Bull. Aust. Math. Soc. **108** (2023), 366–372
doi:10.1017/S0004972723000072

ON NEAR-PERFECT NUMBERS OF SPECIAL FORMS

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(Received 7 November 2022; accepted 2 January 2023; first published online 14 February 2023)

Abstract

We discuss near-perfect numbers of various forms. In particular, we study the existence of near-perfect numbers in the Fibonacci and Lucas sequences, near-perfect values taken by integer polynomials and repdigit near-perfect numbers.

2020 *Mathematics subject classification*: primary 11A25; secondary 11B39.

Keywords and phrases: near-perfect number, repdigit, *ABC*-conjecture, Fibonacci numbers.

1. Introduction

Let $\sigma(n)$ and $\omega(n)$ denote the sum of the positive divisors of n and the number of distinct prime factors of n, respectively. A natural number n is *perfect* if $\sigma(n) = 2n$. More generally, given a fixed integer k, the number n is called *multiperfect* or *k-fold perfect* if $\sigma(n) = kn$. The famous Euclid–Euler theorem asserts that an even number is perfect if and only if it has the form $2^{p-1}(2^p - 1)$, where both p and $2^p - 1$ are primes. It is not known if there are odd perfect numbers.

In 2012, Pollack and Shevelev [10] introduced the concept of near-perfect numbers. A positive integer n is *near-perfect* with redundant divisor d if d is a proper divisor of n and $\sigma(n) = 2n + d$. Note that when $d = 1$, we get a special kind of near-perfect numbers called *quasiperfect*.

Pollack and Shevelev constructed the following three types of even near-perfect numbers.

Type A. $n = 2^{p-1}(2^p - 1)^2$ where both p and $2^p - 1$ are primes and $2^p - 1$ is the redundant divisor.

Type B. $n = 2^{2p-1}(2^p - 1)$ where both p and $2^p - 1$ are primes and $2^p(2^p - 1)$ is the redundant divisor.

Type C. $n = 2^{t-1}(2^t - 2^k - 1)$, $t \geq k + 1$ where $2^t - 2^k - 1$ is prime and 2^k is the redundant divisor.

This research was supported by NSERC Discovery grants RGPIN-2020-06731 of Habiba Kadiri and RGPIN-2020-06032 of Nathan Ng.

In 2013, Ren and Chen [12] proved that all near perfect numbers n with $\omega(n) = 2$ are of types A, B and C together with 40. It is an open problem to classify all even near-perfect numbers. However, from the definition, it is easy to see that all odd near-perfect numbers are squares. Tang *et al.* [14] showed that there is no odd near-perfect number n with $\omega(n) = 3$ and Tang *et al.* [13] proved that the only odd near-perfect number n with $\omega(n) = 4$ is $173369889 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$. Thus, for any other odd near-perfect number n, if it exists, we have $\omega(n) \geq 5$.

There are several papers discussing perfect and multiperfect numbers of various forms. For example, Luca [7] proved that there are no perfect Fibonacci or Lucas numbers, while Broughan *et al.* [2] showed that no Fibonacci number (larger than 1) is multiperfect. Assuming the *ABC*-conjecture, Klurman [5] proved that any integer polynomial of degree ≥ 3 without repeated factors can take only finitely many perfect values. Pollack and Shevelev [9] studied perfect numbers with identical digits in base g, $g \geq 2$. He found that in each base g, there are only finitely many examples and that when $g = 10$, the only example is 6. Later, Luca and Pollack [8] established the same results for multiperfect numbers with identical digits.

In this short note, we study near-perfect numbers in the Fibonacci and Lucas sequences, near-perfect values taken by integer polynomials and near-perfect numbers with identical digits. Recall that the *Fibonacci sequence* $(F_n)_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$ and the *Lucas sequence* $(L_n)_{n\geq 0}$ is given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. A natural number is called a *repdigit in base g* if all of the digits in its base g expansion are equal.

Here we prove the following results.

