov}}_{\mathsf{p}}(X), \alpha_{\mathsf{p}}(X)) \ne (1, 1/2).$$

Then X is K-stable.

Proof. By the main result of [CP17] (cf. Remark 2.25), we have $\alpha^{\mathrm{mov}}_{\mathsf{q}}(X) \ge 1$ for any point $\mathsf{q} \in X$ which is not a maximal center. It follows that the pair $(X, \lambda_{\mathcal{M}}^{-1}\mathcal{M})$ is log canonical for any movable

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linear system $\mathcal{M}$ on X. Combining this with the inequality $\alpha(X) \geq 1/2$ obtained by Theorem 1.8, we see that X is K-semistable by [SZ19, Theorem 1.2].

Suppose that X is not K-stable. Then, by [SZ19, Corollary 3.1], there exists a prime divisor E over X, a movable linear system $\mathcal{M} \sim_{\mathbb{Q}} -nK_X$ and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that E is a log canonical place of $(X, \frac{1}{n}\mathcal{M})$ and $(X, \frac{1}{2}D)$. Note that the center Γ of E on X is necessarily a maximal center, and a maximal center on X is a BI center. Thus, $\Gamma = \mathsf{p}$ is a BI center, and this implies $(\alpha_{\mathsf{p}}^{\mathrm{mov}}(X), \alpha_{\mathsf{p}}(X)) = (1, 1/2)$. This is impossible by the assumption. Therefore, X is K-stable. □

We define

$$\mathsf{I}'_{\mathrm{K}} = \{6, 8, 15, 16, 17, 26, 27, 30, 36, 41, 47, 48, 54, 56, 60, 65, 68\} \subset \mathsf{I}_{\mathrm{BR}}.$$

Theorem 7.12. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}'_{\mathrm{K}}$. Then, for any BI center $\mathsf{p} \in X$, we have*

$$\alpha_{\mathsf{p}}^{\mathrm{mov}}(X) \geq 1 \quad \textit{and} \quad \alpha_{\mathsf{p}}(X) > \frac{1}{2}. \tag{7.1}$$

In particular, X is K-stable.

Proof. Let X be a member of $\mathcal{F}_{\mathsf{i}}$, where $\mathsf{i} \in \mathsf{I}'_{\mathrm{K}}$. We first show that the inequalities (7.12) are satisfied.

Suppose $\mathsf{i} \in \{16, 17, 26, 27, 36, 47, 48, 54, 65\}$. Then the subscript $\diamond$ is given in the fourth column of Table 7 for any BI center on X. By Proposition 5.3, we have $\alpha_{\mathsf{p}}(X) \geq 1$, and hence $\alpha_{\mathsf{p}}^{\mathrm{mov}}(X) \geq 1$, for any BI center $\mathsf{p} \in X$.

Suppose $\mathsf{i} \in \{6, 15, 30, 41, 68\}$. In this case, X admits two QI centers of equal singularity type and does not admit any other BI center. By the proof of Proposition 5.18 (see Section 5.6.c), we have $\alpha_{\mathsf{p}}(X) \geq 1$ for any QI center $\mathsf{p} \in X$. In particular, we have $\alpha_{\mathsf{p}}^{\mathrm{mov}}(X) \geq \alpha_{\mathsf{p}}(X) \geq 1$.

Suppose $\mathsf{i} \in \{8, 56, 60\}$. In this case, X admits a unique BI center and it is a QI center. The inequalities (7.1) follow from Remark 6.2 and Propositions 7.14, 7.15. This completes the verifications for the inequalities (7.1).

The K-stability of X follows from the inequalities (7.1), Theorem 1.8 and Proposition 7.11. □

We define

$$\mathsf{I}_{\mathrm{K}} := \mathsf{I}'_{\mathrm{K}} \sqcup \mathsf{I}_{\mathrm{KE}}.$$

Note that $|\mathsf{I}_{\mathrm{K}}| = 73$. Combining Theorems 7.7, 7.12 and Corollary 1.9, we obtain the K-stability of arbitrary quasi-smooth member for families indexed by I_{K}.

Corollary 7.13. *Let X be a member of a family $\mathcal{F}_{\mathsf{i}}$ with $\mathsf{i} \in \mathsf{I}_{\mathrm{K}}$. Then X is K-stable.*

7.5. Further computations of alpha invariants

In this section, we compute local alpha invariants for a few families in order to give better lower bounds. The results obtained in this section are used only in the proof of Theorem 7.12.

