J. Aust. Math. Soc. **78** (2005), 103–107

HEATH g-FUNCTIONS AND METRIZATION

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(Received 13 November 2002; revised 10 September 2003)

Communicated by S. Gadde

Abstract

In this paper, we present some new metrization theorems in terms of Heath g-functions.

2000 *Mathematics subject classification*: primary 54E30, 54E35.
Keywords and phrases: stratifiability, wM-space, Heath g-function, metrizability.

1. Introduction

In this short note we characterize metrizability in terms of Heath g-functions.

Heath in [3] introduced a method of describing a generalized metric property of a topological space (X, τ) by means of a function $g : \mathbb{N} \times X \to \tau$. Hodel, Fletcher, Lindgren and Nagata have modified this method to obtain important new classes of spaces.

A *Heath g-function* [COC-map (= countable open covering map)] for a topological space X is a function g from $\mathbb{N} \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n+1, x) \subseteq g(n, x)$.

It is well known that many important classes of generalized metrizable spaces can be characterized in terms of a Heath g-function. In particular, X is developable [3] ($w\Delta$-space) if and only if X has a Heath g-function g such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all n, then p is a cluster point of the sequence $\langle x_n \rangle$ (then $\langle x_n \rangle$ has a cluster point).

A space X is called a *$w\dot{M}$-space* [4] if and only if X has a Heath g-function g such that if $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all n then $\langle x_n \rangle$ has a cluster point. Let $\mathscr{G}$ be a collection of sets. We define $st(x, \mathscr{G}) = \bigcup\{G \in \mathscr{G} : x \in G\}$ and $st^2(x, \mathscr{G}) = \bigcup_{y \in st(x, \mathscr{G})} st(y, \mathscr{G})$.

In this paper all spaces will be T_0, unless we state otherwise.

2. Main results

First we consider what additional conditions need to be attached to a space already known to be *stratifiable* (conditions (1)–(3) in the next theorem are the axioms of stratifiability [2, Theorem 5.8]) to make it metrizable. To do this, the function $g : \omega \times X \to \mathcal{T}$ can be strengthened to give it some sort of symmetry as shown by the next theorem.

THEOREM 2.1. *A space X is metrizable if and only if there exists a function $g : \omega \times X \to \mathcal{T}$ such that*

(1) $\{x\} = \bigcap_{n\in\omega} g(n, x)$;

(2) *if $y \in g(n, x_n)$ for all n then $x_n \to y$;*

(3) *for any $y \notin H$ closed, $y \notin \overline{\bigcup\{g(x, n) : x \in H\}}$ for some $n \in \omega$;*

(4) *if $y \in g(n, x)$ then $x \in g(n, y)$.*

PROOF. For any metric space we can define g to satisfy the axioms of stratifiability as given in [2, Theorem 5.8]. The fourth condition of the theorem holds because of the symmetry of a metric.

To prove the converse, we assume that without loss of generality,

$$g(n, x) \subseteq g(n + 1, x) \quad \text{for any } x \in X.$$

If this was not the case, we define the function $g'(n, x) = \bigcap_{k\leqslant n} g(k, x)$. Certainly each $g'(n, x)$ is open as the finite intersection of open sets and the axioms for stratifiability still hold since $g'(n, x) \subseteq g(n, x)$ for each $n \in \omega$ and $x \in X$. Notice also that condition (4) remains true when considering these new open sets.

X can be shown to be a T_1 space by showing $\{x\}$ is closed for each $x \in X$. Suppose $y \notin \{x\}$; that is, $y \neq x$. Then we must have some $n \in \omega$ such that $x \notin g(n, y)$, otherwise $x \in \bigcap_{n\in\omega} g(n, y) = \{y\}$ and so the points are not distinct. Hence there is an open neighbourhood of y which does not meet $\{x\}$ and so $\{x\}$ is closed.

For each $n \in \omega$, we define an open cover $\mathcal{G}_n = \{g(n, x) : x \in X\}$. Suppose that x is in some open set U. If we can show that there exists some $n \in \omega$ such that $st^2(x, \mathcal{G}_n) \subseteq U$ then since X is T_0, the space will be metrizable by the Moore Metrization theorem [1].

