A DUAL FORM OF KURATOWSKI'S THEOREM

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The celebrated criterion of Kuratowski [2] for the planarity of a graph G involves the determination of whether G contains a subgraph homeomorphic to K_5 or $K_{3,3}$ shown in Figure 1.

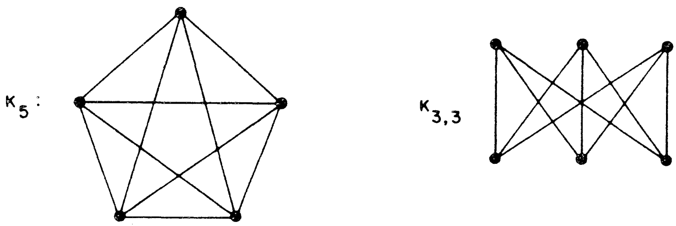

Figure 1

The well-known "Petersen graph", shown in Figure 2, looks suspiciously like K_5 but nevertheless contains no subgraph homeomorphic to K_5. Its nonplanar character may be confirmed by verifying the occurrence of a homeomorph of $K_{3,3}$ in it.

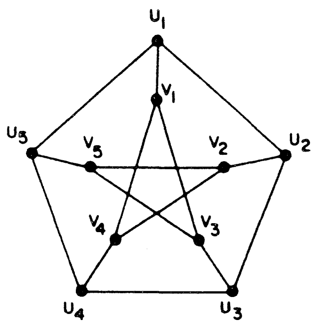

Figure 2

Our object in this note is to present a variation of Kuratowski's Theorem whereby it is possible to test a graph

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for planarity by contracting connected subgraphs into single vertices. The statement of this result includes Kuratowski's criterion since homeomorphic reduction is a special case of contraction. Note that the contraction of each connected subgraph (edge) $u_i v_i$ in Figure 2 into a single vertex results in K_5.

Let $E(G)$ be the set of edges of a graph G and let $V(G)$ be its vertex set. For any set S of edges of G, we write $\overline{S} = E(G) - S$. We write $G:S$ for the subgraph H of G defined by $V(H) = V(G)$ and $E(H) = S$. The reduction $G \cdot S$ of G to S is obtained from $G:S$ by deleting its isolated vertices.

Let $C(S)$ be the set of components of $G:\overline{S}$. The contraction $G \times S$ of G to S is a graph such that $V(G \times S) = C(S)$ and $E(G \times S) = S$. The ends of an edge A of $G \times S$ are the members of $C(S)$ containing an end of A in G. The notation $[u]$, $[v]$, ... will be used later for the vertices of a contraction.

In the case of planar graphs it can be verified that reduction and contraction are dual operations. Even for general graphs they are dual in the sense of matroid theory [3].

A subcontraction of G is a reduction of a contraction of G. Such a graph can also be realized as a contraction of a reduction of G. In [4], formulas are given which are general rules for inverting the orders of contractions and reductions.

THEOREM. A graph is nonplanar if and only if it has K_5 or $K_{3,3}$ as a subcontraction.

The usual version of Kuratowski's theorem asserts that a graph G is nonplanar if and only if it contains a graph H homeomorphic to K_5 or $K_{3,3}$. If this condition is fulfilled it is clear that G has a K_5 or a $K_{3,3}$ as a subcontraction.

Conversely suppose G has a subcontraction $H = (G \cdot S) \times T$ which is a $K_{3,3}$. Consider a vertex $[v]$ of H incident with three edges A_1, A_2 and A_3. Then $[v]$ is a connected subgraph of $G \cdot S$. Let its vertices incident in G with A_1, A_2 and A_3 be v_1, v_2 and v_3 respectively.

If v_1, v_2 and v_3 are distinct we can find a vertex v of $[v]$ joined to v_1, v_2 and v_3 by arcs L_1, L_2 and L_3 respect-

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ively, in $[v]$, so that no two of the L_i intersect except for the common vertex v. We can make the same assertion if two of the v_i coincide, say $v_1 = v_2$. Then $v = v_1 = v_2$, and L_1 and L_2 are trivial graphs, i.e., consist of one vertex only. if $v_1 = v_2 = v_3$ then L_3 is also a trivial graph.

Applying this construction to all the vertices of H we can replace H by a subgraph of $G \cdot S$, with vertices such as v, which is a homeomorph of $K_{3,3}$.

A similar argument applies when G has a subcontraction $H = (G \cdot S) \times T$ which is a K_5. Then we may consider a vertex $[v]$ of H incident with four edges A_1, A_2, A_3 and A_4, with ends v_1, v_2, v_3 and v_4 (in G) belonging to $[v]$. It may be possible to find a vertex v of $[v]$ which can be joined to v_1, v_2, v_3 and v_4 by arcs which meet only at v. If this can be done for all 5 vertices of H, then G contains a homeomorph of K_5.

In the remaining case we may suppose that in one vertex $[v]$ of H the vertices v_i can be joined up as in Figure 3. In this figure, w_1, w_2, w_3 and w_4 represent the other vertices of H, to be considered here as connected subgraphs of $G \cdot S$.

Let A be one of the edges of the arc $u_1 u_2$. Let S_1 be the set obtained from S by deleting the edges $w_1 w_3$ and $w_2 w_4$. Call T_1 the set obtained from T by deleting the edges $w_1 w_3$ and $w_2 w_4$ and then adjoining A. Clearly $(G \cdot S_1) \times T_1$ is a $K_{3,3}$, completing the proof of the theorem.

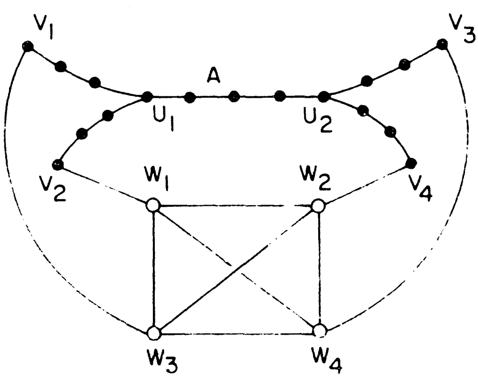

Figure 3

We have illustrated this theorem earlier by applying it to the Petersen graph shown in Figure 2. Note that the following stronger statement may be made: A graph is non planar if and only if it has K_5 or $K_{3,3}$ as a contraction.

The above considerations may perhaps be relevant to the problem of extending Kuratowski's Theorem to surfaces other than the sphere. I. N. Kagno [1] has given incomplete lists of minimal graphs not realizable in the projective plane and the torus. Here the term "minimal" is used in the sense that G is not realizable in S but all its subgraphs are. Perhaps we should replace the word "subgraph" by "subcontraction" in this definition. This would presumably make the list of minimal graphs, if finite, shorter.

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