Proposition 7.14. *Let X be a member of the family $\mathcal{F}_{56}$ and $\mathsf{p} = \mathsf{p}_w \in X$ be the singular point of type $\frac{1}{11}(1, 3, 8)$. Then*

$$\alpha_{\mathsf{p}}(X) \geq \frac{2}{3} \quad \textit{and} \quad \alpha_{\mathsf{p}}^{\mathrm{mov}}(X) \geq 1.$$

Proof. We set $\mathsf{p} = \mathsf{p}_w$. We can write the defining polynomial of X as

$$F = w^2 y + f_{13} + f_{24},$$

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where $f_i = f_i(x, y, z, t)$ is a quasi-homogeneous polynomial of degree i. By the quasi-smoothness of X, we have $t^3 \in F$. It is easy to see that $F(x, 0, z, t, 1) \in (x, z, t)^3$. It follows that $\operatorname{omult}_{\mathsf{p}}(H_y) = 3$, which in particular implies $\operatorname{lct}_{\mathsf{p}}(X; \frac{1}{2}H_y) \geq 2/3$.

Let $D \in |A|_{\mathbb{Q}}$ be an irreducible $\mathbb{Q}$-divisor other than $\frac{1}{2}H_y$. We can take a $\mathbb{Q}$-divisor $T \in |3A|_{\mathbb{Q}}$ such that $\operatorname{omult}_{\mathsf{p}}(T) \geq 1$ and $\operatorname{Supp}(T)$ does not contain any component of the effective 1-cycle $D \cdot H_y$. We have

$$3 \operatorname{omult}_{\mathsf{p}}(D) \leq 11(D \cdot H_y \cdot T) = 3.$$

This shows $\operatorname{lct}_{\mathsf{p}}(X; D) \geq 1$. Therefore, $\alpha_{\mathsf{p}}^{\mathrm{mov}}(X) \geq 1$ and $\alpha_{\mathsf{p}}(X) \geq 2/3$. □

Proposition 7.15. *Let X be a member of the family $\mathcal{F}_{60}$, and let $\mathsf{p} = \mathsf{p}_w$ be the singular point of type $\frac{1}{9}(1, 4, 5)$. Then*

$$\alpha_{\mathsf{p}}(X) = 1.$$

Proof. We set $S = H_x \sim A$, $T = H_z$ and $\Gamma = S \cap T = (x = z = 0)_X$. Let $\rho = \rho_{\mathsf{p}} \colon \breve{U}_{\mathsf{p}} \to U_{\mathsf{p}}$ be the orbifold chart of $\mathsf{p} \in X$, and we set $\breve{\Gamma} = (\breve{x} = \breve{z} = 0) \subset \breve{U}_{\mathsf{p}}$. We can write the defining polynomial of X as

$$F = w^2 t + w f_{15} + f_{24},$$

where $f_i = f_i(x, y, z, t)$ is a quasi-homogeneous polynomial of degree i. By the quasi-smoothness of X, we have $t^4, y^6 \in F$, and we may assume $\operatorname{coeff}_F(t^4) = \operatorname{coeff}_F(y^6) = 1$ by rescaling y and t. We set $\lambda = \operatorname{coeff}_F(t^2 y^3) \in \mathbb{C}$. Then

$$\begin{aligned}
\Gamma &\cong (w^2 t + t^4 + \lambda t^2 y^3 + y^6 = 0) \subset \mathbb{P}(3, 10, 17)_{y,t,w}, \\
\breve{\Gamma} &\cong (\breve{w}^2 \breve{t} + \breve{t}^4 + \lambda \breve{t}^2 \breve{y}^3 + \breve{y}^6 = 0) \subset \mathbb{A}^3_{\breve{y},\breve{t},\breve{w}}.
\end{aligned}$$

It is easy to see that Γ is an irreducible and reduced curve, and $\operatorname{mult}_{\breve{\mathsf{p}}}(\breve{\Gamma}) = 1$, where $\breve{\mathsf{p}} = o \in \mathbb{A}^3$ is the preimage of p via ρ.

We see that H_x is quasi-smooth at p, and hence $\operatorname{lct}_{\mathsf{p}}(X; H_x) = 1$. Therefore, we have $\alpha_{\mathsf{p}}(X) \geq 1$ by Lemma 3.17. □

8. The table

The list of the 93 families together with their basic information are summarized in Table 7, and we explain the contents.

The first two columns indicate basic information of each family and the anticanonical degree $(A^3) = (-K_X)^3$ is indicated in the third column.