THEOREM 1.1

(a) *There are no odd near-perfect Fibonacci or Lucas numbers.*
(b) *There are no near-perfect Fibonacci numbers F_n with $\omega(F_n) \leq 3$.*
(c) *The only near-perfect Lucas number L_n with two distinct prime factors is $L_6 = 18$.*

THEOREM 1.2. *Suppose $P(x) \in \mathbb{Z}[x]$ with $\deg P(x) \geq 3$ has no repeated factors. Then there are only finitely many n such that $P(n)$ is an odd near-perfect number. Furthermore, if we assume that the ABC-conjecture holds, then $P(n)$ takes only finitely many near-perfect values with two distinct prime factors.*

THEOREM 1.3. *Let $2 \leq g \leq 10$.*

(a) *There are only finitely many repdigits in base g which are near-perfect and have two distinct prime factors. All such numbers are strictly less than $(g^3 - 1)/(g - 1)$. In particular, when $g = 10$, the only repdigit near-perfect number with two distinct prime divisors is 88.*
(b) *There are no odd near-perfect repdigits in base g.*

2. Preliminary results

In this section, we collect several auxiliary results. We begin with the famous and remarkable theorem of Bugeaud *et al.* [4] about perfect powers in the Fibonacci and Lucas sequences.

THEOREM 2.1 (Bugeaud–Mignotte–Siksek). *The only perfect powers among the Fibonacci numbers are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$. For the Lucas numbers, the only perfect powers are $L_1 = 1$ and $L_3 = 4$.*

In [11], Pongsriiam gave the description of the Fibonacci numbers satisfying $\omega(F_n) = 3$. We state his results in the following theorems.

THEOREM 2.2. *The only solutions to the equation $\omega(F_n) = 3$ are given by*

(a) $n = 16, 18$ *or* $2p$ *for some prime* $p \geq 19$,
(b) $n = p, p^2$ *or* p^3 *for some prime* $p \geq 5$,
(c) $n = pq$ *for some distinct primes* $p, q \geq 3$.

THEOREM 2.3. *Assume that $\omega(F_n) = 3$ and $n = p_1 p_2$, where $p_1 < p_2$ are odd primes. Then $F_{p_1} = q_1$, $F_{p_2} = q_2$ and $F_n = q_1^{a_1} q_2 q_3^{a_3}$, where q_1, q_2, q_3 are distinct primes, q_3 is a primitive divisor of F_n (that is, a prime divisor which does not divide any F_m for $0 < m < n$), $a_3 \geq 1$ and $a_1 \in \{1, 2\}$. Furthermore, $a_1 = 2$ if and only if $q_1 = p_2$.*

Let us also recall the ABC-conjecture. For $n \in \mathbb{Z} \setminus \{0\}$, the *radical* of n is defined by $\mathrm{rad}(n) = \prod_{p|n} p$.

CONJECTURE 2.4 (ABC-conjecture). For each $\epsilon > 0$, there exists $M_\epsilon > 0$ such that whenever $a, b, c \in \mathbb{Z} \setminus \{0\}$ satisfy the conditions

$$\gcd(a, b, c) = 1 \quad \text{and} \quad a + b = c,$$

then

$$\max\{|a|, |b|, |c|\} \leq M_\epsilon \, \mathrm{rad}(abc)^{1+\epsilon}.$$

The next lemma is important for the proof of Theorem 1.2.

LEMMA 2.5 [5, Corollary 2.4]. *Assume that the ABC-conjecture is true. Suppose that $f(x) \in \mathbb{Z}[x]$ is nonconstant and has no repeated roots. Fix $\epsilon > 0$. Then,*

$$\prod_{p|f(m)} p \gg |m|^{\deg f - 1 - \epsilon}. \tag{2.1}$$

We also need the finiteness result for the solutions of the hyperelliptic equation.

THEOREM 2.6 (Baker [1]). *All solutions in integers x, y of the diophantine equation*

$$y^2 = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

where $n \geq 3$, $a_0 \neq 0$, $a_1, \ldots, a_n$ *are integers and where the polynomial on the right-hand side possesses at least three simple zeros, satisfy*

$$\max(|x|, |y|) < \exp\exp\exp\{(n^{10n}\mathcal{H})^{n^2}\},$$

where $\mathcal{H} = \max_{0 \leq j \leq n} |a_j|$.

The next two theorems characterise those perfect powers which are also repdigits.

THEOREM 2.7 (Bugeaud–Mignotte [3]). *Let a and b be integers with* $2 \leq b \leq 10$ *and* $1 \leq a \leq b - 1$. *The integer N with all digits equal to a in base b is not a pure power, except for* $N = 1, 4, 8$ *or* 9, *for* $N = 11111$ *written in base* $b = 3$, *for* $N = 1111$ *written in base* $b = 7$ *and for* $N = 4444$ *written in base* $b = 7$.

THEOREM 2.8 (Ljunggren [6]). *The only integer solutions* (x, n, y) *with* $|x| > 1$, $n > 2$ *and* $y > 0$ *to the exponential equation*

$$\frac{x^n - 1}{x - 1} = y^2$$

are $(x, n, y) = (7, 4, 20)$ *and* $(x, n, y) = (3, 5, 11)$.