Firstly we notice that there must exist some $n_0 \in \omega$ such that $g(n_0, x) \subseteq U$, otherwise we can define a sequence of points x_n such that $x_n \in g(n, x) \setminus U$ for each $n \in \omega$. Then by our new symmetry condition, $x \in g(n, x_n)$ for each $n \in \omega$, hence $x_n \to x$, contradicting the fact that $x \in U$ since the points x_n all lie in the closed set $X \setminus U$ and so their limit must also lie in $X \setminus U$. Define $U_1 = g(n_0, x)$ and notice that

$$x \notin X \setminus U_1 \Rightarrow x \notin \overline{\bigcup\{g(n_1, y) : y \in X \setminus U_1\}} = X \setminus U_2, \quad \text{some } n_1 \in \omega,$$

$$x \notin X \setminus U_2 \Rightarrow x \notin \overline{\bigcup\{g(n_2, y) : y \in X \setminus U_2\}} = X \setminus U_3, \quad \text{some } n_2 \in \omega,$$

$$x \notin X \setminus U_3 \Rightarrow x \notin \overline{\bigcup\{g(n_3, y) : y \in X \setminus U_3\}} = X \setminus U_4, \quad \text{some } n_3 \in \omega,$$

$$x \notin X \setminus U_4 \Rightarrow x \notin \overline{\bigcup\{g(n_4, y) : y \in X \setminus U_4\}} = X \setminus U_5, \quad \text{some } n_4 \in \omega.$$

Let $n = \max\{n_0, n_1, n_2, n_3, n_4\}$. We now show that $st(x, \mathcal{G}_n) \subseteq U_3$. If x_2 is any point in $st(x, \mathcal{G}_n)$ then there is some x_1 such that $x \in g(n, x_1)$ and $x_2 \in g(n, x_1)$, hence $x_1 \in g(n, x_2)$ and $x_1 \in g(n_3, x_2)$. If we assume that $x_2 \notin U_3$, then $x_2 \in X \setminus U_3$, so $x_1 \in \overline{\bigcup\{g(n_3, y) : y \in X \setminus U_3\}} = X \setminus U_4$. Similarly, since we have $x_1 \notin U_4$ and $x \in g(n, x_1)$ (hence $x \in g(n_4, x_1)$), then $x \in \overline{\bigcup\{g(n_4, y) : y \in X \setminus U_4\}} = X \setminus U_5$ which contradicts the fact that $x \in U_5$. This means that $x_2 \in U_3$ and so $st(x, \mathcal{G}_n) \subseteq U_3$.

The final stage of the proof is to show that $st^2(x, \mathcal{G}_n) \subseteq U$ by showing that $st^2(x, \mathcal{G}_n) \subseteq U_1$. Consider $x_4 \in st^2(x, \mathcal{G}_n)$. This means we have some point x_3 such that $x_2 \in g(n, x_3)$ and $x_4 \in g(n, x_3)$ (for some $x_2 \in st(x, \mathcal{G}_n)$), hence $x_3 \in g(n, x_4)$ and $x_3 \in g(n_1, x_4)$. If we assume that $x_4 \notin U_1$, then $x_3 \in \overline{\bigcup\{g(n_1, y) : y \in X \setminus U_1\}} = X \setminus U_2$. Similarly, since we have $x_3 \notin U_2$ and $x_2 \in g(n, x_3)$ (hence $x_2 \in g(n_2, x_3)$), then $x_2 \in \overline{\bigcup\{g(n_2, y) : y \in X \setminus U_2\}} = X \setminus U_3$ which contradicts the fact that $x_2 \in U_3$. This means that $x_4 \in U_1$ and so $st^2(x, \mathcal{G}_n) \subseteq U_1 \subseteq U$. □

We now consider some similar results where, instead of requiring convergence of sequences, we only require clustering.

THEOREM 2.2. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *if $\{x, x_n\} \subset g(n, y_n)$ for all n then x is a cluster point of the sequence $\langle x_n \rangle$.*

PROOF. Necessity is clear. For sufficiency: since the condition (2) gives developability to the space X, we need only to prove that X is a regular and wM-space (every regular, developable, wM-space is metrizable [5]). We first prove X is regular. Let $x \in U$ be open in X. Suppose $x_n \in \overline{g(n, x)} - U$ for all $n \in \mathbb{N}$. Then $y_n \in g(n, x) \cap g(n, x_n)$ for each n. So $x \in g(n, y_n)$ and $x_n \in g(n, y_n)$. Therefore, we have $\{x, x_n\} \subset g(n, y_n)$, so x is a cluster point of the sequence $\langle x_n \rangle$. But $x \in U$ is open and $x_n \notin U$ for each n, which contradicts that x is a cluster point of the sequence $\langle x_n \rangle$. Therefore, $\overline{g(n, x)} \subset U$ for some n and X is regular.