In the fourth column, the number and the singularities of X are described. The symbol $\frac{1}{r}[a, r-a]$ stands for the cyclic quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where $1 \leq a \leq r/2$. Moreover, the symbols $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ stand for singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 1, 3)$. The superscripts QI and EI indicate that the corresponding singular point p is a QI center and EI center, respectively (see Section 2.3.b for definitions). The meaning of the subscripts is explained as follows.

- The subscript $\heartsuit$ indicates that $\alpha_{\mathsf{p}}(X) \geq 1$ is proved by Proposition 5.2.
- The subscript $\diamondsuit$ (resp. $\diamondsuit'$) indicates that $\alpha_{\mathsf{p}}(X) \geq 1$ (resp. $\alpha_{\mathsf{p}}(X) \geq 2/3$) is proved by Proposition 5.3.
- The subscript $\clubsuit$ indicates that $\alpha_{\mathsf{p}}(X) \geq 1/2$ is proved by Proposition 5.4.
- The subscript $\spadesuit$ indicates that $\alpha_{\mathsf{p}}(X) \geq 1/2$ is proved by Proposition 5.5.

In Theorem 1.8, any birational superrigid member of each of the 95 families is proved. Apart from this main result, we have results on the existence of KE metrics or K-stability for any quasi-smooth member

