AN EXAMPLE CONCERNING EXPOSED POINTS

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In his paper [1] V. L. Klee gave an example of a smooth bounded convex body C, in E^n, with the property that extC is closed and extC - expC is dense in extC. As in [1] extC and expC denote the sets of extreme and exposed points of C respectively. It is the purpose of this note to exhibit a similar example in a general separable Banach space using a direct construction which involves Minkowski summation of convex sets.

We state several lemmas which will be used in establishing our main result. Proofs of these lemmas are quite simple and omitted.

LEMMA 1. Let $\{A_i\}$ be a sequence of sets in a Banach space and suppose $\sum_1^\infty \sup_{A_i} \|a_i\| < \infty$. Then $\sum_1^\infty a_i$ exists for all choices of $a_i \in A_i$.

LEMMA 2. Let $\{A_i\}$ be as in lemma 1 and suppose, further, that all A_i are compact. Then $A = \Sigma A_i$ $= \{a \mid a = \sum_1^\infty a_i, a_i \in A_i\}$ is compact.

LEMMA 3. Let A_0 be closed. Then, if $\{A_i\}$ is as in lemma 2, $A_0 + \Sigma A_i$ is closed.

LEMMA 4. Let $\{A_i\}$ be a sequence of convex sets satisfying the hypothesis of lemma 1. Then ΣA_i is convex.

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LEMMA 5. Let $A = \Sigma A_i$ with A_i satisfying the assumptions of lemma 4. Let f be a continuous linear functional with $\sup_A f(x) = f(a) = \alpha$ for some $a \in A$. Suppose further that $\sup_{A_i} f(x) = f(a_i) = \alpha_i$ for some $a_i \in A_i$ $(i = 1, 2, \ldots)$. Then $f^{-1}(\alpha) \cap A = \Sigma f^{-1}(\alpha_i) \cap A_i$.

LEMMA 6. Let all assumptions of lemma 5 hold and let g be another continuous linear functional with properties analogous to those of f (with β and β_i instead of α and α_i). Then $f^{-1}(\alpha) \cap A \cap g^{-1}(\beta) = \Sigma f^{-1}(\alpha_i) \cap A_i \cap g^{-1}(\beta_i)$.

COROLLARIES:

(1) For A to be strictly convex it is necessary and sufficient that all A_i have the same property.

(2) For A to be smooth it is necessary and sufficient that at least one of the A_i have this property.

LEMMA 7. Let C_0 denote the closed unit ball in a normed linear space X, and let $a \in X$, $a \neq 0$. Let f be a continuous linear functional satisfying (i) $\sup_{C_0} f(x) = f(y)$ for some $y \in FrC_0$; and (ii) $f(a) > 0$. Then a neighborhood U of y in FrC_0 exists such that any linear functional g with $\sup_{C_0} g(x) = g(\xi)$ for some $\xi \in U$ satisfies $g(a) > 0$.

THEOREM. In any separable Banach space X of dimension ≥ 2 there exists a smooth bounded convex body C with the property that $extC$ is closed and $extC - expC$ is dense in $extC$.

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Proof. Let C_0 be the closed unit ball in an arbitrary separable Banach space of dimension ≥ 2. Since a space isomorphism maps $\text{ext}C_0$ and $\text{exp}C_0$ onto corresponding sets in the image of C_0 we may, by a result of M. M. Day [2], assume that C_0 is both smooth and strictly convex. Thus each point of $\text{Fr}C_0$ is exposed and belongs to exactly one hyperplane of support of C_0. Let $\{x_n\}$ be a dense sequence in $\text{Fr}C_0$ and let H_n $(n = 1, 2, \ldots)$ be the hyperplane through the origin with the property that $H_n + x_n$ supports C_0 at x_n. Let I_1 be an arbitrary, but fixed, line segment of length 2, in H_1, having its midpoint at the origin, and suppose I_{n-1} is already defined. Then $I_n = I_i$ if H_n is parallel to H_i for some $i < n$; otherwise I_n is a line segment of length 2 in H_n, having its midpoint at the origin and such that I_n is not contained in $\bigcup_{i=1}^{i=n-1} H_i$. Clearly $\{I_n\}$ is defined for an arbitrary separable Banach space X provided, as assumed, $\dim X \geq 2$. Let $C = C_0 + \sum_1^{\infty} 2^{-n} I_n$. By our lemmas we know that C is a closed smooth convex body. Obviously it is bounded too. It is easily seen that $\text{ext}C$ is closed. (Indeed if $\xi = \xi_0 + \sum_1^{\infty} 2^{-i} \xi_i$ is a point of C with $\xi_0 \in C_0$ and $\xi_i \in I_i$, which is not in $\text{ext}C$, then, clearly, ξ_j is an inner point of I_j for some j; but then $C_0 + \sum_{n \neq j} 2^{-n} I_n + 2^{-j} I'_j$, where I'_j is any open segment in I_j containing ξ_j, is a subset of $C - \text{ext}C$, open in C and containing ξ.) Suppose now that $\eta = \sum_0^{\infty} 2^{-i} \eta_i$ with $\eta_0 \in C_0$ and $\eta_i \in I_i$, $i > 0$, is a point of $\text{exp}C$ and let ε be an arbitrary positive number. The fact that η is exposed can be restated to say

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that a continuous linear functional f exists such that $\sup_{C_0} f(x) = f(\eta_0)$ and $\sup_{I_n} f(x) = f(\eta_n) > 0$ $(n = 1, 2, \ldots)$.
It follows from lemma 7 that a suitably small neighborhood V_k of η_0, on FrC_0, exists so that if $v \in V_k$ and g is a continuous linear functional with $\sup_{C_0} g(x) = g(v)$ then

$\sup_{I_n} g(x) = g(\eta_n)$ $(n = 1, 2, \ldots, k)$.

Let $x_m \in \{x_n\} \cap V_k$ with $\|\eta_0 - x_m\| < \frac{\varepsilon}{2}$ and let k be taken large enough so as to have $2^{-k+2} < \varepsilon$. By our construction $H_m + x_m$ is a support hyperplane of C which contains $I_m + x_m$. Since C is smooth no point of $K = C \cap (H_m + x_m)$ is exposed. Now a point $\xi \in K$ is clearly of the form

$$x_m + \sum_{i}^{k} 2^{-i}\eta_i + \sum_{k+1}^{\infty} 2^{-i}\xi_i$$

for some $\xi_i \in I_i$ $(i = k+1, k+2, \ldots)$. We have

$$\|\eta - \xi\| = \|\eta_0 + \sum_{1}^{\infty} 2^{-i}\eta_i - x_m - \sum_{1}^{k} 2^{-i}\eta_i - \sum_{k+1}^{\infty} 2^{-i}\xi_i\|$$

$$\leq \|\eta_0 - x_m\| + \|\sum_{k+1}^{\infty} 2^{-i}(\eta_i - \xi_i)\|$$

$$< \frac{\varepsilon}{2} + 2^{-k+1} < \varepsilon .$$

Since K is, again by our construction, a compact convex set (a translate of $\sum_{k+1}^{m} 2^{-i}\xi_i$), there exists a point $z \in \mathrm{ext}K$. Clearly z is an extreme nonexposed point of C of distance $< \varepsilon$ from η. This shows that extC - expC is dense in extC, completing the proof of the theorem.

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REFERENCES

1. V. L. Klee, Extremal structure of convex sets II, Math. Zeitscrift 69 (1958), 90-114.

2. M. M. Day, Strict convexity and smoothness of normed spaces, Trans. Am. Math. Soc. 78 (1955), 516-528.

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