3. Proofs

PROOF OF THEOREM 1.1. (a) Since any odd near-perfect number is square, the result follows from Theorem 2.1.

(b) It is easy to show that there are no near-perfect numbers of the form p^k, $k \geq 0$, where p is prime. Suppose that there exists an even near-perfect number of type A belonging to the Fibonacci sequence. For $p = 2$ or $p = 3$, one gets the numbers 18 and 196 which do not belong to the Fibonacci sequence.

Assume now that $p \geq 5$. The equation $F_n = 2^{p-1}(2^p - 1)^2$ implies that $16 \mid F_n$. From this, it follows that $12 \mid n$. Hence, $3 = F_4 \mid F_n = 2^{p-1}(2^p - 1)^2$, which is impossible because $p \geq 5$ and $2^p - 1$ is prime. A similar argument can be used to show that there are no type B or type C near-perfect Fibonacci numbers.

Suppose now that F_n is a near-perfect Fibonacci number with $\omega(F_n) = 3$. Since F_n is even, by Theorems 2.2 and 2.3, $n = 3p$, $p > 3$ and $F_n = 2q_1q_2^{\alpha}$, where $F_p = q_1$ and q_2 is a primitive divisor of F_n and $\alpha \geq 1$. If $q_1 \geq 7$, then

$$2 = \frac{\sigma(F_n)}{F_n} - \frac{d}{F_n} < \frac{3}{2} \cdot \frac{q_1 + 1}{q_1} \cdot \frac{q_2}{q_2 - 1} < \frac{3}{2} \cdot \frac{8}{7} \cdot \frac{11}{10} < 2,$$

which is impossible. Thus, $q_1 = 5$. Then $F_n = F_{15} = 2 \cdot 5 \cdot 61$, which is not a near-perfect number.

(c) Clearly $L_6 = 18$ is a near-perfect number of type A. Using the fact that no member of the Lucas sequence is divisible by 8, it is easy to verify that there are no other near-perfect Lucas numbers with two distinct prime divisors. □

PROOF OF THEOREM 1.2. For odd near-perfect numbers, the result follows unconditionally from Baker's Theorem 2.6. Note that if m is a sufficiently large near-perfect number with $\omega(m) = 2$, then $\mathrm{rad}(m) \ll \sqrt{m}$. Assume $P(n)$ is a near-perfect number with a large value of n, $\deg P = d \geq 3$ and $\omega(P(n)) = 2$. Fix $\epsilon > 0$. Applying (2.1),

$$n^{d-1-\epsilon} \ll \mathrm{rad}(P(n)) \ll n^{d/2},$$

which gives

$$\tfrac{1}{2}d \geq d - 1 - \epsilon$$

or $d \leq 2 + \epsilon < 3$. This contradiction implies the result. □

Proof of Theorem 1.3. Fix $g \geq 2$. Let $U_n = (g^n - 1)/(g - 1)$.

(a) First we consider the near-perfect numbers of type A. We may assume that $g > 2$ (since every binary repdigit is odd). Thus, to find repdigit near-perfect numbers, we need to solve the equation

$$N = aU_n = 2^{p-1}(2^p - 1)^2, \quad \text{where } a \in \{1, \dots, g-1\} \text{ and } 2^p - 1 \text{ is prime.}$$

For the sake of contradiction, assume that $n \geq 3$. It is clear that $2^p - 1 \mid U_n$ for otherwise $(2^p - 1)^2 \mid a$ and then

$$g > a \geq (2^p - 1)^2 > \sqrt{N} \geq \Big(\frac{g^n - 1}{g - 1}\Big)^{1/2} = \sqrt{g^{n-1} + \cdots + 1} > g^{(n-1)/2} \geq g,$$

which is impossible. Thus, $U_n = 2^b(2^p - 1)^2$ or $U_n = 2^b(2^p - 1)$ for some nonnegative integer b. Consider the first case. If g is even, then U_n is odd, therefore $b = 0$. Hence, $U_n = (2^p - 1)^2$ which has no solutions for $n \geq 3$ by Theorem 2.8. Thus, g must be odd and n must be even. Write $n = 2m$. We then get

$$2^b(2^p - 1)^2 = \frac{g^{2m} - 1}{g - 1} = (g^m + 1)\Big(\frac{g^m - 1}{g - 1}\Big).$$

Note that $g^m + 1 > (g^m - 1)/(g - 1)$ and $2^p - 1 > 2^b$. Moreover,

$$\gcd\Big(g^m + 1, \frac{g^m - 1}{g - 1}\Big) \leq 2.$$

Therefore, $g^m + 1 = 2(2^p - 1)^2$ and $(g^m - 1)/(g - 1) = 2^{b-1}$. The latter equation has no solutions in view of our assumption $2 \leq g \leq 10$ and Theorem 2.7.