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Table 7. *The 93 families.*

No.	$X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$	(A^3)	Singular points	
2	$X_5 \subset \mathbb{P}(1,1,1,1,2)$	$\frac{5}{2}$	$\frac{1}{2}^{\mathrm{QI}}$	
4	$X_6 \subset \mathbb{P}(1,1,1,2,2)$	$\frac{3}{2}$	$3\times\frac{1}{2}^{\mathrm{QI}}$	
5	$X_7 \subset \mathbb{P}(1,1,1,2,3)$	$\frac{7}{6}$	$\frac{1}{2}^{\mathrm{QI}},\ \frac{1}{3}^{\mathrm{QI}}$	
6	$X_8 \subset \mathbb{P}(1,1,1,2,4)$	1	$2\times\frac{1}{2}^{\mathrm{QI}}$	K
7	$X_8 \subset \mathbb{P}(1,1,2,2,3)$	$\frac{2}{3}$	$4\times\frac{1}{2}^{\mathrm{EI}},\ \frac{1}{3}^{\mathrm{QI}}$	
8	$X_9 \subset \mathbb{P}(1,1,1,3,4)$	$\frac{3}{4}$	$\frac{1}{4}^{\mathrm{QI}}$	K
9	$X_9 \subset \mathbb{P}(1,1,2,3,3)$	$\frac{1}{2}$	$\frac{1}{2}_{\heartsuit},\ 3\times\frac{1}{3}^{\mathrm{QI}}$	
10	$X_{10} \subset \mathbb{P}(1,1,1,3,5)$	$\frac{2}{3}$	$\frac{1}{3}_{\clubsuit}$	KE
11	$X_{10} \subset \mathbb{P}(1,1,2,2,5)$	$\frac{1}{2}$	$5\times\frac{1}{2}_{\heartsuit}$	KE
12	$X_{10} \subset \mathbb{P}(1,1,2,3,4)$	$\frac{5}{12}$	$2\times\frac{1}{2}_{\spadesuit},\ \frac{1}{3}^{\mathrm{QI}},\ \frac{1}{4}^{\mathrm{QI}}$	
13	$X_{11} \subset \mathbb{P}(1,1,2,3,5)$	$\frac{11}{30}$	$\frac{1}{2}_{\spadesuit},\ \frac{1}{3}^{\mathrm{QI}},\ \frac{1}{5}[2,3]^{\mathrm{QI}}$	
14	$X_{12} \subset \mathbb{P}(1,1,1,4,6)$	$\frac{1}{2}$	$2\times\frac{1}{2}_{\heartsuit}$	KE
15	$X_{12} \subset \mathbb{P}(1,1,2,3,6)$	$\frac{1}{3}$	$2\times\frac{1}{2}_{\diamondsuit},\ 2\times\frac{1}{3}^{\mathrm{QI}}_{\diamondsuit}$	K
16	$X_{12} \subset \mathbb{P}(1,1,2,4,5)$	$\frac{3}{10}$	$3\times\frac{1}{2}_{\heartsuit},\ \frac{1}{5}[1,4]^{\mathrm{QI}}_{\diamondsuit}$	K
17	$X_{12} \subset \mathbb{P}(1,1,3,4,4)$	$\frac{1}{4}$	$3\times\frac{1}{4}^{\mathrm{QI}}_{\diamondsuit}$	K
18	$X_{12} \subset \mathbb{P}(1,2,2,3,5)$	$\frac{1}{5}$	$6\times\frac{1}{2}_{\heartsuit},\ \frac{1}{5}[2,3]^{\mathrm{QI}}$	
19	$X_{12} \subset \mathbb{P}(1,2,3,3,4)$	$\frac{1}{6}$	$3\times\frac{1}{2}_{\heartsuit},\ 4\times\frac{1}{3}_{\heartsuit}$	KE
20	$X_{13} \subset \mathbb{P}(1,1,3,4,5)$	$\frac{13}{60}$	$\frac{1}{3}^{\mathrm{EI}},\ \frac{1}{4}^{\mathrm{QI}},\ \frac{1}{5}[1,4]^{\mathrm{QI}}$	
21	$X_{14} \subset \mathbb{P}(1,1,2,4,7)$	$\frac{1}{4}$	$3\times\frac{1}{2}_{\heartsuit},\ \frac{1}{4}_{\diamondsuit}$	KE
22	$X_{14} \subset \mathbb{P}(1,2,2,3,7)$	$\frac{1}{6}$	$7\times\frac{1}{2}_{\heartsuit},\ \frac{1}{3}_{\heartsuit}$	KE
23	$X_{14} \subset \mathbb{P}(1,2,3,4,5)$	$\frac{7}{60}$	$3\times\frac{1}{2}_{\clubsuit},\ \frac{1}{3}_{\clubsuit},\ \frac{1}{4}^{\mathrm{EI}}_{\clubsuit},\ \frac{1}{5}[2,3]^{\mathrm{QI}}$	
24	$X_{15} \subset \mathbb{P}(1,1,2,5,7)$	$\frac{3}{14}$	$\frac{1}{2}_{\spadesuit},\ \frac{1}{7}[2,5]^{\mathrm{QI}}$	
25	$X_{15} \subset \mathbb{P}(1,1,3,4,7)$	$\frac{5}{28}$	$\frac{1}{4}^{\mathrm{QI}},\ \frac{1}{7}[3,4]^{\mathrm{QI}}$	