Finally, we prove X is a wM-space. Let $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$. Now we want to show that $\langle x_n \rangle$ has a cluster point. Let $p_n \in g(n, z_n) \cap g(n, y_n)$. Since $p_n \in g(n, z_n)$ and $x \in g(n, z_n)$, $\{x, p_n\} \subset g(n, z_n)$. Therefore, x is a cluster point of the sequence $\langle p_n \rangle$. There is a subsequence $\langle m(n) \rangle$ of the sequence $\langle n \rangle$ such that $p_{m(n)} \in g(n, x)$, which implies that $x \in g(n, p_{m(n)})$. We have $p_{m(n)} \in g(n, y_{m(n)})$, so $y_{m(n)} \in g(m(n), p_{m(n)}) \subset g(n, p_{m(n)})$. Now $\{x, y_{m(n)}\} \subset g(n, p_{m(n)})$, so x is a cluster point of the sequence $\langle y_{m(n)} \rangle$. Therefore, there is a subsequence $\langle m(n)(k) \rangle$ of the sequence $\langle m(n) \rangle$ such that $y_{m(n)(k)} \in g(k, x)$ for all k and hence $x \in g(k, y_{m(n)(k)})$ for all k. Since

$$x_{m(n)(k)} \in g(m(n)(k), y_{m(n)(k)}) \subset g(k, y_{m(n)(k)}),$$

$\{x, x_{m(n)(k)}\} \subset g(n, y_{m(n)(k)})$ for all k and hence x is the cluster point of the sequence $\langle x_{m(n)(k)} \rangle$. Therefore, x is the cluster point of the sequence $\langle x_n \rangle$. □

We define $g^1(n, x) = g(n, x)$, and $g^{k+1}(n, x) = \bigcup \{g(n, y) : y \in g^k(n, x)\}$ for $k \geq 1$.

THEOREM 2.3. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *if $x \in g^2(n, x_n)$ for all n then x is a cluster point of the sequence $\langle x_n \rangle$.*

PROOF. Let X be metrizable space with a sequence $\{\mathscr{G}_n\}_{n \in \mathbb{N}}$ of open covers of X satisfying that $\{st^2(x, \mathscr{G}_n)\}$ is a local base at x for all $x \in X$. Put $g(n, x) = st(x, \mathscr{G}_n)$ for each $x \in X$ and for each n. Then g is a COC-map which satisfies (1) and (2), because $g^2(n, x_n) = st^2(x, \mathscr{G}_n)$.

For the converse, we can prove by induction on k that if $\langle x_n \rangle$ is a sequence in X and $x \in X$ with $x_n \in g^k(n, x)$ for all n then x is a cluster point of $\langle x_n \rangle$. From this it follows that if U open with $x \in U$ then there is some n with $g^4(n, x) \subset U$. Put $\mathscr{G}_n = \{g(n, x) : x \in X\}$ for $n \in \mathbb{N}$. Then $\{st^2(x, \mathscr{G}_n)\} = g^4(n, x)$, so $\{\mathscr{G}_n\}_{n \in \mathbb{N}}$ is a sequence of open covers such that $\{st^2(x, \mathscr{G}_n) : n \in \mathbb{N}\}$ is a local base at x for all $x \in X$. Hence, by By the Moore Metrization theorem [1], X is metrizable. This completes the proof. □

COROLLARY 2.4. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *$\{g^2(n, x_n) : n \in \mathbb{N}\}$ is a local basis at x for all $x \in X$.*

THEOREM 2.5. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *$\bigcap_{n \in \mathbb{N}} \overline{g^2(n, x)} = \{x\}$;*
(3) *if $\{x, x_n\} \subset g(n, y_n)$ then the sequence $\langle x_n \rangle$ has a cluster point.*

https://doi.org/10.1017/S1446788700015585 Published online by Cambridge University Press

PROOF. It is easy to prove necessity. To prove sufficiency, we need to prove that x is a cluster point of the sequence $\langle x_n \rangle$. Let q be a cluster point of $\langle x_n \rangle$. Suppose that $q \neq x$. Then there are infinitely many integer $m \geq n$ such that $x_m \in g(n, q)$. Now we have $\{x, x_m\} \subset g(n, y_m)$. By conditions (1) and (2) we get $x \in g(n, y_m)$ and $y_m \in g(n, x_m)$. Therefore, $\{x_m : m \geq n\} \subset g^2(n, x)$, so $q \in \overline{\{x_m : m \geq n\}} \subset \overline{g^2(n, x)}$. Thus $q \in \bigcap_{n \in \mathbb{N}} \overline{g^2(n, x)} = \{x\}$. Hence $q = x$, as required. □

Acknowledgement

The author is grateful to Dr Chris Good for his kind help and valuable comments and suggestions on this paper, especially for the proof of Theorem 2.1. The author also thanks the referee for a number of suggestions which have improved the presentation of this paper.

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https://doi.org/10.1017/S1446788700015585 Published online by Cambridge University Press