Now suppose that $U_n = 2^b(2^p - 1)$. If g is even, then U_n is odd, therefore $b = 0$. Hence,

$$a = 2^{p-1}(2^p - 1) > 2^p - 1 = \frac{g^n - 1}{g - 1} = g^{n-1} + \cdots + 1 > g^{n-1} > g,$$

which contradicts the assumption $1 \leq a \leq g - 1$. Thus, g must be odd and n must be even. Put $n = 2m$. We then obtain

$$2^b(2^p - 1) = \frac{g^{2m} - 1}{g - 1} = (g^m + 1)\Big(\frac{g^m - 1}{g - 1}\Big).$$

Since $g^m + 1 > (g^m - 1)/(g - 1)$ and $2^p - 1 > 2^b$, it follows that $2^p - 1 \mid g^m + 1$, and we get $g^m + 1 = 2(2^p - 1)$ and $(g^m - 1)/(g - 1) = 2^{b-1}$. Since $(g^m - 1)/(g - 1)$ is even and g is odd, we see that m is even. Hence, $m = 2m_1$ and so $2(2^p - 1) = g^m + 1 = g^{2m_1} + 1 \equiv 2 \pmod 8$. Then $2^p - 1 \equiv 1 \pmod 4$, but this is impossible for any prime $p \geq 2$. Observe that for this case, we did not use the assumption $2 \leq g \leq 10$.

Suppose now aU_n is near-perfect of type B, where $1 \leq a < g$ and $n \geq 3$. We may write

$$aU_n = 2^{2p-1}(2^p - 1).$$

Suppose first that U_n is odd. Since $1 < U_n \mid 2^{2p-1}(2^p - 1)$, it follows that $U_n = 2^p - 1$. Thus, $a = 2^{2p-1}$. However, since $n \geq 3$,

$$g^2 < U_3 \leq U_n = 2^p - 1 < 2^p, \quad \text{whence} \quad g < 2^{p/2} < 2^{2p-1} = a,$$

which contradicts $a < g$. If U_n is even, then since $U_n = 1 + g + \cdots + g^{n-1}$, it follows that g is odd and n is even. Write $n = 2m$. We have

$$(g^m + 1)\left(\frac{g^m - 1}{g - 1}\right) = U_n \mid 2^{2p-1}(2^p - 1). \tag{3.1}$$

If $2 \mid m$, then $g^m + 1$ has a prime divisor $q \equiv 1 \pmod 4$ contradicting (3.1). Hence, $2 \nmid m$. Thus, U_m is odd. Since $m > 1$ and $2^p - 1$ is prime, (3.1) implies that $U_m = 2^p - 1$. Hence, $g^m + 1 \mid 2^{2p-1}$. So $g^m + 1$ is a power of 2. However,

$$g^m + 1 = (g + 1)(g^{m-1} - g^{m-2} + \cdots + 1).$$

The second factor here is odd, so must equal 1. Thus, $m = 1$, which is a contradiction.

In a similar manner, one can show finiteness of repdigits in base g among near-perfect numbers of type C.

(b) The result is an immediate consequence of Theorem 2.7. □

Theorem 1.3 asserts that repdigit near-perfect numbers of types A, B and C have at most two digits in base g, $2 \leq g \leq 10$. For $g \in \{2, 3, 4, 6\}$, there are no repdigit near-perfect numbers with two distinct prime factors. For $g = 5$, the only repdigit near-perfect numbers with two distinct prime factors are 12, 18 and 24. For $g = 7$, the only repdigit near-perfect numbers with two distinct prime factors are 24 and 40. For $g = 8$, the only repdigit near-perfect number with two distinct prime factors is 18. For $g = 9$, the only repdigit near-perfect numbers with two distinct prime factors are 20 and 40. Finally, in base $g = 10$, the only repdigit near-perfect number with two distinct prime factors is 88.

Acknowledgement

The author would like to thank the anonymous referee for the helpful comments.

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https://doi.org/10.1017/S0004972723000072 Published online by Cambridge University Press