26	$X_{15} \subset \mathbb{P}(1,1,3,5,6)$	$\frac{1}{6}$	$2\times\frac{1}{3}_{\heartsuit},\ \frac{1}{6}[1,5]^{\mathrm{QI}}_{\diamondsuit}$	K
27	$X_{15} \subset \mathbb{P}(1,2,3,5,5)$	$\frac{1}{10}$	$\frac{1}{2}_{\spadesuit}, 3\times\frac{1}{5}[2,3]^{\mathrm{QI}}_{\diamondsuit}$	K
28	$X_{15} \subset \mathbb{P}(1,3,3,4,5)$	$\frac{1}{12}$	$5\times\frac{1}{3}_{\heartsuit},\ \frac{1}{4}_{\heartsuit}$	KE
29	$X_{16} \subset \mathbb{P}(1,1,2,5,8)$	$\frac{1}{5}$	$2\times\frac{1}{2}_{\clubsuit},\ \frac{1}{5}[2,3]_{\clubsuit}$	KE
30	$X_{16} \subset \mathbb{P}(1,1,3,4,8)$	$\frac{1}{6}$	$\frac{1}{3}_{\heartsuit},\ 2\times\frac{1}{4}^{\mathrm{QI}}$	K
31	$X_{16} \subset \mathbb{P}(1,1,4,5,6)$	$\frac{2}{15}$	$\frac{1}{2}_{\clubsuit},\ \frac{1}{5}[1,4]^{\mathrm{QI}},\ \frac{1}{6}[1,5]^{\mathrm{QI}}$	
32	$X_{16} \subset \mathbb{P}(1,2,3,4,7)$	$\frac{2}{21}$	$4\times\frac{1}{2}_{\heartsuit},\ \frac{1}{3}_{\spadesuit},\ \frac{1}{7}[3,4]^{\mathrm{QI}}$	
33	$X_{17} \subset \mathbb{P}(1,2,3,5,7)$	$\frac{17}{210}$	$\frac{1}{2}_{\spadesuit},\ \frac{1}{3}_{\clubsuit},\ \frac{1}{5}[2,3]^{\mathrm{QI}},\ \frac{1}{7}[2,5]^{\mathrm{QI}}$	
34	$X_{18} \subset \mathbb{P}(1,1,2,6,9)$	$\frac{1}{6}$	$3\times\frac{1}{2}_{\heartsuit},\ \frac{1}{3}_{\heartsuit}$	KE
35	$X_{18} \subset \mathbb{P}(1,1,3,5,9)$	$\frac{2}{15}$	$2\times\frac{1}{3}_{\heartsuit},\ \frac{1}{5}[1,4]_{\diamondsuit}$	KE
36	$X_{18} \subset \mathbb{P}(1,1,4,6,7)$	$\frac{3}{28}$	$\frac{1}{2}_{\heartsuit},\ \frac{1}{4}^{\mathrm{EI}}_{\diamondsuit},\ \frac{1}{7}[1,6]^{\mathrm{QI}}_{\diamondsuit}$	K
37	$X_{18} \subset \mathbb{P}(1,2,3,4,9)$	$\frac{1}{12}$	$4\times\frac{1}{2}_{\heartsuit},\ 2\times\frac{1}{3}_{\clubsuit},\ \frac{1}{4}_{\heartsuit}$	KE
38	$X_{18} \subset \mathbb{P}(1,2,3,5,8)$	$\frac{3}{40}$	$2\times\frac{1}{2}_{\heartsuit},\ \frac{1}{5}[2,5]^{\mathrm{QI}}, \frac{1}{8}[3,5]^{\mathrm{QI}}$	
39	$X_{18} \subset \mathbb{P}(1,3,4,5,6)$	$\frac{1}{20}$	$\frac{1}{2}_{\heartsuit},\ 3\times\frac{1}{3}_{\clubsuit},\ \frac{1}{4}_{\clubsuit},\ \frac{1}{5}[1,4]_{\heartsuit}$	KE
40	$X_{19} \subset \mathbb{P}(1,3,4,5,7)$	$\frac{19}{420}$	$\frac{1}{3}_{\spadesuit},\ \frac{1}{4}_{\clubsuit},\ \frac{1}{5}[2,3]^{\mathrm{EI}}_{\clubsuit},\ \frac{1}{7}[3,4]^{\mathrm{QI}}$	
41	$X_{20} \subset \mathbb{P}(1,1,4,5,10)$	$\frac{1}{10}$	$\frac{1}{2}_{\heartsuit},\ 2\times\frac{1}{5}[1,4]^{\mathrm{QI}}_{\diamondsuit}$	K
42	$X_{20} \subset \mathbb{P}(1,2,3,5,10)$	$\frac{1}{15}$	$2\times\frac{1}{2}_{\heartsuit},\ \frac{1}{3}_{\heartsuit},\ 2\times\frac{1}{5}[2,3]^{\mathrm{QI}}$	KE
43	$X_{20} \subset \mathbb{P}(1,2,4,5,9)$	$\frac{1}{18}$	$5\times\frac{1}{2}_{\heartsuit},\ \frac{1}{9}[4,5]^{\mathrm{QI}}$	
44	$X_{20} \subset \mathbb{P}(1,2,5,6,7)$	$\frac{1}{21}$	$3\times\frac{1}{2}_{\heartsuit},\ \frac{1}{6}[1,5]^{\mathrm{EI}}_{\diamondsuit},\ \frac{1}{7}[2,5]^{\mathrm{QI}}_{\diamondsuit}$	KE

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Table 7. *(Continued).*

No.	$X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$	(A^3)	Singular points	
45	$X_{20} \subset \mathbb{P}(1, 3, 4, 5, 8)$	$\frac{1}{24}$	$\frac{1}{3}_\heartsuit,\ 2\times\frac{1}{4}_\heartsuit,\ \frac{1}{8}[3,5]^{\mathrm{QI}}_\diamondsuit$	KE
46	$X_{21} \subset \mathbb{P}(1, 1, 3, 7, 10)$	$\frac{1}{10}$	$\frac{1}{10}[3,7]^{\mathrm{QI}}$	
47	$X_{21} \subset \mathbb{P}(1, 1, 5, 7, 8)$	$\frac{3}{40}$	$\frac{1}{5}[2,3]_\spadesuit,\ \frac{1}{8}[1,7]^{\mathrm{QI}}_\diamondsuit$	K
48	$X_{21} \subset \mathbb{P}(1, 2, 3, 7, 9)$	$\frac{1}{18}$	$\frac{1}{2}_\spadesuit,\ 2\times\frac{1}{3}_\heartsuit,\ \frac{1}{9}[2,7]^{\mathrm{QI}}_\diamondsuit$	K
49	$X_{21} \subset \mathbb{P}(1, 3, 5, 6, 7)$	$\frac{1}{30}$	$3\times\frac{1}{3}_\heartsuit,\ \frac{1}{5}[2,3]_\spadesuit,\ \frac{1}{6}[1,5]_\heartsuit$	KE
50	$X_{22} \subset \mathbb{P}(1, 1, 3, 7, 11)$	$\frac{2}{21}$	$\frac{1}{3}_\heartsuit,\ \frac{1}{7}[3,4]_\clubsuit$	KE
51	$X_{22} \subset \mathbb{P}(1, 1, 4, 6, 11)$	$\frac{1}{12}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{6}[1,5]_\diamondsuit$	KE
52	$X_{22} \subset \mathbb{P}(1, 2, 4, 5, 11)$	$\frac{1}{20}$	$5\times\frac{1}{2}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit$	KE
53	$X_{24} \subset \mathbb{P}(1, 1, 3, 8, 12)$	$\frac{1}{12}$	$2\times\frac{1}{3}_\heartsuit,\ \frac{1}{4}_\heartsuit$	KE
54	$X_{24} \subset \mathbb{P}(1, 1, 6, 8, 9)$	$\frac{1}{18}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{9}[1,8]^{\mathrm{QI}}_\diamondsuit$	K
55	$X_{24} \subset \mathbb{P}(1, 2, 3, 7, 12)$	$\frac{1}{21}$	$2\times\frac{1}{2}_\heartsuit,\ 2\times\frac{1}{3}_\heartsuit,\ \frac{1}{7}[2,5]_\diamondsuit$	KE
56	$X_{24} \subset \mathbb{P}(1, 2, 3, 8, 11)$	$\frac{1}{22}$	$3\times\frac{1}{2}_\heartsuit,\ \frac{1}{11}[3,8]^{\mathrm{QI}}$	K
57	$X_{24} \subset \mathbb{P}(1, 3, 4, 5, 12)$	$\frac{1}{30}$	$2\times\frac{1}{3}_\heartsuit,\ 2\times\frac{1}{4}_\heartsuit,\ \frac{1}{5}[2,3]_\heartsuit$	KE
58	$X_{24} \subset \mathbb{P}(1, 3, 4, 7, 10)$	$\frac{1}{35}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{7}[3,4]^{\mathrm{QI}},\ \frac{1}{10}[3,7]^{\mathrm{QI}}$	
59	$X_{24} \subset \mathbb{P}(1, 3, 6, 7, 8)$	$\frac{1}{42}$	$\frac{1}{2}_\heartsuit,\ 4\times\frac{1}{3}_\heartsuit,\ \frac{1}{7}[1,6]_\heartsuit$	KE
60	$X_{24} \subset \mathbb{P}(1, 4, 5, 6, 9)$	$\frac{1}{45}$	$2\times\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{9}[4,5]^{\mathrm{QI}}$	K
61	$X_{25} \subset \mathbb{P}(1, 4, 5, 7, 9)$	$\frac{5}{252}$	$\frac{1}{4}_\clubsuit,\ \frac{1}{7}[2,5]^{\mathrm{EI}}_\diamondsuit,\ \frac{1}{9}[4,5]_\diamondsuit$	KE
62	$X_{26} \subset \mathbb{P}(1, 1, 5, 7, 13)$	$\frac{2}{35}$	$\frac{1}{5}[2,3]_\spadesuit,\ \frac{1}{7}[1,6]_\diamondsuit$	KE
63	$X_{26} \subset \mathbb{P}(1, 2, 3, 8, 13)$	$\frac{1}{24}$	$3\times\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{8}[3,5]_\clubsuit$	KE
64	$X_{26} \subset \mathbb{P}(1, 2, 5, 6, 13)$	$\frac{1}{30}$	$4\times\frac{1}{2}_\heartsuit,\ \frac{1}{5}[2,3]_\clubsuit,\ \frac{1}{6}[1,5]_\heartsuit$	KE
65	$X_{27} \subset \mathbb{P}(1, 2, 5, 9, 11)$	$\frac{3}{110}$	$\frac{1}{2}_\spadesuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{11}[2,9]^{\mathrm{QI}}_\diamondsuit$	K
66	$X_{27} \subset \mathbb{P}(1, 5, 6, 7, 9)$	$\frac{1}{70}$	$\frac{1}{3}_\heartsuit,\ \frac{1}{5}[1,4]_\clubsuit,\ \frac{1}{6}[1,5]_\heartsuit,\ \frac{1}{7}[2,5]_\heartsuit$	KE
67	$X_{28} \subset \mathbb{P}(1, 1, 4, 9, 14)$	$\frac{1}{18}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{9}[4,5]_\spadesuit$	KE
68	$X_{28} \subset \mathbb{P}(1, 3, 4, 7, 14)$	$\frac{1}{42}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\clubsuit,\ 2\times\frac{1}{7}[3,4]^{\mathrm{QI}}_\diamondsuit$	K
69	$X_{28} \subset \mathbb{P}(1, 4, 6, 7, 11)$	$\frac{1}{66}$	$2\times\frac{1}{2}_\heartsuit,\ \frac{1}{6}[1,5]_\heartsuit,\ \frac{1}{11}[4,7]^{\mathrm{QI}}_\diamondsuit$	KE
70	$X_{30} \subset \mathbb{P}(1, 1, 4, 10, 15)$	$\frac{1}{20}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit$	KE
71	$X_{30} \subset \mathbb{P}(1, 1, 6, 8, 15)$	$\frac{1}{24}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{8}[1,7]_\diamondsuit$	KE
72	$X_{30} \subset \mathbb{P}(1, 2, 3, 10, 15)$	$\frac{1}{30}$	$3\times\frac{1}{2}_\heartsuit,\ 2\times\frac{1}{3}_\heartsuit,\ \frac{1}{5}[2,3]_\heartsuit$	KE
73	$X_{30} \subset \mathbb{P}(1, 2, 6, 7, 15)$	$\frac{1}{42}$	$5\times\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{7}[1,6]_\heartsuit$	KE
74	$X_{30} \subset \mathbb{P}(1, 3, 4, 10, 13)$	$\frac{1}{52}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{13}[3,10]^{\mathrm{QI}}_\diamondsuit$	KE
75	$X_{30} \subset \mathbb{P}(1, 4, 5, 6, 15)$	$\frac{1}{60}$	$2\times\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ 2\times\frac{1}{5}[1,4]_\heartsuit$	KE
76	$X_{30} \subset \mathbb{P}(1, 5, 6, 8, 11)$	$\frac{1}{88}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{8}[3,5]^{\mathrm{EI}}_\diamondsuit,\ \frac{1}{11}[5,6]^{\mathrm{QI}}_\diamondsuit$	KE
77	$X_{32} \subset \mathbb{P}(1, 2, 5, 9, 16)$	$\frac{1}{45}$	$2\times\frac{1}{2}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{9}[2,7]_\diamondsuit$	KE
78	$X_{32} \subset \mathbb{P}(1, 4, 5, 7, 16)$	$\frac{1}{70}$	$2\times\frac{1}{4}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{7}[2,5]_\heartsuit$	KE
79	$X_{33} \subset \mathbb{P}(1, 3, 5, 11, 14)$	$\frac{1}{70}$	$\frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{14}[3,11]^{\mathrm{QI}}_\diamondsuit$	KE
80	$X_{34} \subset \mathbb{P}(1, 3, 4, 10, 17)$	$\frac{1}{60}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\clubsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{10}[3,7]_\diamondsuit$	KE
81	$X_{34} \subset \mathbb{P}(1, 4, 6, 7, 17)$	$\frac{1}{84}$	$2\times\frac{1}{2}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{6}[1,5]_\heartsuit,\ \frac{1}{7}[3,4]_\heartsuit$	KE
82	$X_{36} \subset \mathbb{P}(1, 1, 5, 12, 18)$	$\frac{1}{30}$	$\frac{1}{5}[2,3]_\spadesuit,\ \frac{1}{6}[1,5]_\heartsuit$	KE
83	$X_{36} \subset \mathbb{P}(1, 3, 4, 11, 18)$	$\frac{1}{66}$	$\frac{1}{2}_\heartsuit,\ 2\times\frac{1}{3}_\heartsuit,\ \frac{1}{11}[4,7]_\diamondsuit$	KE
84	$X_{36} \subset \mathbb{P}(1, 7, 8, 9, 12)$	$\frac{1}{168}$	$\frac{1}{3}_\heartsuit,\ \frac{1}{4}_\heartsuit,\ \frac{1}{7}[2,5]_\spadesuit,\ \frac{1}{8}[1,7]_\heartsuit$	KE
85	$X_{38} \subset \mathbb{P}(1, 3, 5, 11, 19)$	$\frac{2}{165}$	$\frac{1}{3}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{11}[3,8]_\diamondsuit$	KE
86	$X_{38} \subset \mathbb{P}(1, 5, 6, 8, 19)$	$\frac{1}{120}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{5}[1,4]_\heartsuit,\ \frac{1}{6}[1,5]_\heartsuit,\ \frac{1}{8}[3,5]_\heartsuit$	KE
87	$X_{40} \subset \mathbb{P}(1, 5, 7, 8, 20)$	$\frac{1}{140}$	$\frac{1}{4}_\heartsuit,\ 2\times\frac{1}{5}[2,3]_\heartsuit,\ \frac{1}{7}[1,6]_\heartsuit$	KE
88	$X_{42} \subset \mathbb{P}(1, 1, 6, 14, 21)$	$\frac{1}{42}$	$\frac{1}{2}_\heartsuit,\ \frac{1}{3}_\heartsuit,\ \frac{1}{7}[1,6]_\heartsuit$	KE

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Table 7. *(Continued).*

No.	$X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$	(A^3)	Singular points	
89	$X_{42} \subset \mathbb{P}(1, 2, 5, 14, 21)$	$\frac{1}{70}$	$3\times\frac{1}{2}{}_{\heartsuit},\ \frac{1}{5}[1,4]_{\heartsuit},\ \frac{1}{7}[2,5]_{\heartsuit}$	KE
90	$X_{42} \subset \mathbb{P}(1, 3, 4, 14, 21)$	$\frac{1}{84}$	$\frac{1}{2}{}_{\heartsuit},\ 2\times\frac{1}{3}{}_{\heartsuit},\ \frac{1}{4}{}_{\heartsuit},\ \frac{1}{7}[3,4]_{\heartsuit}$	KE
91	$X_{44} \subset \mathbb{P}(1, 4, 5, 13, 22)$	$\frac{1}{130}$	$\frac{1}{2}{}_{\heartsuit},\ \frac{1}{5}[2,3]_{\heartsuit},\ \frac{1}{13}[4,9]_{\diamondsuit}$	KE
92	$X_{48} \subset \mathbb{P}(1, 3, 5, 16, 24)$	$\frac{1}{120}$	$2\times\frac{1}{3}{}_{\heartsuit},\ \frac{1}{5}[1,4]_{\heartsuit},\ \frac{1}{8}[3,5]_{\heartsuit}$	KE
93	$X_{50} \subset \mathbb{P}(1, 7, 8, 10, 25)$	$\frac{1}{280}$	$\frac{1}{2}{}_{\heartsuit},\ \frac{1}{5}[2,3]_{\heartsuit},\ \frac{1}{7}[3,4]_{\clubsuit},\ \frac{1}{8}[1,7]_{\heartsuit}$	KE
94	$X_{54} \subset \mathbb{P}(1, 4, 5, 18, 27)$	$\frac{1}{180}$	$\frac{1}{2}{}_{\heartsuit},\ \frac{1}{4}{}_{\heartsuit},\ \frac{1}{5}[2,3]_{\heartsuit},\ \frac{1}{9}[4,5]_{\heartsuit}$	KE
95	$X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$	$\frac{1}{330}$	$\frac{1}{2}{}_{\heartsuit},\ \frac{1}{3}{}_{\heartsuit},\ \frac{1}{5}[2,3]_{\clubsuit},\ \frac{1}{11}[5,6]_{\heartsuit}$	KE

of suitable families. In the right-most column the mark 'KE' and 'K' are given and their meanings are as follows.

- ◦ The mark 'KE' in the right-most column means that any quasi-smooth member admits a KE metric and is K-stable (see Section 7.3).
- ◦ The mark 'K' in the right-most column means that any quasi-smooth member is K-stable (see Section 7.4.b).

Remark 8.1. We explain what is left about K-stability of quasi-smooth Fano 3-fold weighted hypersurfaces of index 1.

As it is explained in Section 1.4, the result [LXZ22] obtained after this paper is written in particular implies that the K-stability of a quasi-smooth Fano 3-fold weighted hypersurface is equivalent to the existence of a KE metric. It follows that the meaning of the mark 'KE' and 'K' in the right-most column of Table 7 are the same: It indicates that any quasi-smooth member is K-stable (and admits a KE metric). All in all, we obtain the following results in this article:

- ◦ Any quasi-smooth member in a family $\mathcal{F}_{\mathrm{i}}$ with a mark 'K' or 'KE' in the right-most column of Table 7 is K-stable.
- ◦ Any quasi-smooth and birationally superrigid member in a family $\mathcal{F}_{\mathrm{i}}$ with a blank right-most column in Table 7 is K-stable.

Therefore, it remains to determine K-stability for quasi-smooth members in a family $\mathcal{F}_{\mathrm{i}}$ with a blank right-most column that are not birationally superrigid.

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Competing interest. The authors have no competing interest to declare.

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